# Ubiquity of algebraic Stieltjes transforms

# (and free coin-tossing)

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Workshop on Free Probability, Random Matrices and Planar Algebras

Fields Institute, Toronto

September 2007

The limiting empirical distribution of eigenvalues $\mu$ for many random matrix models has an algebraic Stieltjes transform $S(\lambda) = \int \frac{\mu(dx)}{\lambda - x}$.

Why algebraic?

What can be proved about this in general?

## Talk outline

### Part I

Some RM models yielding an algebraic Stieltjes transform.

### Part II

The algebraicity criterion

### Part III

Free coin-tossing

Part I and II concern joint work with O. Zeitouni.

**Part I**

Some RM models yielding an algebraic Stieltjes transform

(a) The basic band matrix model

(b) An enhancement allowing finite-range dependence

(c) Free convolutions

## The basic band matrix model

Let $\mathcal{F}$ be an algebra of sets in $[0,1]$ generated by finitely many intervals. Let $\sigma : [0,1]^2 \to [0,\infty)$ be $\mathcal{F} \times \mathcal{F}$-measurable and symmetric: $\sigma(x,y) = \sigma(y,x)$. Let $[Z_{ij}]_{1 \leq i \leq j < \infty}$ be i.i.d. with moments of all orders.

Consider the $N$-by-$N$ random symmetric matrix $X^{(N)}$ with entries

$$X_{ij}^{(N)} = \underbrace{\frac{Z_{ij}\sqrt{\sigma\left(\frac{i}{N}, \frac{j}{N}\right)}}{\sqrt{N}} \text{ if } i \leq j, \quad \frac{Z_{ji}\sqrt{\sigma\left(\frac{i}{N}, \frac{j}{N}\right)}}{\sqrt{N}} \text{ if } j < i.}$$

Let's call $\{X^{(N)}\}_{N=1}^{\infty}$ the **basic band matrix model**. Put

$$\lambda_1^{(N)} \leq \cdots \leq \lambda_N^{(1)} : \text{ eigenvalues of } X^{(N)},$$

$$L^{(N)} = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i^{(N)}} : \text{ empirical distribution.}$$

## The law of large numbers

Through work of many authors, it has long been known that in this and similar models with weaker moment assumptions

$$L^{(N)} \Rightarrow_{N\to\infty} \mu$$

where

$$\int_0^1 \frac{\sigma(x,y)dy}{\lambda - \Psi(y,\lambda)} = \Psi(x,\lambda),$$

$$S(\lambda) = \int \frac{\mu(dx)}{\lambda - x} = \int_0^1 \Psi(x,\lambda)dx.$$

Here $\Psi(x,\lambda)$ is $\mathcal{F}$-measurable in $x$, defined for complex numbers $\lambda$ such that $|\lambda| \gg 0$, and depends analytically on $\lambda$.

I emphasize that these sorts of LLN results and functional equations are well-known.

For many references and background on this type of model, as well as for generalizations, and development in the CLT direction, see:

G. Anderson, O. Zeitouni:

**A CLT for a band matrix model.**

PTRF **134**(2006),283-338.

In any relatively simple case of the basic band matrix model you can prove $S(\lambda)$ is algebraic "by hand". To say that $S(\lambda)$ is **algebraic** means there exists some two-variable polynomial equation $F(\lambda, S(\lambda)) \equiv 0$ where $F(x,y)$ does not vanish identically.

Here are the two most important examples:

- If $\sigma \equiv 1$, then $X^{(N)}$ is a Wigner matrix, $\mu$ is the semicircle law with density $\sqrt{4-x^2}\mathbf{1}_{|x|\leq 2}$ with respect to Lebesgue measure and

$$(\lambda - S(\lambda))S(\lambda) = 1.$$

- If

$$\sigma = \mathbf{1}_{((a,1]\times[0,a])\cup([0,a]\times(a,1])}$$

for some $0 < a < 1$, then $\mu$ is related to the Pastur-Marchenko law by a simple transformation, and again $S(\lambda)$ is algebraic.

