Null form estimates for the wave equation

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Joint work with Sanghyuk Lee (Seoul National University)
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Short Courses

R. DeVore	S. Hofmann	C. Kenig	G. Pisier
U. South Carolina	U. Missouri-Columbia	U. Chicago	Texas A&M, U. Paris VI
The Mathematical Foundations of Compressed Sensing	*Local Tb Theorems and Applications in Partial Differential Equations*	*Some Recent Developments in Nonlinear Dispersive Equations*	*Complex Interpolation between Banach, Hilbert and Operator Spaces*

Main Speakers

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C. Muscalu Cornell U.	J. Ortega-Cerdá U. Barcelona	S. Petermichl U. Bordeaux	A. Seeger U. Wisconsin	S. Thangavelu Indian Statistical Inst.
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Strichartz's estimates

Let ϕ be a solution of the homogeneous wave equation in $\mathbb{R}^{n+1}$, $n \geq 2$;

$$\partial_t^2 \phi(x,t) - \Delta_x \phi(x,t) = 0, \quad x \in \mathbb{R}^n, t \in \mathbb{R}.$$

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(Strichartz (1977), Ginibre-Velo (1985), Keel-Tao (1998))

$$\|\phi\|_{L_t^q L_x^r} \leq C[\|\phi(0)\|_{\dot{H}^s} + \|\partial_t \phi(0)\|_{\dot{H}^{s-1}}]$$

if $1/q + n/r = n/2 - s$ and $2/q + (n-1)/r \leq (n-1)/2$ with exception $(q,r) = (2,\infty)$ when $n = 3$.

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$$\phi(x,t) = \frac{1}{2}\left(\int_{R^n} e^{i(x\cdot\xi + t|\xi|)} \widehat{\phi(\cdot,0)}(\xi)\, d\xi + \int_{R^n} e^{i(x\cdot\xi - t|\xi|)} \widehat{\phi(\cdot,0)}(\xi)\, d\xi \right)$$

$$+\frac{1}{2}\left(\int_{R^n} e^{i(x\cdot\xi + t|\xi|)} \frac{\widehat{\partial_t \phi(\cdot,0)}(\xi)}{2i|\xi|}\, d\xi - \int_{R^n} e^{i(x\cdot\xi - t|\xi|)} \frac{\widehat{\partial_t \phi(\cdot,0)}(\xi)}{2i|\xi|}\, d\xi \right).$$

Strichartz's estimates

Notation :

$$e^{\pm it\sqrt{\Delta}}f(x) := \int_{R^n} e^{i(x\cdot\xi \pm t|\xi|)}\widehat{f}(\xi)\,d\xi.$$

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Restriction of the Fourier transform to the cone, adjoint version : (Stein 70's)

$$\Gamma = \{(\xi,\tau) \in \mathbf{R}^n \times \mathbf{R} : \ \ \tau = |\xi|\}$$

$$\|e^{\pm it\sqrt{\Delta}}f\|_{L^q(\mathbb{R}^{n+1})} \leq C\|\widehat{f}\|_{L^p(\mathbb{R}^n)}$$

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B. Barceló (1985) : sharp estimates ($q = r$) for $n = 2$.
T. Wolff (2001) : sharp estimates in higher dimensions.

Surfaces with non-vanishing curvature (Schrödinger eq.)

$n = 2$: Feffermann (1972), Zygmund (1974)

Higher dimensions, $p = 2$: Tomas (1975), Strichartz (1977), Stein (1986)

$p \neq 2$: Bourgain (90's), Wolff, Moyua, V, Vega, Tao (2003) (surfaces with positive curvature)

V, Sanghyuk Lee : the saddle (negative curvature) $\tau = \xi_1^2 - \xi_2^2$.

Bilinear restriction estimates

$\frac{1}{q} \leq \frac{n-1}{2(n+1)}$, $\widehat{f}$ compactly supported : $\|e^{\pm it\sqrt{\Delta}} f\|_{L^q(\mathbf{R}^{n+1})} \leq C\|f\|_{L^2(\mathbf{R}^n)}$.

