Estimates for the X-ray transform

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f - a function on $\mathbb{R}^d$

l - a line in $\mathbb{R}^d$

The X-ray transform of f at l:

$$X[f](l) = \int_l f(y)\, dy$$

Will assume f is supported on a fixed ball

Part 1: Unrestricted directions

$\mathbb{S}^{d-1}$ - unit sphere in $\mathbb{R}^d$

$\xi \in \mathbb{S}^{d-1}$, $x \in \xi^\perp$

$l(\xi, x)$ = the line in the direction ξ passing through x

Mixed-norms:

$$\|X[f]\|_{L^q(L^r)} := \left(\int_{\mathbb{S}^{d-1}} \left(\int_{\xi^\perp} |X[f](l(\xi, x))|^r \, dx \right)^{\frac{q}{r}} \, d\xi \right)^{\frac{1}{q}}$$

Want estimates:

$$\|X[f]\|_{L^q(L^r)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

Conjecture

If $1 \le p < d$, $r = \frac{(d-1)p}{d-p}$, and $q \le (d-1)p'$

$$\|X[f]\|_{L^q(L^r)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

For local problem, may also interpolate with trivial $L^\infty \to L^\infty(L^\infty)$ estimate to obtain "non-sharp (p, r)" type estimates.

The (d, k) Kakeya problem

Definition

$E \subset \mathbb{R}^d$ is a (d, k) set if E contains a translate of every k-dimensional disc of diameter 1.

$(d, 1)$ set $\Leftrightarrow$ Kakeya set

Question

How small can (d, k) sets be?

Example

Any ball of diameter 1 is a (d, k) set for every d, k.

$k = 1$

Besicovitch - There are $(d, 1)$ sets of Lebesgue measure zero.

Conjecture

Every $(d, 1)$ set has Minkowski/Hausdorff dimension d

Davies - true when $d = 2$

Open for $d > 2$ (WR - Katz/Łaba/Tao)

$k > 1$

Conjecture

(d, k) sets have positive Lebesgue measure, $k > 1$.

Marstrand $\quad (d, k) = (3, 2)$

Falconer $\quad d - k < k$

Bourgain $\quad d - k \leq 2^{k-1}, \quad k \geq 2.$

L^p estimates vs. dimension

Would have:

$$\|X[f]\|_{L^q(L^\infty)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$
$$\Downarrow$$
$(d, 1)$ sets have positive Lebesgue measure

$X_\delta[f](l) :=$ average over δ-neighborhood of l

$$\|X_\delta[f]\|_{L^q(L^\infty)} \lesssim \delta^{-\frac{\alpha}{p}} \|f\|_{L^p(\mathbb{R}^d)}$$
$$\Downarrow$$
$(d, 1)$ sets have Hdim $\geq d - \alpha$

$r < \infty$

$$\|X[f]\|_{L^q(L^r)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

$$\Downarrow$$

$$\|X_\delta[f]\|_{L^q(L^\infty)} \lesssim \delta^{-\frac{\alpha}{p}} \|f\|_{L^p(\mathbb{R}^d)}$$
$$\alpha = (d-1)\frac{p}{r}$$

$$\Downarrow$$

$(d,1)$ sets have Hdim $\geq d - (d-1)\frac{p}{r}$

Mixed-norm Kakeya sets

$0 \le \gamma \le d - 1$

Definition

$E \subset \mathbb{R}^d$ is a γ-Kakeya set if for each $\xi \in \mathbb{S}^{d-1}$ there is a nonempty $H_\xi \subset \xi^\perp$ with $\mathrm{Hdim}(H_\xi) \ge \gamma$ such that for every $x \in H_\xi$, E contains a segment of $l(\xi, x)$.