I haven't stated Theorem 1 yet, but here is one of its corollaries:

**Corollary 1**(G.A. and O. Zeitouni) *The Stieltjes transform* $S(\lambda)$ *associated to the basic band matrix model is algebraic. Moreover, for each fixed* $x$*,* $\Psi(x, \lambda)$ *is an algebraic function of* $\lambda$*.*

In generality, algebraicity of $S(\lambda)$ and $\Psi(x, \lambda)$ in the basic band matrix model seems to be new, even though the model and the equations have been known a long time.

**Remarks**

1. We cannot produce the polynomial equation satisfied by $\lambda$ and $S(\lambda)$ explicitly. The proof of Theorem 1 is "soft", i. e., nonconstructive.

2. Algebraicity of $S(\lambda)$ implies by the inversion formula for Stieltjes transforms a considerable degree of regularity of $\mu$.

## The enhanced band matrix model

In recent work

G.A. and O. Zeitouni: **A law of large numbers for finite-range dependent random matrices** (arXiv:math/0609364)

we considered a generalization of the basic band matrix model in which dependence of entries is allowed at short range. Let's refer to this by the catchphrase "enhanced band matrix model."

For example, starting with a Wigner matrix $X$ and then replacing each entry with the sum of its four neighbors to northwest, northeast, southeast, southwest,

$$\begin{array}{cc} \searrow & \swarrow \\ & \\ \nearrow & \nwarrow \end{array}, \quad Y_{i,j} = X_{i+1,j+1} + X_{i-1,j+1} + X_{i+1,j-1} + X_{i-1,j-1}$$

one obtains a matrix $Y$ of the class we studied. More generally one may apply any finitely supported "filtering procedure" with suitable symmetry to a Wigner matrix to get a random matrix belonging to the class of enhanced band matrix models.

See the preprint of G.A. and O.Z. for details of the model and for its LLN.

**A few references**

RM models with dependent entries have been considered by many authors. Here are just a few references on the subject, coming from several rather different directions:

A. Boutet de Monvel, A. Khorunzhy, and V. Vasilchuck, *Markov Proc. Rel. Fields* **2** (1996), pp. 607–636.

D. Shlyakhtenko, *Int. Math. Res. Notices* **20** (1996), pp. 1013–1025.

R. Speicher, *Mem. Amer. Math. Soc.* **132** (1998).

W. Hachem, P. Loubaton, J. Najim, *Markov Proc. Rel. Fields* **11** (2005), pp. 629–648.

See the preprint of G.A. and O.Z. for more references.

## Equations for the Stieltjes transform

We omit further details of the enhanced band matrix model, but we do want to look at the equations for $S(\lambda)$ in detail.

These are as follows:

$$\int_0^1 \int_0^1 \frac{s(t,\theta;t',\theta')}{\lambda - \Psi(t',\theta';\lambda)} dt' d\theta' = \Psi(t,\theta;\lambda),$$

$$S(\lambda) = \int_0^1 \int_0^1 \frac{dt d\theta}{\lambda - \Psi(t,\theta;\lambda)} = \int \frac{\mu(dx)}{\lambda - x}.$$

Superficially this system of equations looks almost exactly the same as in the "basic" case. The difference arises from the type of function $s$ allowed here.

## The variance profiling function $s$

- As before: the function $s : [0,1]^4 \to [0,\infty)$ is required to be symmetric: $s(x,\theta;x',\theta') = s(x',\theta';x,\theta)$.

- As before: let $\mathcal{F}$ be the algebra of sets generated by some finite collection of subintervals of $[0,1]$. The function $s$ is required to be $\mathcal{F}$-measurable in each variable $x$ and $x'$.

- In contrast to the previous setup: $s$ is required to admit finite (terminating) expansions in its variables $\theta$ and $\theta$' in powers of $\exp(2\pi i\theta)$ and $\exp(2\pi i\theta')$.

The latter type of information "codes" the finite-range correlations of matrix entries allowed in the model.

**Corollary 2**(G.A. and O. Zeitouni) *The Stieltjes transform* $S(\lambda)$ *associated to the enhanced band matrix model is algebraic.*

This is another corollary to the as-of-yet unstated Theorem 1. The function $\Psi(x, y; \lambda)$ also turns out to be algebraic in a suitable sense.

Corollary 2 includes Corollary 1 as a special case. Actually Theorem 1 was a byproduct of the proof Corollary 2.

The method codified by Theorem 1 appears to be widely applicable.

