On radial Fourier multipliers

I. **Gustavo Garrigós**, A.S.:

Characterizations of Hankel multipliers

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II. **Fyodor L. Nazarov**, A.S.:

Radial Fourier multipliers in high dimensions

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Consider convolution operators

$$Tf = K * f$$

with **radial** kernels, and thus radial Fourier multipliers

$$\widehat{K}(\xi) = m(|\xi|).$$

There is a large literature and many good results on specific multipliers
(in particular $\eta(\lambda(|\xi|-1))$, Bochner-Riesz, ...).

Here we are interested in **characterizations** of L^p boundedness of T in terms of checkable conditions on K or m. W.l.o.g. $1 < p < 2$.

Want nec. and suff. condition for

$$\|K * f\|_{L^p(\mathbb{R}^d)} \lesssim \mathcal{C}(K)\|f\|_{L^p(\mathbb{R}^d)}. \qquad (*)$$

More modest question: Characterize boundedness on the subspace of radial functions; I.e.

$$\|K * f\|_{L^p(\mathbb{R}^d)} \lesssim \mathcal{C}_{\text{rad}}(K)\|f\|_{L^p_{\text{rad}}(\mathbb{R}^d)} \qquad (**)$$

Equivalently , for $g \in L^p(\mathbb{R}^+, \mu_d)$, $d\mu_d = r^{d-1}dr$ we wish to characterize

$$\|m(\sqrt{-\mathcal{L}})g\|_{L^p(\mu_d)} \lesssim \|g\|_{L^p(\mu_d)}$$

where

$$\mathcal{L} = D_r^2 + \frac{d-1}{r}D_r.$$

Note: For $d = 1$ the question of characterizing L^p_{rad} boundedness is equivalent to the characterization of all multipliers in $M^p(\mathbb{R})$.

Necessary conditions for L^p_{rad}-boundedness.

- If $\widehat{K}$ has compact support then $K \in L^p(\mathbb{R}^d)$.

- For general K: If Φ is *any* radial Schwartz function then $m \in \mathfrak{M}^p_d$ implies that

$$\sup_{t>0} \left\| \Phi * t^{-d} K(t^{-1} \cdot) \right\|_p < \infty$$

An equivalent necessary condition is that the $\mathbb{R}^d$-Fourier transforms of $\phi m(t|\cdot|)$ have uniform L^p norms.

Remark: For $\frac{2d}{d+1} \leq p \leq 2$ these conditions cannot possibly be sufficient conditions for L^p_{rad} boundedness since they don't even imply boundedness of the multiplier.

Characterizations of L^p_{rad} boundedness

Let ϕ be a nontrivial $C_c^\infty(\mathbb{R}^+)$ function. Let $\eta(x) = \mathcal{F}^{-1}[\phi(|\cdot|)]$. $\widehat{K} =: m(|\cdot|)$; $Tf = K * f$.

Thm. [G. Garrigós - S]

Let $1 < p < \frac{2d}{d+1}$. **TFAE:**

(i) T is bounded on $L^p_{\text{rad}}(\mathbb{R}^d)$.

(ii) T maps $L^{p,1}_{\text{rad}}$ to $L^p(\mu_d)$.

(iii) $\sup_{s>0} s^{d/p} \|T[\eta(s\cdot)]\|_{L^p(\mathbb{R}^d)}$.

(iv) $\sup_{t>0} \left\| \mathcal{F}_{\mathbb{R}^d}[\phi(|\cdot|)m(t|\cdot|)] \right\|_{L^p(\mathbb{R}_d)} < \infty$

(v) For $\kappa_t := \mathcal{F}^{-1}_{\mathbb{R}}[\phi m(t\cdot)]$ we have

$$\sup_{t>0} \left(\int |\kappa_t(r)|^p (1+|r|)^{(d-1)(1-p/2)} dr \right)^{1/p} < \infty.$$

Corollaries

Assumption $1 < p < 2d/(d+1)$. $\mathfrak{M}^p_d$: multiplier space for L^p_{rad} problem.

