Sharp weighted endpoint estimates for Calderón-Zygmund Operators

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Report on the paper

“Sharp A_1 bounds for Calderón-Zygmund operators and the relationship with a problem of Muckenhoupt and Wheeden” , IMRN, 2008.

with A. Lerner and S. Ombrosi

and also **some work in progress** with the same authors

Three conjectures:

1) **Muckenhoupt-Wheeden (M.-W.) conjecture**

2) **”weak” (M.-W.) conjecture**

3) **The A_2 conjecture**

previous motivating work

K. Astala, T. Iwaniec, E. Saksman, **Beltrami operators in the plane**, Duke Mathematical Journal. **107**, no. 1, (2001), 27-56.

S. Petermichl and A. Volberg, **Heating of the Ahlfors-Beurling operator: weakly quasiregular maps on the plane are quasiregular**, Duke Math. J. **112** (2002), no. 2, 281–305.

S. Petermichl, **The sharp bound for the Hilbert transform on weighted Lebesgue spaces in terms of the classical A_p characteristic**, American Journal of Mathematics 129, no. 5 (2007): 1355–75.

S. Petermichl, **The sharp weighted bound for the Riesz transforms**, Proceedings of the American Mathematical Society 136 (2008): 1237–49.

S.M. Buckley, **Estimates for operator norms on weighted spaces and reverse Jensen inequalities**, Trans. Amer. Math. Soc., **340** (1993), no. 1, 253–272.

motivation

The **Hardy–Littlewood** maximal function

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|\, dy$$

C. Fefferman and E.M. Stein (early 70's)

extension of the classical weak-type $(1,1)$ estimate:

$$\|Mf\|_{L^{1,\infty}(w)} \leq C \int_{\mathbb{R}^n} |f(x)|\, Mw(x)dx$$

if $w \geq 0$

This is a sort of **duality** for M.

The proof is by a covering classical argument.

Hence

$$M : L^1(w) \longrightarrow L^{1,\infty}(w)$$

if and only if

$$M(w) \le C\, w$$

the A_1 condition.

we denote by

$$[w]_{A_1}$$

the best of these C (the A_1 constant, characteristic or norm)

consequences

if $1 < p < \infty$:

$$\int_{\mathbb{R}^n} (Mf)^p \, w \, dx \le c_p \int_{\mathbb{R}^n} |f|^p \, M(w) \, dx.$$

Proof by interpolation

Why F-S considered this question?

vector-valued extension: for every $1 < p, q < \infty$:

$$\left\| \left(\sum_j (Mf_j)^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} \le C \left\| \left(\sum_j |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)}$$

model

We view M as the model example to understand Singular Integrals.

$$\|M\|_{L^{1,\infty}(w)} \leq c_n \, [w]_{A_1}$$

for 1 $< p < \infty$:

$$\|M\|_{L^{p,\infty}(w)} \leq c_n \, [w]_{A_1}^{1/p}$$

but

$$\|M\|_{L^{p}(w)} \leq c_n p' \, [w]_{A_1}^{1/p}$$

$p' = \frac{p}{p-1}$

Singular Integrals

To fix ideas we think of the case

$$Tf(x) = v.p. \int_{\mathbb{R}^n} K(x-y)\, f(y)\, dy$$

The critical size

$$|K(x)| \leq \frac{C}{|x|^n}$$

and the regularity assumption

$$|\nabla K(x)| \leq \frac{C}{|x|^{n+1}}$$

The results are new for the **Hilbert transform**:

$$Hf(x) = \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy$$

But they hold for **Calderón-Zygmund singular integral operators**.

Muckenhoupt-Wheeden conjecture

Natural question: Suppose that $w \geq 0$ Is it true

Conjecture 1 (Muckenhoupt-Wheeden conjecture)

$$\|Tf\|_{L^{1,\infty}(w)} \leq C \int_{\mathbb{R}^n} |f(x)|\, Mw(x)dx$$

It was studied during the **70**'s by **B. Muckenhoupt and R. Wheeden**.

Best result: let $\varepsilon > 0$ very tiny and $w \geq 0$, then

$$\|Tf\|_{L^{1,\infty}(w)} \leq C_\varepsilon \int_{\mathbb{R}^n} |f|\, M_{L(\log L)^\varepsilon}(w)\, dx.$$

C. P. $\approx$ 1994

Consequence

if $w \in A_1$

$$T : L^1(w) \to L^{1,\infty}(w)$$

but with a **“rough”** bound

$$\|T\|_{L^1(w)\to L^{1,\infty}(w)} \leq c\,[w]_{A_1}^2$$

since

$$M_{L(\log L)^\varepsilon}(w) \leq c\,M_{L\log L}(w) \approx M^2 w$$

PROBLEM: Can we improve this estimate?