Then

$$\|X_\delta[f]\|_{L^q(L^r)} \lesssim \delta^{-\frac{\alpha}{p}} \|f\|_{L^p(\mathbb{R}^d)}$$
$$\Downarrow$$
$$\gamma\text{-Kakeya sets have Hdim} \ge d - (d - 1 - \gamma)\frac{p}{r} - \alpha$$

Estimates for (d, k) sets

$$\|X[f]\|_{L^{q_d}(L^{r_d})} \lesssim \|f\|_{L^{p_d}(\mathbb{R}^d)}$$

$\frac{r_d}{p_d}$ independent of d

$$\Downarrow$$

(d, k) sets have positive measure when

$$d - k < \left(\frac{r}{p}\right)^{k-1}$$

Proof by induction

$(d-(k-1),1)$ sets have Hdim $\geq d-(k-1)-(d-k)\frac{p}{r}$

Suppose $(\tilde{d}-1,\tilde{k}-1)$ sets have Hdim $\geq \tilde{d}-1-\beta$

Observe this implies $(\tilde{d},\tilde{k})$ sets are γ-Kakeya sets with $\gamma=\tilde{d}-1-\beta$

$$\Downarrow$$

$(\tilde{d},\tilde{k})$ sets have Hdim $\geq \tilde{d}-\beta\frac{p}{r}$

Known $L^p \to L^q(L^r)$ estimates

Drury/Christ:

$$\|X[f]\|_{L^{d+1}(L^{d+1})} \lesssim \|f\|_{L^{\frac{d+1}{2}}(\mathbb{R}^d)}$$
$$\frac{r}{p} = 2$$

Łaba/Tao/Wolff:

$$\|X_\delta[f]\|_{L^q(L^r)} \lesssim \delta^{-\frac{\alpha}{p}} \|f\|_{L^p(\mathbb{R}^d)}$$
$$p = \frac{d+2}{2},\ q = \frac{(d-1)(d+2)}{d},\ r = 2(d+2),\ \alpha = \frac{d-3}{4} + \epsilon$$

Katz/Tao:

$$\|X_\delta[f]\|_{L^q(L^\infty)} \lesssim \delta^{-\frac{\alpha}{p}} \|f\|_{L^p(\mathbb{R}^d)}$$
$$p = \frac{4d+3}{7},\ q = \frac{4d+3}{4},\ \alpha = \frac{3(d-1)}{7} + \epsilon.$$

Using method from Katz/Tao maximal operator bound:

$$\|X[f]\|_{L^q(L^r)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$
$$p = \frac{4d+3}{7}, q = \frac{4d+3}{4}, r = \frac{4d+3}{3} - \epsilon$$

Using method from Katz/Tao Hdim estimate:

Let $\epsilon > 0$, then there exist $p_\epsilon, q_\epsilon, r_\epsilon$ such that

$$\|X[f]\|_{L^{q_\epsilon}(L^{r_\epsilon})} \lesssim \|f\|_{L^{p_\epsilon}(\mathbb{R}^d)}$$
$$\frac{r_\epsilon}{p_\epsilon} \geq 1 + \sqrt{2} - \epsilon$$

Corollary

(d, k) sets have positive measure if $d - k < (1 + \sqrt{2})^{k-1}$

Part 2: Restricted directions

Joint work with Burak Erdoğan

$H \subset \mathbb{R}^d$ a hyperplane orthogonal to e_d

$1 \leq k < d - 1$

$\theta(z) : B \subset \mathbb{R}^k \to H$ parameterizes submanifold of H

$x \in H, z \in B$

$l(z, x) :=$ the line in the direction $\theta(z) + e_d$, passing through the point x.

Restricted Mixed-Norms:

$$\|X[f]\|_{L^q(L^r),\theta} := \left(\int_{B \subset \mathbb{R}^k} \left(\int_H |X[f](l(z,x))|^r \; dx \right)^{\frac{q}{r}} \; dz \right)^{\frac{1}{q}}$$

Question

For which p, q, r do we have

$$\|X[f]\|_{L^q(L^r),\theta} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

To satisfy non-trivial estimates, θ should not be "flat"

i.e. should have $d-1$ linearly independent derivatives at each point.