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Bilinear version : $\|e^{\pm it\sqrt{\Delta}} f_1 e^{\pm it\sqrt{\Delta}} f_2\|_{L^{q/2}(\mathbf{R}^{n+1})} \le C\|f_1\|_{L^2(\mathbf{R}^n)}\|f_2\|_{L^2(\mathbf{R}^n)}$.

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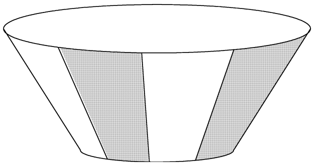
Bilinear restriction theorem (Wolff, Tao). $\widehat{f_1}$ supported in $\{\xi \in \mathbf{R}^n : |\xi| \sim 1,\ 0 < \xi \cdot e_1 < 1/4\}$ and $\widehat{f_2}$ supported in $\{\xi \in \mathbf{R}^n : |\xi| \sim 1,\ 0 < \xi \cdot e_2 < 1/4\}$, then

$$\|e^{\pm it\sqrt{\Delta}} f_1 e^{\pm it\sqrt{\Delta}} f_2\|_{L^{q/2}(\mathbf{R}^{n+1})} \leq C\|f_1\|_{L^2(\mathbf{R}^n)}\|f_2\|_{L^2(\mathbf{R}^n)}$$

for $\frac{1}{q} \leq \frac{n+1}{2(n+3)}$.

Bilinear restriction estimates



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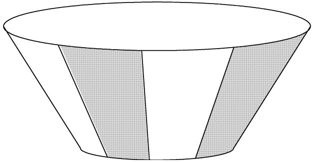

There were previous results by Bourgain (1995), Tao-V (2000)

Null form estimates

$$\|\phi\psi\|_{L_t^q L_x^r} \leq C(\|\phi(0)\|_{\dot{H}^s} + \|\partial_t \phi(0)\|_{\dot{H}^{s-1}}) \times (\|\psi(0)\|_{\dot{H}^s} + \|\partial_t \psi(0)\|_{\dot{H}^{s-1}})$$

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Null form estimates :

$$\|D_0^{\beta_0} D_+^{\beta_+} D_-^{\beta_-}(\phi\psi)\|_{L^q_t L^r_x}$$

$$\leq C(\|\phi(0)\|_{\dot{H}^{\alpha_1}} + \|\partial_t\phi(0)\|_{\dot{H}^{\alpha_1-1}}) \times (\|\psi(0)\|_{\dot{H}^{\alpha_2}} + \|\partial_t\psi(0)\|_{\dot{H}^{\alpha_2-1}})$$

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$r = q = 2$: Beals (1983), Klainerman-Machedon (1993...), Klainerman-Selberg (1997), Klainerman-Tataru (1999), Foschi-Klainerman (2000).
General values of q, r : Tao-V (2000), Tao (2001), Tataru (2001).

Null form estimates

If

$$(\xi, \tau) = (\xi^{(1)}, |\xi^{(1)}|) + (\xi^{(2)}, |\xi^{(2}|) \in \operatorname{supp} \widehat{\phi} + n \operatorname{supp} \widehat{\psi}$$

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If $\xi^{(1)} = \xi^{(2)}$, then

$$|\xi| - |\tau| = |\xi^{(1)} + \xi^{(2)}| - (|\xi^{(1)}| + |\xi^{(2)}| = 2|\xi^{(1)}| - 2|\xi^{(1)}|) = 0.$$

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About D_0, D_+ : If $|\xi^{(1)}| \sim 2^k >> |\xi^{(2)}|$, then

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About D_0, D_+ :
If $\xi^{(1)} = -\xi^{(2)}$, then

$$|\xi| = 0, \qquad |\xi| + |\tau| = 2|\xi^{(1)}|, \qquad\qquad D_0 << D_+$$

Null form estimates

$$Q_0(\phi, \psi) = \partial_t \phi \partial_t \psi - \nabla_x \phi \cdot \nabla_x \psi$$

$$Q_{0j}(\phi, \psi) = \partial_t \phi \partial_{x_j} \psi - \partial_{x_j} \phi \partial_t \psi$$

$$Q_{ij}(\phi, \psi) = \partial_{x_i} \phi \partial_{x_j} \psi - \partial_{x_j} \phi \partial_{x_i} \psi$$