- Putting dyadic pieces together

$$\|m\|_{\mathfrak{M}^p_d} \lesssim \sup_{t>0} \|\phi m(t\cdot)\|_{\mathfrak{M}^p_d}.$$

- Interpolation (using one of Calderón's complex methods)

$$[\mathfrak{M}^{p_0}_{d_0}, \mathfrak{M}^{p_1}_{d_1}]^\vartheta = \mathfrak{M}^p_d;$$

$$(1-\vartheta)(\tfrac{1}{p_0}, d_0) + \vartheta(\tfrac{1}{p_1}, d_1) = (\tfrac{1}{p}, d).$$

  In contrast $M^p(\mathbb{R})$ is no interpolation space of $M^{p_0}(\mathbb{R})$ and $M^{p_1}(\mathbb{R})$. (Zafran, ...)

- Sharp multiplier theorem involving the condition $\sup_t \|\phi m(t\cdot)\|_{B^2_{d(\frac{1}{p}-\frac{1}{2}),p}} < \infty$.

Result for radial Fourier multipliers

(in high dimensions)

Let $d \geq 5$. $Tf = K * f$, K radial.

Thm. A [F. Nazarov -S.] Let

$$1 < p < \frac{2(d^2 - 2d - 3)}{d^2 - 5} =: p_d.$$

Fix an arbitrary Schwartz function η that is not identically 0. Then

$$\left\|T\right\|_{L^p \to L^p} \approx \sup_{t>0} t^{d/p} \left\|T[\eta(t\cdot)]\right\|_{L^p}.$$

This means:

- Equivalence of L^p- and L^p_{rad}-boundedness!

It has been observed (e.g. in [Müller-S.]) that sharp 'local smoothing' results for the wave equation yield good results for multipliers (ϵ loss).

Sogge's question (1990): Let I be a compact time interval. Does the inequality

$$\Big(\int_I \|e^{it\sqrt{-\Delta}}f\|_q^q\,dt\Big)^{1/q} \lesssim \|f\|_{L^q_\beta}$$

hold for

$$\beta \equiv \alpha(q) - \frac{1}{q} = d(\frac{1}{2} - \frac{1}{q}) - \frac{1}{2},$$

in the optimal range $q > \frac{2d}{d-1}$?.

Not known. Almost true in the category of $L^q_{\mathrm{rad}}(L^2_{\mathrm{sph}})$ spaces [MS].

Partial results ($L^q(\mathbb{R}^d)$ problem, for $\beta > \beta(q)$):

Wolff: [GAFA2000] for $d = 2$, $q > 74$.

Łaba-Wolff [JdA2002]: $q > 2 + \min\{\frac{32}{3d-7}, \frac{8}{d-3}\}$.

Further improvements by Garrigós, S. [05] and Garrigós, Schlag, S. [08]: $d = 2$, $q \geq 20$ and $d \geq 3$, $q > 2 + \frac{8}{d-2}\frac{2d+1}{2d+2}$.

All this work relies on versions of **Wolff's inequality for plate decompositions** (which has other important applications).

Let $d \geq 5$ and $p_d' = \frac{2(d^2-2d-3)}{d^2-4d-1} = 2 + \frac{4}{d} + O(\frac{1}{d^2})$.

Thm. B [Nazarov-S.] For $q > q_d$, $\beta = \beta(q) = d(\frac{1}{2} - \frac{1}{q}) - \frac{1}{2}$,

$$\Big(\int_I \| e^{it\sqrt{-\Delta}} f \|_q^q \, dt \Big)^{1/q} \lesssim_I \|f\|_{L^q_\beta}.$$

Remark: The proof does not use Wolff's inequality and the method does not improve on its range. It can be used to prove some endpoint versions (no induction on scales).

For the multiplier question concentrate on the case where $\widehat{K}$ is supported in $\{1 < |\xi| < 2\}$. For the wave eqn question, we look at the case case where $\widehat{f}$ is supported in an annulus $\{|\xi| \approx 2^k\}$ for large k. This leads to the easier Besov space version. We consider this in the dualized version ($p < p_d$).

$$\Big\| \int_I \eta(\tfrac{\sqrt{-\Delta}}{2^k}) e^{it\sqrt{-\Delta}} f(\cdot,t)dt \Big\|_p \lesssim 2^{k(\frac{d}{p}-\frac{d+1}{2})} \Big(\int_I \|f(\cdot,t)\|_p^p \, dt \Big)^{1/p}.$$

or, after rescaling

$$\Big\| \int_{r\approx 2^k} 2^{k(d-1)/2} \eta(\sqrt{-\Delta}) e^{ir\sqrt{-\Delta}} f(\cdot,r)dr \Big\|_p \lesssim \Big(\int_{r\approx 2^k} \|f(\cdot,r)\|_p^p \, r^{d-1} \, dr \Big)^{1/p}.$$

Let σ_r is surface measure on the sphere of radius r (not normalized, so that $\|\sigma_r\|_{meas} \approx r^{d-1}$).