Expect best possible estimates when θ is "well-curved"

$(d-2)$-surface, well-curved means non-vanishing Gaussian curvature

1-surface, well-curved means first $d-1$ derivatives lin. indep.

$1 < k < d-2$, well-curved means ??

Wolff: $\theta = \mathbb{S}^{d-2}$

$$\|X[f]\|_{L^q(L^r),\theta} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

$d = 3, 4:$

$$1 \leq p \leq \frac{d^2-2d+2}{d}, \frac{d}{p} - \frac{d-1}{r} < 1, \frac{d-2}{q} > \frac{d}{p} - r$$

$d \geq 5$:

$$1 \leq p \leq \frac{d+1}{2},\ q, r \text{ as above}$$

Almost sharp when $d = 3, 4$

Almost sharp in all dimensions would imply Kakeya

Erdoğan: $\theta(t) = (t, t^2, \ldots, t^{d-1})$

$d = 4, 5$: Almost sharp range of p, q, r

Christ/Erdoğan: $\theta(t) = (t, t^2, \ldots, t^{d-1})$

All d: Almost sharp range of p, q, r

Morally, these all use geometric combinatorial bush/hairbrush arguments

Wolff,Erdoğan: Discrete counting arguments

Christ/Erdoğan: Continuous counting argument via "iterated T^*T" method. Intersections of lines controlled by analyzing sublevel sets of a certain Jacobian.

Wolff gets almost sharp result when $\mathbb{S}^{d-2}$ is a 1 or 2-surface.

Question

Can these methods be used to give almost sharp results for 2-surfaces in higher dimensions?

Answer

At least sometimes.

First open case: $d = 5$

θ is a 2-surface in $\mathbb{R}^4$

$$\theta(u, v) = (u, v, \overline{\theta}(u, v)), \text{ where } \overline{\theta} : \mathbb{R}^2 \to \mathbb{R}^2$$

$$A = \det(\overline{\theta}_{uu}, \overline{\theta}_{uv})$$
$$B = \det(\overline{\theta}_{uu}, \overline{\theta}_{vv})$$
$$C = \det(\overline{\theta}_{uv}, \overline{\theta}_{vv})$$

Christ's codimension 2 curvature condition:
θ is nondegenerate if $B^2 - 4AC \neq 0$ everywhere.

Example: $\theta(u, v) = (u, v, u^2 - v^2, 2uv)$

Theorem

Suppose the entries of $\overline{\theta}$ are quadratic polynomials and θ is nondegenerate. Then

$$\|X[f]\|_{L^q(L^r),\theta} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

when

$$1 + \frac{4}{r} > \frac{5}{p}, \quad \frac{2}{q} + \frac{6}{r} > \frac{6}{p}, \quad \frac{6}{r} > \frac{4}{p}.$$

Except for endpoints, no better $L^p \to L^q(L^r)$ estimates are satisified for any 2-surface in $\mathbb{R}^4$.

Higher Dimensions

When $d = 7, 8, 9$ we obtain similarly optimal $L^p \to L^q(L^r)$ estimates for the following surfaces:

d	$\theta(u, v)$
7	$(u, v, u^2, uv, v^2, u^3 + v^3)$
8	$(u, v, u^2, uv, v^2, u^2v, uv^2)$
9	$(u, v, u^2, uv, v^2, u^3 + v^3, u^2v, uv^2)$

When $d = 6$ we obtain almost sharp $L^p \to L^q(L^q)$ estimates for

$$\theta(u, v) = (u, v, u^2, uv, v^2)$$

The iterated T^*T method

$$E \subset \mathbb{R}^d, \quad F \subset \mathbb{R}^k \times \mathbb{R}^{d-1}$$

Want:

$$\langle X[1_E], 1_F \rangle \lesssim |E|^{\frac{1}{p}} \|1_F\|_{L^{q'}(L^{r'}),\theta}$$