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$$-2Q_0(\phi,\psi) = D_+D_-(\phi,\psi).$$
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$$\|D_0^{\beta_0}D_+^{\beta_+}D_-^{\beta_-}Q(\phi,\psi)\|_{L_t^qL_x^r}$$
$$\leq C(\|\phi(0)\|_{\dot H^{\gamma_1}} + \|\partial_t\phi(0)\|_{\dot H^{\gamma_1-1}}) \times (\|\psi(0)\|_{\dot H^{\gamma_2}} + \|\partial_t\psi(0)\|_{\dot H^{\gamma_2-1}})$$

Motivation

Study semilinear wave equations with quadratic non-linearities, arising from **wave maps :**

Given a Riemannian manifold (M, g) consider functions $\phi : \mathbf{R} \times \mathbf{R}^n \to M$. Wave maps are critical points of the Lagrangian

$$L(\phi) = \int_{\mathbf{R}^n} \left(|\partial_x \phi|_g^2 - |\partial_t \phi|_g^2 \right) dx.$$

In local coordinates,

$$\partial_t^2 \phi^\ell - \Delta_x \phi^\ell = \Gamma_{j,k}^\ell(\phi) \left[\partial_t \phi^j \partial_t \phi^k - \sum \partial_{x_i} \phi^j \partial_{x_i} \phi^k \right] = \Gamma_{j,k}^\ell(\phi) D_+ D_- (\phi^j \phi^k).$$

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To simplify, let us consider $n = 2$

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Local "smoothing" estimates for waves : $\phi^{\pm} = e^{\pm i\sqrt{-\Delta}t} f$

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For fixed t, the sharp estimate (J. C. Peral (1980)) is

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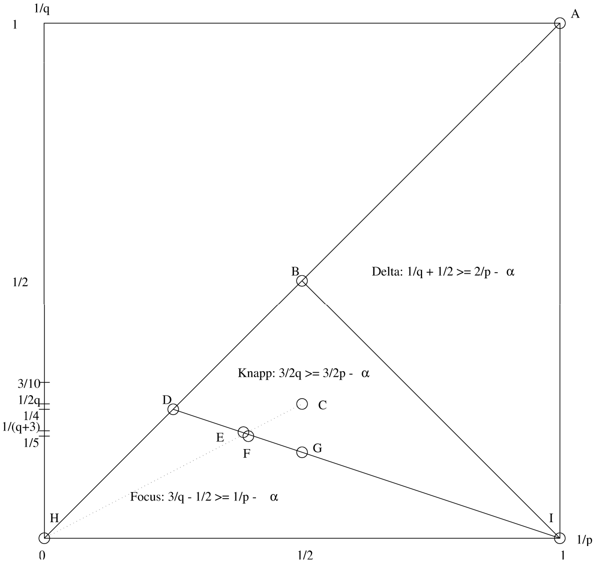
Conjecture (b) : Holds for $q \geq 4$, $p = (q/3)'$ and all $\alpha > 3/2 - 6/q$.

Motivation


1/q
1
A
1/2
B
Delta: 1/q + 1/2 >= 2/p - α
3/10
1/2q
1/4
1/(q+3)
1/5
D
C
Knapp: 3/2q >= 3/2p - α
E
F
G
Focus: 3/q - 1/2 >= 1/p - α
H
I
1/p
0
1/2
1

Motivation

Theorem (Tao-V (2000) : Suppose that $q_0 < 2$ is such that the null form estimate is true in the $(++)$ case for $q = q_0$, $\beta_0 = \beta_+ = 0$, $\beta_- = \frac{3}{2q_0} - \frac{1}{2} + 2\epsilon$, and $\alpha_1 = \alpha_2 = \frac{3}{4q_0'} + \epsilon$ for arbitrarily small $\epsilon > 0$. Then the conjecture (b) is true for all $q \geq q_0 + 3$.

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The conjecture (b) holds for for $q > 5 - 1/3$

The multiplier of the cone

$$m^{\alpha}(\xi,\tau)=\phi(\tau)\left(1-\frac{|\xi|}{\tau}\right)_{+}^{\alpha},$$

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Conjecture $(n = 2)$: For all $\alpha > 0$, the operator T_α is bounded in $L^4(\mathbb{R}^2 \times \mathbb{R})$.