## Free convolutions

Many examples of random variables with algebraic Stieltjes transform are produced by applying the theories of $R$- and $S$-transforms to sums (resp., products) of freely independent random variables with simple distributions, e.g., Bernoulli random variables. We mention this just in passing now.

We discuss an example moving beyond $R$- and $S$-transforms later in the talk in some detail.

## Part II

The algebraicity criterion

(a) Germs

(b) Formulation of the result

(c) Hints of proof

(d) Application to the basic band matrix model

## Germs of holomorphic functions

Define a **function element** $\phi : U \to \mathbb{C}$ to be a holomorphic function defined in an open neighborhood $U$ of the origin in $\mathbb{C}^n$. Declare two function elements $\phi_i : U_i \to \mathbb{C}$ to be **equivalent** if there exists some open neighborhood of the origin $V \subset U_1 \cap U_2$ such that $\phi_1|_V = \phi_2|_V$. Equivalence classes under this relation we call **germs** of holomorphic functions at the origin in $\mathbb{C}^n$. We define $\mathcal{O}_n$ to be the set of germs of holomorphic functions at the origin.

## Composition of germs

Given $\phi_1, \ldots, \phi_N \in \mathcal{O}_n$ vanishing at the origin and $F \in \mathcal{O}_N$, the composition

$$F(\phi_1, \ldots, \phi_N) \in \mathcal{O}_n$$

makes sense as a germ, even if not globally.

**Remarks**

1. If a holomorphic function $\phi$ defined in a connected open neighborhood $U$ of the origin has identically vanishing germ at the origin, then $\phi$ vanishes identically in $U$. Essentially no information is lost in passing to the germ.

2. In fact $\mathcal{O}_n$ is a ring, because addition and multiplication of function elements is compatible with equivalence.

3. The main advantage of working with germs is to make it easy to talk about functional composition without having to make qualifications about ranges and domains of functions.

## Polynomial germs

Let $z_1, \ldots, z_n \in \mathcal{O}_n$ denote the (germs of) the standard coordinates in $\mathbb{C}^n$. A **polynomial germ** is an element of $\mathcal{O}_n$ represented by a polynomial in $z_1, \ldots, z_n$. Let $\mathcal{O}_n^{\mathrm{poly}}$ denote the ring of such. By Remark 1 this is just a copy of the usual polynomial ring in $n$ variables embedded in $\mathcal{O}_n$.

## Rational germs

A **rational germ** is an element of $\mathcal{O}_n$ represented by a quotient of polynomial germs where the denominator does not vanish at the origin. Let $\mathcal{O}_n^{\mathrm{rat}} \subset \mathcal{O}_n$ denote the ring of such. By Remark 1 this is just a copy of the subring of the ring of rational functions in $n$ variables consisting of functions which have well-defined finite values at the origin.

## Algebraic germs

We say that $\phi \in \mathcal{O}_n$ is **algebraic** if there exists $F \in \mathcal{O}_n^{\text{poly}}$ not identically vanishing such that $F(z_1, \ldots, z_n, \phi) \equiv 0$, and we denote the set of such by $\mathcal{O}_n^{\text{alg}}$. In fact $\mathcal{O}_n^{\text{alg}}$ is a ring, i. e., closed under addition and multiplication. Existence of $F \in \mathcal{O}_n^{\text{rat}}$ not identically vanishing such that $F(z_1, \ldots, z_n, \phi) \equiv 0$ is sufficient for algebraicity.

## Composition stability of germ classes

If $\phi_1, \ldots, \phi_N \in \mathcal{O}_n$ and $F \in \mathcal{O}_N$ are polynomial (resp., rational, algebraic) germs, then $F(\phi_1, \ldots, \phi_N) \in \mathcal{O}_n$ is likewise a polynomial (resp., rational, algebraic) germ.