The L^p case

The proof is based on the following L^p strong estimate:

Let $1 < p < \infty$ and let $\varepsilon > 0$ be tiny.
Also let $w \geq 0$. Then

$$\int_{\mathbb{R}^n} |Tf|^p \, w \, dx \leq C \int_{\mathbb{R}^n} |f|^p \, M_{L(\log L)^{p-1+\varepsilon}}(w) \, dx$$

C. P. 1994

M. Wilson some previous partial results for smooth, convolution and small p.

Hence if $w \in A_1$

$$\|T\|_{L^p(w)} \leq c_p \, [w]_{A_1}^{1+1/p}$$

Weak Muckenhoupt-Wheeden conjecture

If we assume $w \in A_1$

$$\|Tf\|_{L^{1,\infty}(w)} \leq c_n [w]_{A_1} \int_{\mathbb{R}^n} |f| w \, dx.$$

i. e.

Conjecture 2 (Weak Muckenhoupt-Wheeden conjecture)

$$\|T\|_{L^1(w)\to L^{1,\infty}(w)} \leq c_n \, [w]_{A_1}$$

To understand the weak estimate we need to understand the L^p case first

the case $p = 2$

R. Fefferman and J. Pipher (aprox. in 1997) proved

$$\|H\|_{L^2(w)} \leq c\,[w]_{A_1}$$

where H is the Hilbert transform.

The proof is based on sharp A_1 bounds for square functions on $L^2(w)$ from a **well known estimate of Chang-Wilson-Wolff**:

$$\int_{\mathbb{R}^n} (Sf)^2\, w\, dx \leq C \int_{\mathbb{R}^n} |f|^2\, M(w)\, dx$$

the general case

Natural question: find the best exponent α for

$$\|T\|_{L^p(w)} \leq c_{n,p}[w]_{A_1}^{\alpha} \tag{1}$$

Same approach as in the case $p = 2$ yields $\alpha = 1$ when $p > 2$.

However, if $1 < p < 2$

$$\alpha \leq 1/2 + 1/p$$

which is not sharp.

another drawback of the method: the approach works only for **smooth convolution** classical singular integrals.

First theorem: the sharp strong case

(with A. Lerner and S. Ombrosi)

Theorem 3 (the linear growth) *Let* $w \in A_1$ *and* $1 < p < \infty$. *Then, there is a constant* $c = c(n, T)$ *such that:*

$$\|T\|_{L^p(w)} \leq c\, p' \, [w]_{A_1}$$

New approach that works for any $1 < p < \infty$

Observe that **the result is sharp in both** p **and** $[w]_{A_1}$

in the previous work we had obtained

$$p' \log p'$$

so the proof of the conjecture cannot follow from this.

A consequence: the logarithmic growth in the weak case

The **main result of the talk**

Theorem 4 (the logarithmic growth)
Let $w \in A_1$. *Then, there is a constant* $c = c(n,T)$ *such that:*

$$\|T\|_{L^1(w)\to L^{1,\infty}(w)} \le c\,\varphi\Big([w]_{A_1}\Big)$$

where $\varphi(t) = t(1+\log^+ t)$.

of course is related to the weak M-W conjecture.

The behavior of $\|T\|_{L^p(w)}$

Recall that he behavior of the constant

$$\|T\|_{L^p(w)} \approx p'[w]_{A_1}$$

as $p \to 1$ turns out to be crucial, and it is reflected in the function

$$\varphi(t) = t(1 + \log^+ t)$$

In the previous work we had obtained

$$\|T\|_{L^p(w)} \approx p' \log p'[w]_{A_1}$$

which was **NOT SHARP** by the unweighted theory

which leads to

$$\|T\|_{L^1(w) \to L^{1,\infty}(w)} \leq c\, \varphi\Big([w]_{A_1}\Big)$$

where $\varphi(t) = t(1 + \log^+ t)(1 + \log^+ \log^+ t)$

A consequence for the A_p class

Recall that $w \in A_p$:

$$[w]_{A_p} \equiv \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1}$$

Corollary 5 *Let* $1 < p < \infty$ *and* $w \in A_p$
Then

$$\|T\|_{L^{p,\infty}(w)} \leq c\,\varphi(\|w\|_{A_p})$$

where $\varphi(t) = t(1 + \log^+ t)$.

conjecture

However, the conjecture is the following

Conjecture 6 (linear growth) *Let* $1 < p < \infty$ *and* $w \in A_p$ *Then*

$$\|T\|_{L^{p,\infty}(w)} \leq c\,\|w\|_{A_p}$$

This would follow from the weak M-W conjecture were true, namely if we had $\varphi(t) = t$ as well.