Note: $2^{k(d-1)/2}\eta(\sqrt{-\Delta})e^{ir\sqrt{-\Delta}}$ 'is like' convolution with $\eta * \sigma_r$.

Because of the assumption of compact Fourier support we can discretize the kernel and the multiplier. This is related to the Plancherel-Polya theorem. General case by straightforward averaging argument.

Let $\psi \in \mathcal{S}$ be a radial function compactly supported in $\{|y| \leq 1/2\}$ so that $\widehat{\psi}$ is 'almost supported' in $\{|\xi| \approx 1\}$, i.e. $\widehat{\psi}$ vanishes of very high order at 0.

Assume

$$f(y) = \sum_{z \in \mathbb{Z}^d} a(z)\psi(y-z)$$

where $\alpha \in \ell^p(\mathbb{Z}^d)$.

Assume that

$$K = \sum_{r \in \mathbb{N}} b(r)\psi * \sigma_r$$

where $\sum_{r\in\mathbb{N}} |b(r)|^p r^{d-1} < \infty$.

Recall σ_r is surface measure on the sphere of radius r.

Model inequality for Thm A. (local)

$$\Big\|\sum_r b(r)\psi * \sigma_r * \sum_z a(z)\psi(\cdot - z)\Big\|_p \lesssim \Big(\sum_r |b(r)|^p r^{d-1}\Big)^{1/p}\Big(\sum_z |a(z)|^p\Big)^{1/p}$$

Model inequality for Thm. B (Besov-rescaled)

$\eta(|\xi|)r^{(d-1)/2}e^{ir|\xi|}$ ‘is like’ $\widehat{\sigma_r * \psi}$ $\Longrightarrow$

$$\Big\|\sum_{r\approx 2^k}\sum_z c(z,r)\psi * \sigma_r * \psi(\cdot - z)\Big\|_p \lesssim \Big(\sum_{r\approx 2^k} |c(z,r)|^p r^{d-1}\Big)^{1/p}$$

Unify: If $F_{z,r} = \sigma_r * \Psi(\cdot - z)$ then

$$\Big\|\sum_{r,z} c(z,r)F_{z,r}\Big\|_p \lesssim \Big(\sum_{z,r} |c(z,r)|^p r^{d-1}\Big)^{1/p}.$$

Notice for $r \geq 1$

$$\|\sigma_r\|_{meas} \approx r^{d-1}$$
$$\|\widehat{\sigma}_r\|_\infty \approx r^{(d-1)/2}$$

Thus

$$\Big\| \sum_{r,z} c(z,r)\sigma_r * \Psi(\cdot - z) \Big\|_p \lesssim \Big(\sum_{z,r} |c(z,r)|^p r^{d-1} \Big)^{1/p}.$$

is trivial for $p = 1$.

If the $F_{z,r} = \sigma_r * \Psi(\cdot - z)$ were (almost) orthogonal then the inequality would be true for $p = 2$ and we would obtain a result which is too good to be true.

Only for *fixed* r we have a good L^2 estimate:

$$\Big\| \sum_{z} c(z,r)\sigma_r * \Psi(\cdot - z) \Big\|_2 \lesssim \Big(\sum_{z} |c(z,r)|^2 r^{d-1} \Big)^{1/2}$$

The $F_{z,r}$ for different r, r' are not almost orthogonal, but we do have a limited amount of orthogonality:

Lemma:

$$|\langle F_{z,r}, F_{z',r'}\rangle| \lesssim \frac{(rr')^{(d-1)/2}}{(1+|r-r'|+|z-z'|)^{(d-1)/2}}$$

LHS is equal to

$$\int \widehat{\sigma}_r(\xi)\widehat{\sigma}_{r'}(\xi)|\psi(\xi)|^2 e^{i\langle z'-z,\xi\rangle} d\xi$$

$$= c'(rr')^{d-1}$$

$$\times \int B_d(r\rho)B_d(r'\rho)B_d(|z-z'|\rho)|a(\rho)|^2\rho^{d-1}d\rho.$$

The decay properties of B_d and the behavior of a at 0 imply

$$\left|\left\langle F_{z,r}, F_{z',r'}\right\rangle\right| \lesssim \frac{(rr')^{\frac{d-1}{2}}}{(1+|z-z'|)^{\frac{d-1}{2}}}$$