Important quantities:

$$\alpha := \frac{\langle X[1_E], 1_F \rangle}{|F|} \qquad \beta := \frac{\langle 1_E, X^*[1_F] \rangle}{|E|}$$

$$(z, x) \in \mathbb{R}^k \times \mathbb{R}^{d-1},\ t \in \mathbb{R}$$

set

$$\gamma(z, x, t) := x + t(\theta(z) + e_d)$$

so that

$$X[1_E](z, x) = \int 1_E(\gamma(z, x, t))\ dt$$

$$y \in \mathbb{R}^d,\ z \in \mathbb{R}^k$$

set

$$\gamma^*(y, z) := (z, y_H - y_t \theta(z))$$

so that

$$X^*[1_F](y) = \int 1_F(\gamma^*(y, z))\ dz$$

Choose a typical $y_0 \in E$

Should have $X^*[1_F](y_0) \approx \beta$

i.e. $\exists$ a measure β set of z's with $\gamma^*(y_0, z) \in F$

For each of these points $\gamma^*(y_0, z) \in F$

Expect $X[1_E](\gamma^*(y_0, z)) \approx \alpha$

i.e. $\exists$ a measure α set of t's with $\gamma(\gamma^*(y_0, z), t) \in E$

Assume this point in E is typical and iterate

Obtain

$$\Omega = \{(z_1, t_1, \ldots, z_n, t_n)\}$$

for each $(z_1, t_1, \ldots, z_i)$, $|\{t_i\}| \approx \alpha$

for each $(z_1, t_1, \ldots, z_i, t_i)$, $|\{z_{i+1}\}| \approx \beta$

$\Gamma(z_1, t_1, \ldots, z_n, t_n) := \gamma(\gamma^*(\ldots\gamma(\gamma^*(y_0, z_1), t_1 \ldots), z_n), t_n)$

$\Gamma(\Omega) \subset E$

Choose n so that $n(k+1) = d$ to obtain a lower bound for $|E|$

End up with

$$|E| \gtrsim J|\Omega| \gtrsim J\alpha^n\beta^n$$

where J is a lower bound for $\det(\frac{\partial\Gamma}{\partial z_i, t_i})$ which holds on a refinement of Ω

Yields R.W.T. estimate

Finding lower bound for J

If each $\{t_i\}$ was evenly distributed over $[0,1]$ and each $\{z_i\}$ was evenly distributed over $B \subset \mathbb{R}^k$, then we would be in business.

$k = 1$:
Tao/Wright-

Lemma

Suppose $S \subset [0,1]$, $0 < \epsilon \ll 1$. Then there is an interval $I \subset [0,1]$ so that

$$|S \cap I| \gtrsim |S|^{1+\epsilon}$$

and

$$|S \cap J| \lesssim \left(\frac{|J|}{|I|}\right)^{\epsilon} |S \cap I|$$

for any interval J with $|J| \ll |I|$.

Replacement lemmas in higher dimensions

Would like to use Christ's:

Lemma

Suppose $S \subset B \subset \mathbb{R}^k$, and $0 < \epsilon \ll 1$. Then there is a parallelotope P so that

$$|S \cap P| \gtrsim |S|^{1+\epsilon}$$

and

$$|S \cap Q| \lesssim \left(\frac{|Q|}{|P|}\right)^{\epsilon} |S \cap P|$$

for any parallelotope Q with $|Q| \ll |P|$

Above, the P's and Q's can have any orientation

Difficulty

P, P' same size eccentricity, different orientations,
$P \cap P' \neq \emptyset \nRightarrow P \subset C \cdot P'$

Instead:

Lemma

Suppose $S \subset B \subset \mathbb{R}^k$, and $0 < \epsilon \ll 1$. Fix axes $w_1, \ldots, w_k$. Then there is a parallelotope P with axes parallel to $w_1, \ldots, w_k$ so that

$$|S \cap P| \gtrsim |S|^{1+\epsilon}$$

and

$$|S \cap Q| \lesssim \left(\frac{|Q|}{|P|}\right)^{\epsilon} |S \cap P|$$

for any parallelotope Q with $|Q| \ll |P|$ whose axes are parallel to $w_1, \ldots, w_k$.