Compare with the strong case when: when $1 < p < 2$:

$$\|H\|_{L^p(w)} \leq c\,\|w\|_{A_p}^{\frac{1}{p-1}}$$

(non linear growth). S. Petermichl

Proof by "Extrapolation" ideas

Lemma 1 *Let* $1 < q < \infty$ *and let* $w \in A_q$. *Then there exists a nonnegative sublinear operator* D *bounded on* $L^{q'}(w)$ *such that for any nonnegative* $h \in L^{q'}(w)$:

(a) $h \leq D(h)$

(b) $\|D(h)\|_{L^{q'}(w)} \leq 2\,\|h\|_{L^{q'}(w)}$

(c) $D(h) \cdot w \in A_1$ *with*

$$[D(h) \cdot w]_{A_1} \leq c_n\, q\, \|w\|_{A_q}$$

Idea of the Proof: "Rubio de Francia iteration scheme´´:

$$D(h) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{S_w^k(h)}{\|S_w\|_{L^{q'}(w)}^k}$$

where

$$S_w(f) = \frac{M(fw)}{w}$$

which is bounded on $L^{q'}(w)$ by Muckenhoupt's theorem.

However, we need the sharp version due to Buckley:

$$\|S_w\|_{L^{q'}(w)} \leq cq\,\|w^{1-q'}\|_{A_{q'}}^{q-1} = cq\,\|w\|_{A_q}$$

2° part of the proof

We linearize the problem $\Omega_t = \{x \in \mathbb{R}^n : |Tf(x)| > t\}$,

$$w(\Omega_t)^{1/p} = \|\chi_{\Omega_t}\|_{L^p(w)} = \int_{\Omega_t} h\, w dx.$$

where $h \in L^{p'}(w)$ such that $\|h\|_{L^{p'}(w)} = 1$.

Hence by the lemma and the hypothesis

$$w(\Omega_t)^{1/p} \leq \int_{\Omega_t} D(h)\, w\, dx = (D(h)\, w)(\Omega_t)$$

$$\leq c\, \varphi([D(h)\, w]_{A_1}) \int_{\mathbb{R}^n} \frac{|f|}{t} D(h)\, w\, dx$$

$$\leq \frac{c}{t}\varphi(p[w]_{A_p})\Big(\int_{\mathbb{R}^n} |f|^p\, w\, dx\Big)^{1/p}\Big(\int_{\mathbb{R}^n} D(h)^{p'} w\, dx\Big)^{\frac{1}{p'}}$$

$$\leq \frac{c}{t}\varphi([w]_{A_p})\Big(\int_{\mathbb{R}^n} |f|^p\, w\, dx\Big)^{1/p}.$$

This completes the proof.

THE A_2 CONJECTURE

Suppose that the weak M-W conjecture holds, hence as consequence we have

$$\begin{cases} \|T\|_{L^{p,\infty}(w)} \leq c\,[w]_{A_p} \\ \|T\|_{L^{p',\infty}(\sigma)} \leq c\,[\sigma]_{A_{p'}} = c\,[w]_{A_p}^{\frac{1}{p-1}}, \end{cases}$$

here as usual $\sigma = w^{1-p'}$

Hence, we can state a "metatheorem":

The worst of both exponents: $\max\{1, \frac{1}{p-1}\}$

is governing the strong case:

Conjecture 7 (the A_2 conjecture)

$$\|T\|_{L^p(w)} \leq c\,\|w\|_{A_p}^{\max\{1,\frac{1}{p-1}\}}$$

Important reduction: it is enough to take $p = 2$

Thanks to the sharp extrapolation theorem due to by Dragičević, Grafakos, Pereyra, and Petermichl.