So ok, if $|r-r'| \le C(1+|z-z'|)$. But if $|r-r'| \gg (1+|z-z'|)$ then $F_{z,r}$ and $F_{z',r'}$ have disjoint support. $\square$

Want for

$$F_{z,r} = \sigma_r * \psi(\cdot - z)$$

the inequality

$$\| \sum_{z,r} c(z,r) F_{z,r} \|_p \lesssim \Big(\sum |c(z,r)|^p r^{d-1} \Big)^{1/p}.$$

'For simplicity' we shall only consider the case where radii are restricted to $[R, 2R]$, $R = 2^k$.

By interpolation we just need a restricted (strong) type estimate with $E \subset \mathbb{Z}^d \times (\mathbb{N} \cap [R, 2R])$:

$$\Big\| \sum_{(z,r) \in E} F_{z,r} \Big\|_p^p \lesssim R^{d-1} \# E$$

A favorable situation for L^2.

While the L^2 inequality does not hold in general it does hold for certain sparse sets. Namely if we *assume that for any* $\rho \geq 1$ *every ball of radius* ρ *contains only* $\lesssim \rho^{\frac{d-1-\varepsilon}{2}}$ *points* (z, r) then

$$\Big\| \sum_{(z,r)\in E} F_{z,r} \Big\|_2^2 \lesssim \sum_{(z,r)\in E} \sum_{(z',r')\in E} |\langle F_{z,r}, F_{z',r'} \rangle|$$

$$\lesssim R^{d-1} \sum_{(z,r)\in E} \sum_{(z',r')} (1 + |z - z'| + |r - r'|)^{-\frac{d-1}{2}}$$

$$\lesssim R^{d-1} \# E.$$

by our assumption.

Recall the other favorable situation for L^2

For fixed r

$$\Big\| \sum_{z:(z,r)\in E} F_{z,r} \Big\|_2^2 \lesssim R^{d-1} \# \{z : (z, r) \in E\}.$$

An L^2 bound combining these two cases
Let $h = \frac{d-1-\varepsilon}{2}$.

Def. Fix $R \geq 1$ and $u \geq 1$. Let E be a (finite) 1-separated subset of $\mathbb{R}^d \times [R, 2R)$. We say that E is of *h-density type* (u, R) if

$$\#(B \cap E) \lesssim \max\{u, \rho^h\}$$

for any ball B of radius $\rho \lesssim R$.

Essentially by combining the two observations on the last page with Cauchy-Schwarz ... one can prove

Lemma. Suppose $h < \frac{d-1}{2}$ and let $u \geq 1$. Let $E \subset \mathbb{Z}^d \times (\mathbb{N} \cap [R, 2R]$ be a set of h-density type (u, R). Then

$$\Big\| \sum_{(z,r) \in E} F_{z,r} \Big\|_2^2 \lesssim_h u^{\frac{2}{d+1}} R^{d-1} \# E.$$

We shall need to decompose E into subsets of h-density type (u, R) for values of $u = 1, 2, 4, 8, \ldots$. For large u the L^2 bounds become worse but there will be **crowding effect resulting in a support which is smaller than expected**. Thus we shall gain in L^p for $p \ll 2$.

Let $R_u = \min\{R, u^{1/h}\}$.

Lemma:
(i) $E = \bigcup_{u=1,2,4,\ldots} E(u)$ (disjoint).
(ii) $E(u)$ can be covered by $\lesssim u^{-1}\#E$ balls of radius $\approx R_u$ ($\lesssim u^{1/h}$) each.
(iii) $E(u)$ is a set of h-density type (u, R) i.e., for every $\rho \lesssim R$ the ball of radius ρ contains $\lesssim \max\{u, \rho^h\}$ points in $E(u)$.