The "soft" method is summarized as follows:

**Theorem 1** (G. A. and O. Zeitouni)
*Let $F_1, \ldots, F_N \in \mathcal{O}_{n+N}^{\mathrm{rat}}$ be germs vanishing at the origin such that*

$$\left(\det_{i,j=1}^{N} \frac{\partial F_i}{\partial z_{j+n}}\right)(0) \neq 0. \tag{1}$$

*Let*

$$\phi_1, \ldots, \phi_N \in \mathcal{O}_n$$

*be germs vanishing at the origin and satisfying*

$$F_i(z_1, \ldots, z_n, \phi_1, \ldots, \phi_N) \equiv 0 \tag{2}$$

*for $i = 1, \ldots, N$. Then*

$$\phi_1, \ldots, \phi_N \in \mathcal{O}_n^{\mathrm{alg}}.$$

Theorem 1 is a reformulation in slightly more sophisticated language of the result proved in the preprint arXiv:math/0609364 of G.A. and O.Z. mentioned above.

## Remarks

1. Given $F_1, \ldots, F_N \in \mathcal{O}_{n+N}$ vanishing at the origin and satisfying condition (1), the implicit function theorem in its version for multi-variable holomorphic functions implies existence and uniqueness of $\phi_1, \ldots, \phi_N \in \mathcal{O}_n$ vanishing at the origin and satisfying condition (2). Theorem 1 should be viewed as a sort of amplification of the implicit function theorem.

2. As a fact of commutative algebra, the novelty of Theorem 1 is absolutely zero. The techniques to prove it can be found in standard graduate level commutative algebra texts, e.g., Atiyah-Macdonald and Matsumura. The point is the scope of applications on the RMT side, not the fact of algebra itself.

3. A trivial but necessary remark: in applications usually one has some germs $\psi_1, \ldots, \psi_N$ which do not vanish at the origin, and one must replace them by $\phi_i = \psi_i - \psi_i(0)$ to use the theorem.

## Brief hint of proof

What we need to show that is that the equations

$$F_i(z_1, \ldots, z_n, \phi_1, \ldots, \phi_N) \equiv 0 \;\; (i = 1, \ldots, N)$$

have consequences

$$G_i(z_1, \ldots, z_n, \phi_i) \equiv 0 \;\; (i = 1, \ldots, N)$$

where $0 \neq G_i \in \mathcal{O}_{N+1}^{\text{rat}}$. Showing existence of the special ring elements $G_i$ (without making them explicit) is a task handled well by the classical dimension theory of noetherian local rings. We omit further details.

## Scholium

If we want to prove that a given Stieltjes tranform $S(\lambda)$ is algebraic, we write

$$\Sigma(z) = zS(1/z),$$

thus defining (since the singularity at $z = 0$ is removable) a germ $\Sigma \in \mathcal{O}_1$, and then we try to use the germ machinery plus Theorem 1 to show the equivalent statement $\Sigma \in \mathcal{O}_1^{\mathrm{alg}}$. Even though the goal is to prove the one-dimensional statement "$\Sigma \in \mathcal{O}_1^{\mathrm{alg}}$", to use Theorem 1 we have to work in the multidimensional environment of $\mathcal{O}_n^{\mathrm{rat}}$, $\mathcal{O}_n^{\mathrm{alg}}$ and $\mathcal{O}_n$. But this is natural: relatively simple systems of rational equations (often merely quadratic polynomial equations) holding among large numbers of variables seem to be what's on offer in this game.

## Application to the basic band matrix model

After some evident manipulations we omit, proving algebraicity of $S(\lambda)$ in the basic band matrix model comes down to to proving the following essentially algebraic fact:

**Proposition 1** *Given an $n$-by-$n$ matrix*

$$\sigma = [\sigma_{ij}]_{i,j=1}^{n}$$

*of complex numbers and*

$$\psi_1, \ldots, \psi_n \in \mathcal{O}_1$$

*vanishing at the origin such that*

$$\psi_i(z) = z^2 \sum_{j=1}^{n} \frac{\sigma_{ij}}{1 - \psi_j(z)},$$

*necessarily*

$$\psi_1, \ldots, \psi_n \in \mathcal{O}_1^{\mathrm{alg}}.$$

**Proof** Take

$$F_i(z_1, \ldots, z_{n+1}) = z_{i+1} - z_1^2 \sum_{j=1}^{n} \frac{\sigma_{ij}}{1 - z_{j+1}}$$

for $i = 1, \ldots, N$, noting that

$$\det_{i,j=1}^{n} \frac{\partial F_i}{\partial z_{j+1}}(0) = 1,$$

and

$$F_i(z, \psi_1(z), \ldots, \psi_n(z)) \equiv 0$$

for $i = 1, \ldots, N$. Now apply Theorem 1. QED

The application to the enhanced band matrix model is similar but more complicated. We omit those details.