The sharp Reverse Hölder reverse inequality

Classical situation:

Let $w \in A_1$, then there is $r > 1$ such that

$$\left(\frac{1}{|Q|}\int_Q w^r\right)^{1/r} \leq \frac{c}{|Q|}\int_Q w$$

problem: there is a bad dependence on the constant $c = c(r, [w]_{A_1})$

Lemma. Let $w \in A_1$, then

$$\left(\frac{1}{|Q|}\int_Q w^{r_w}\right)^{1/r_w} \leq \frac{2}{|Q|}\int_Q w$$

i.e.

$$M_{r_w}(w) \leq 2\,[w]_{A_1}\,w$$

where

$$r_w = 1 + \frac{1}{2^{n+1}[w]_{A_1}}$$

as usual $M_r w = M(w^r)^{1/r}$

The key (tricky) lemma

Classical situation: Theorem of Coifman-Fefferman

Let $0 < p < \infty$ and $w \in A_\infty$. Then, there is a constant c depending of the A_∞ constant of w such that

$$\|Tf\|_{L^p(w)} \leq c\,\|Mf\|_{L^p(w)}$$

However, we need a more precise result for very specific weights

Lemma 2 (tricky) *Let w be **any weight** and let $1 \leq p, r < \infty$. Then, there is a constant $c = c(n, T)$ such that:*

$$\|Tf\|_{L^p((M_r w)^{1-p})} \leq cp\,\|Mf\|_{L^p((M_r w)^{1-p})}$$

In the previous paper we had obtained logarithmic growth:

$$C(p) \approx p \log p$$

observations

The classical proof by good λ Coifman-Fefferman is not sharp i.e gives:

$$C(p) \approx 2^p$$

because

$$[(M_r w)^{1-p})]_{A_p} \approx (r')^{p-1}$$

The proof by Bagby-Kurtz (with rearrangements) given in the mid 80's is more optimal from the point of view of the L^p constant but NOT in terms of the weight constant..

proof

The proof is tricky, it uses another variation **of Rubio de Francia algorithm** and relies surprisingly on a **sharp** L^1 **Coifman-Fefferman estimate**

Lemma

Let $w \in A_q$, $1 \leq q < \infty$. Then, there is a dimensional constant c such that:

$$\|Tf\|_{L^1(w)} \leq c[w]_{A_q}\|Mf\|_{L^1(w)}$$

The original proof we had was based on an idea by Fefferman-Pipher using a sharp version of the good-λ inequality of S. Buckley

However we have a better proof

$$\|f\|_{L^p(w)} \leq cp[w]_{A_q}\|M_\delta^{\#}f\|_{L^p(w)}$$

Main lemma

Lemma

Let w be **any weight** and let $1 < p < \infty$, $1 < r < 2$.
Then, there is a $c = c_n$ such that:

$$\|Tf\|_{L^p(w)} \leq cp' \left(\frac{1}{r-1}\right)^{1-1/pr} \|f\|_{L^p(M_r w)}$$

The proof is by duality:

$$\|T^*f\|_{L^{p'}(M_r w)^{1-p'})} \leq cp' \left(\frac{1}{r-1}\right)^{1-1/pr} \|f\|_{L^{p'}(w^{1-p'})}$$

then use the **key lemma**

$$\|T^*f\|_{L^{p'}(M_r w)^{1-p'})} \leq p'\, c\, \|Mf\|_{L^{p'}(M_r w)^{1-p'})}$$

sketch of the proof

Applying the Calderón-Zygmund decomposition to f at level λ,

$$f = g + b$$

usually, g is the "good" part and b is the "bad" part,

However, b is **"excellent"**

but g is really **"ugly´´**.

Applying as usual Chebyshev for the bad part, for any $p > 1$ we have

$$\|Tg\|_{L^p(w)} \leq cp' \left(\frac{1}{r-1}\right)^{1-\frac{1}{pr}} \|g\|_{L^p(M_r w)},$$

Choosing here, the r optimal, namely $r \approx 1 + \frac{1}{[w]_{A_1}}$ we have

$$\|Tg\|_{L^p(w)} \leq cp' [w]_{A_1} \|g\|_{L^p(w)}$$

Raising the power p and if we pick $p = 1 + \frac{1}{\log(1+\|w\|_{A_1})}$ we have finally

$$w\{x \in (\widetilde{\Omega})^c : |Tg(x)| > \lambda/2\} \leq \frac{c\varphi(\|w\|_{A_1})}{\lambda} \int_{\mathbb{R}^n} |f| w dx$$

Final conjecture

If we could prove the following weak type (p,p) estimate

$$\|Tf\|_{L^{p,\infty}(w)} \leq c_n\, p\, [w]_{A_1} \|f\|_{L^p(w)},$$

then we could prove the weak M-W conjecture

The version of the ”tricky” lemma for these weak norms does exist

$$\left\|\frac{Tf}{M_r w}\right\|_{L^{p,\infty}(M_r w))} \leq\ cp' \left\|\frac{Mf}{M_r w}\right\|_{L^{p,\infty}(M_r w)}$$

The proof is even trickier, but it is useless for the problem because of the bad constant p'

THANK YOU VERY MUCH