Corollary on support size of $G_u := \sum_{E(u)} F_{z,r}$

$$\text{meas}\Big(\text{supp}(G_u\Big) \lesssim u^{-1}(\#E)\; u^{1/h} R^{d-1}$$

Consequently

$$
\begin{aligned}
\|G_u\|_p &\le \Big[\text{meas}(\text{supp}\, G_u)\Big]^{\frac{1}{p}-\frac{1}{2}} \\
&\le \Big(u^{-1}\#E\ u^{1/h}R^{d-1}\Big)^{\frac{1}{p}-\frac{1}{2}}\|G_u\|_2 \\
&\lesssim \Big(u^{-\frac{h-1}{h}}R^{d-1}\#E\Big)^{\frac{1}{p}-\frac{1}{2}}\Big(u^{\frac{2}{d+1}}R^{d-1}\#E\Big)^{1/2} \\
&\lesssim u^{-\delta(p)}(R^{d-1}\#E)^{1/p}
\end{aligned}
$$

with the exponent of u being negative if

$$-\frac{h-1}{h}(\frac{1}{p}-\frac{1}{2})+\frac{1}{d+1}<0.$$

Taking h close to $\frac{d-1}{2}$ this leads to the condition

$$p<\frac{2(d^2-2d-3)}{d^2-5}$$

as asserted. Now sum in u.

The density decomposition of E.
Recall $R_u = \min\{R, u^{1/h}\}$.

Cover $\mathbb{R}^{d+1}$ with a grid $\mathcal{Q}(R_u)$ of cubes of side-length R_u.

Let $\widehat{E}(u)$ be the union over all $E \cap Q$ where Q ranges over those $Q \in \mathcal{Q}(R_u)$ for which

$$\#(E \cap Q) \geq u.$$

Let

$$E(u) = \widehat{E}(u) \setminus \bigcup_{u : u' > u} \widehat{E}(u').$$

(i) To prove $E = \cup_u E(u)$ observe that $\widehat{E}(0) = E$ and $\widehat{E}(u) = \varnothing$ for $u \geq CR^{d+1}$.

(ii) Of course the number of $Q \in \mathcal{Q}(R_u)$ for which $\mathrm{card}(E \cap Q) \geq u$ is $\lesssim u^{-1}\#E$. Thus $E(u)$ is covered by the $O(u^{-1}\#E)$ balls of radius R_u.

(iii) Now verify that $E(u)$ is of h-density type (u, R); i.e. every ball of radius $\rho \lesssim R$ contains $\lesssim \max\{u, \rho^h\}$ points in $E(u)$.

Now let B a ball of center $\rho \lesssim R$.

Case 1. If $\rho \lesssim R_u$ then $B \cap E(u)$ is contained in $O(1)$ cubes in $\mathcal{Q}(R_u)$. However in each such cube $E(u)$ does not intersect $\widehat{E}(2u)$. Thus $\#(B \cap E) \lesssim u$.

Case 2. If $\rho \gg R_u$ but $\rho \lesssim R$: Pick $\widetilde{u}$ so that $\rho^h \approx \widetilde{u}$. Then $B \cap E(u)$ is contained in $O(1)$ cubes in $\mathcal{Q}(R_{\widetilde{u}})$ and it does not meet the portion of any cube in $\mathcal{Q}(R_{2\widetilde{u}})$ that contains more than $2\widetilde{u}$ points in E. This is because $E(u)$ does not intersect $\widehat{E}(2\widetilde{u})$. Thus $\#(B \cap E) \lesssim \widetilde{u} \approx \rho^h$.

What else to do?

(A) We need to combine the sets with radii in $[2^k, 2^{k+1})$; call these E_k.

For every k we decompose E_k into sets $E_k(u)$ of h-density type $(u, 2^k)$ and combine $E(u) = \cup_k E_k(u)$.

For the L^2 estimates we need to consider the interaction between different E_k, $E_{k'}$. This is ok if $|k - k'| \gg \log u$, otherwise get only a factor of $\log u$ by Cauchy-Schwarz.

(B) Getting rid of the assumption that m is supported in $\{|\xi| \approx 1\}$.

This starts the second part of the proof which is based on the first and **atomic decompositions** in L^p.

• A useful inequality for putting scales together.

Let ψ as before and assume that K is radial and *supported in* $\{|x| > 2^\ell\}$.

Let $\mathcal{Q}^\ell$ be a tiling of $\mathbb{R}^d$ with cubes of length 2^ℓ.

Prop. For $p < p_d$:

$$\Big\|\psi * K * g\Big\|_p \lesssim \|K\|_p \, 2^{-\ell\varepsilon} \Big(\sum_{Q \in \mathcal{Q}^\ell} |Q| \, \|g\chi_Q\|_\infty^p \Big)^{\frac{1}{p}}$$