# Part III

Free coin-tossing

To make the case for ubiquity of algebraic Stieltjes transforms, we investigate another relatively simple model, but this time one of a character quite different from the basic band matrix model.

More precisely we are going to investigate the distribution of an arbitrary noncommutative polynomial in freely independent Bernoulli random variables of parameter $1/2$. (For technical reasons it will be more convenient to study random variables of the form $2X - 1$ where $X$ is Bernoulli(1/2).) We will show that the Stieltjes transform of such is algebraic by applying Theorem 1. The inspiration to look at such a thing we got from a paper of B. Collins (arXiv:math/0406560).

We will find methods of random walk on groups and trees useful.

## The free coin-tossing model

Let

$$F = F(\xi_1, \ldots, \xi_\ell)$$

be an arbitrary polynomial in noncommuting variables $\xi_1, \ldots, \xi_\ell$ with complex coefficients. Let

$$U_1 = U_1^{(N)}, \ldots, U_\ell = U_\ell^{(N)}$$

be independent $2N$-by-$2N$ Haar-distributed unitary matrices. Let

$$D = D^{(N)} = \mathrm{diag}(\underbrace{1, \ldots, 1}_{N}, \underbrace{-1, \ldots, -1}_{N}).$$

Consider the random matrix

$$X = X^{(N)} = F(U_1 D U_1^*, \ldots, U_\ell D U_\ell^*).$$

We are interested in the spectrum of $X$ in the limit as $N \to \infty$.

## Formulation of the result

Now consider the formal Stieltjes transform

$$\Sigma(z) \;=\; 1+\sum_{n=1}^{\infty}\left(\lim_{N\to\infty}\frac{1}{2N}\mathrm{trace}\,(X^{(N)})^n\right)z^n \in \mathbb{C}[[z]].$$

From the theory of asymptotic freeness we know the limits on $N$ exist, and so at least $\Sigma$ makes sense as a formal power series.

We claim the following:

**Theorem 2**. $\Sigma \in \mathcal{O}_1^{\mathrm{alg}}$.

A proof will be sketched.

We begin with a reduction that gets rid of the random matrices.

## Group-theoretical description of $\Sigma$

Let

$$G = \langle x_1, \ldots, x_\ell \mid x_1^2 = \cdots = x_\ell^2 = 1_G \rangle.$$

Let

$$\mathrm{trace}_G = \left( \sum_g c_g G \mapsto c_{1_G} \right) : \mathbb{C}[G] \to \mathbb{C}.$$

Using asymptotic freeness, we have

$$\Sigma(z) = 1 + \sum_{n=1}^{\infty} (\mathrm{trace}_G\, F(x_1, \ldots, x_\ell)^n) z^n.$$

Thus Theorem 2 is really "just algebra".

Note also that

$$\mathrm{trace}_G\, F(x_1, \ldots, x_\ell)^n = O(\rho^n)$$

for some $\rho > 0$, and hence $\Sigma \in \mathcal{O}_1$.

## The Cayley tree

The Cayley graph for $G$ with respect to the generators $x_1, \ldots, x_\ell$ is an $\ell$-valent tree, with each edge labelled by a letter $x_i$. In this tree the distance of $g \in G$ from the origin equals the length of the shortest word in the $x_i$ representing $g$. We will approach the problem of proving $\Sigma \in \mathcal{O}_1^{\mathrm{alg}}$ using the intuitions of random walk on trees and groups.

## The transition function $p$

Define $p : G \to \mathbb{C}$ by the expansion

$$\sum_{g \in G} p(g) g = F(x_1, \ldots, x_\ell) \in \mathbb{C}[G].$$

Without loss of generality we assume that

$$p(1_G) = 0,$$

which simplifies many formulas below.

## The punctured ball $S$

Let $S$ be a ball of some fixed finite radius about the origin $1_G$ in the Cayley tree from which $1_G$ is excluded. Choose $S$ large enough so that the transition function $p$ is supported in $S$.

## $(V, S)$-paths

Let $V \subset G$ be any set. We define a $(V, S)$-**path** of length $n$ to be a sequence $v_0, \ldots, v_n \in G$ such that

$$v_0, \ldots, v_{n-1} \in V, \quad v_n \in V^c, \quad v_0^{-1} v_1, \ldots, v_{n-1}^{-1} v_n \in S.$$

In other words, $(V, S)$-paths are walks in $G$ which take steps in $S$ and which stay in $V$ until exiting at the last step.

## Generating functions

For any subset $V \subset G$ and $g_1, g_2 \in G$ we define

$$\Phi^V_{g_1,g_2} = \Phi^V_{g_1,g_2}(z) = \sum_{n=2}^{\infty} \left( \sum_{(v_0,\ldots,v_n)} \prod_{i=1}^{n} p(v_{i-1}^{-1} v_i) \right) z^{n-2},$$

where the sum is extended over $(V, S)$-paths of length $n \geq 2$. Crucially, we have the equivariance

$$\Phi^{gV}_{gg_1,gg_2} = \Phi^V_{g_1,g_2}$$

for all $g, g_1, g_2 \in G$.

## Expression of $\Sigma$ in terms of $\Phi$

Trivially, we have

$$1-\Sigma^{-1} \;=\; z^2 \sum_s p(s)\left(p(s^{-1}) + z\Phi_{s,1_G}^{G\setminus\{1_G\}}\right).$$

This simply says that each "loop" in $G$ of positive length contributing to $\Sigma$ can be broken into excursions away from the origin. Our assumption $p(1_G)=0$ simplifies this formula since walks starting at the origin must initially step away.

## Remark

Notice that all the information on algebraicity of $\Sigma$ is present in $\Phi$. We concentrate on $\Phi^V$ for well-chosen sets $V$ hereafter.

## Matrix formalism

The formulas we need to handle become bearable only if we can express them in terms of finitely supported $G$-by-$G$ matrices. Here are the rules of the latter game.

Given a $G$-by-$G$ matrix $X$ and a set $T \subset G \times G$, put

$$(X[T])_{g_1,g_2} = \begin{cases} X_{g_1,g_2} & \text{if } (g_1, g_2) \in T, \\ 0 & \text{otherwise.} \end{cases}$$

Given also $g \in G$ put

$$(X^g)_{g_1,g_2} = X_{g^{-1}g_1,g^{-1}g_2}.$$

Note that

$$(X[T])^g = (X^g)[(g \times g)T].$$

We multiply finitely supported $G$-by-$G$ matrices by the usual rule.

## Special matrices

For all $g, h \in G$ put

$$\Theta(g,h) = \begin{cases} p(g^{-1}h) & \text{if } g^{-1}h \in S, \\ 0 & \text{otherwise,} \end{cases}$$

thus defining a $G$-by-$G$ matrix $\Theta$ with complex number entries. Note that

$$\Theta^a = \Theta$$

for all $a \in G$. Given a set $V \subset G$, let $\Phi^V$ be the $G$-by-$G$ matrix with entries $\Phi^V_{g_1,g_2} \in \mathcal{O}_1$. Crucially, we have the symmetry

$$\Phi^{gV}[gV_1 \times gV_2] = (\Phi^V[V_1 \times V_2])^g$$

holding for all $g \in G$ and sets $V_1, V_2 \subset G$.

## Boundaries

Given $V \subset G$ let

$$\begin{aligned} \partial V &= V \cap \left( \bigcup_{s \in S} V^c s \right) = V \cap \left( \bigcup_{s \in S} V^c s^{-1} \right), \\ \partial V^c &= \partial(V^c). \end{aligned}$$

The set $\partial V$ is the "foyer" through which every path contributing to $\Phi^V_{g_1,g_2}$ must exit, and $\partial V^c$ is the set in which every such path ends.

Note the equivariance

$$\partial(gV) = g(\partial V) \quad (g \in G).$$

Note that

$$\#V = 1 \Rightarrow \begin{cases} \partial V = 1, \\ \partial V^c = VS, \\ \Phi^V \equiv 0. \end{cases}$$

This trivial remark simplifies formulas below.

## Self-similarity

We have

$$G \setminus \{1_G\} = \coprod_{i=1}^{\ell} G_i$$

where $G_i$ is the set of words beginning with $x_i$. We have

$$\partial G_i = G_i \cap S, \ \ \partial G_i^c \subset \bigcup_{j \neq i} \partial G_j.$$

For suitable finite sets $A_{ij} \subset G$ we have

$$G_i \setminus \partial G_i = \coprod_{j=1}^{\ell} \coprod_{a \in A_{ij}} a G_j,$$

which is easy to see from the Cayley tree point of view. This **self-similarity** is the key to getting recursions, as in many problems about random walks on groups and trees.

## Glueing equation

Let $V \subset G$ be a subset, expressed as a finite disjoint union

$$V = \coprod_i V_i.$$

Assume that

$$\partial V^c \cup (\cup \partial V_i)$$

is a finite set. We have the **glueing equation**

$$\Phi^V \left[(\cup_i \partial V_i) \times \partial V^c\right] - \Sigma_i \Phi^{V_i} \left[\partial V_i \times (\partial V_i^c \setminus V)\right]$$

$$= \left(\Sigma_i \left(\Theta + z\Phi^{V_i}\right) \left[\partial V_i \times \partial V_i^c\right]\right) \left(\Theta + z\Phi^V\right) \left[(\cup_i \partial V_i) \times \partial V^c\right].$$

The proof is easy. Break each $(V, S)$-path of length $\geq 2$ into "legs" which are $(V_i, S)$-paths. Use the fact that entries to $V_i$ must come by way of $\partial V_i$ and exits from $V$ must go by way of $\partial V^c$. This is where we make the heaviest use of random walk technology.

## Relating $\Phi^{G\setminus\{1_G\}}$ to $\Phi^{G_i}$

We have the following special case of the glueing equation:

$$\Phi^{G\setminus\{1_G\}}\,[S \times \{1_G\}] - \Sigma_i \Phi^{G_i}\,[\partial G_i \times (\partial G_i^c \cap \{1_G\})]$$

$$= \left(\Sigma_i \left(\Theta + z\Phi^{G_i}\right)[\partial G_i \times \partial G_i^c]\right)\left(\Theta + z\Phi^{G\setminus\{1_G\}}\right)[S \times \{1_G\}]$$

Recall that $S = \bigcup_i \partial G_i$.

## Remark

We have seen that algebraicity of $\Sigma$ reduces to algebraicity of entries of $\Phi^{G\setminus\{1_G\}}[S \times \{1_G\}]$. The preceding equation makes a further reduction to algebraicity of entries of $\Phi^{G_i}[\partial G_i \times \partial G_i^c]$.

## The master recursion

Put

$$
\begin{aligned}
T_i &= \{(g, gs) \mid g \in \partial G_i, \;\; s \in S\}, \\
\partial\partial G_i &= \partial G_i \cup (\cup_j \cup_{a \in A_{ij}} a \partial G_j).
\end{aligned}
$$

We have the **master recursion**

$$
\begin{aligned}
&\Phi^{G_i}[\partial\partial G_i \times \partial G_i^c] - \sum_j \sum_{a \in A_{ij}} (\Phi^{G_j}\left[\partial G_j \times (\partial G_j^c \setminus a^{-1} G_i)\right])^a \\
= &\left( \Theta[T_i] + \sum_j \sum_{a \in A_{ij}} \left( \left( \Theta + z\Phi^{G_j} \right) [\partial G_j \times \partial G_j^c] \right)^a \right) \\
&\times \left( \Theta + z\Phi^{G_i} \right) [\partial\partial G_i \times \partial G_i^c].
\end{aligned}
$$

by glueing and self-similarity. These quadratic equations for the entries of the finitely supported matrices

$$
\Phi^{G_i}[\partial\partial G_i \times \partial G_i^c]
$$

close and they are after only very slight massage of the form to which Theorem 1 applies.

## End of the proof of Theorem 2

Working backwards through the glueing equations, one checks that $\Sigma$ can be expressed as a rational function of entries of the matrices $\Phi^{G_i}\left[\partial\partial G_i \times \partial G_i^c\right]$, and hence indeed $\Sigma \in \mathcal{O}_1^{\mathrm{alg}}$. QED

## Concluding remark

The "arboreal" style of analysis for proving Theorem 2 should be adaptable to proving many other statements which one feels are reasonable, e.g., that an arbitrary noncommutative polynomial in independent Wigner matrices has (asymptotically) an algebraic formal Stieltjes transform.