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Mockenhaupt, Seeger, Sogge, Bourgain, Wolff, Tao, V, Laba, Pramanik, Schlag, Garrigós

Necessary conditions :

Null form estimates :

$$\|D_0^{\beta_0} D_+^{\beta_+} D_-^{\beta_-}(\phi\psi)\|_{L_t^q L_x^r} \leq C(\|\phi(0)\|_{\dot{H}^{\alpha_1}} + \|\partial_t \phi(0)\|_{\dot{H}^{\alpha_1 - 1}})$$
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Scaling invariance :

$$\beta_0 + \beta_+ + \beta_- = \alpha_1 + \alpha_2 + \frac{1}{q} - n(1 - \frac{1}{r}). \tag{1}$$

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Scaling invariance :

$$\beta_0 + \beta_+ + \beta_- = \alpha_1 + \alpha_2 + \frac{1}{q} - n(1 - \frac{1}{r}). \tag{1}$$

Geometry of the cones :

$$\frac{1}{q} \leq \frac{n+1}{2}(1 - \frac{1}{r}), \quad \frac{1}{q} \leq \frac{n+1}{4}. \tag{2}$$

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Concentration near null directions :

$$\beta_- \geq \frac{1}{q} - \frac{n-1}{2}(1 - \frac{1}{r}). \tag{3}$$

Necessary conditions :

Low frequency interactions $(++)$ *:*

$$\beta_0 \geq \frac{1}{q} - n(1 - \frac{1}{r}), \tag{4}$$

$$\beta_0 \geq \frac{2}{q} - (n+1)(1 - \frac{1}{r}) \tag{5}$$

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Low frequency interactions $(+-)$:

$$\alpha_1 + \alpha_2 \geq \frac{1}{q}, \tag{6}$$

$$\alpha_1 + \alpha_2 \geq \frac{3}{q} - n(1 - \frac{1}{r}). \tag{7}$$

Necessary conditions :

Interaction between high and low frequency :

$$\alpha_i \leq \beta_- + \frac{n}{2}, \tag{8}$$

$$\alpha_i \leq \beta_- + \frac{n}{2} - \frac{1}{q} + \frac{n-1}{2}(\frac{1}{2} - \frac{1}{r}), \tag{9}$$

$$\alpha_i \leq \beta_- + \frac{n}{2} - \frac{1}{q} + n(\frac{1}{2} - \frac{1}{r}), \tag{10}$$

$$\alpha_i \leq \beta_- + \frac{n}{2} - \frac{1}{q} + n(\frac{1}{2} - \frac{1}{r}) + (\frac{1}{2} - \frac{1}{q}). \tag{11}$$

New necessary conditions

Interaction between high and low frequency :

$$\alpha_i \leq \beta_- + \frac{n}{2} - \frac{1}{q} + \frac{1}{2}, \tag{12}$$

$$\alpha_i \leq \beta_- + \frac{n}{2} - \frac{1}{q} + n(\frac{1}{2} - \frac{1}{r}) + \frac{1}{r} - \frac{1}{q}. \tag{13}$$

Low frequency interactions $(++)$ *:*

$$\beta_0 \geq \frac{2}{q} - n(1 - \frac{1}{r}) - \frac{1}{2}. \tag{14}$$

An (almost) sharp theorem

Null form estimates :

$$\|D_0^{\beta_0} D_+^{\beta_+} D_-^{\beta_-}(\phi\psi)\|_{L_t^q L_x^r} \leq C(\|\phi(0)\|_{\dot{H}^{\alpha_1}} + \|\partial_t \phi(0)\|_{\dot{H}^{\alpha_1 - 1}})$$

$$\times(\|\psi(0)\|_{\dot{H}^{\alpha_2}} + \|\partial_t \psi(0)\|_{\dot{H}^{\alpha_2 - 1}})$$

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Theorem (S. Lee-V)

Suppose (1) holds and that (2)– (15) hold with strict inequalities.

a) If $n \geq 4$, the null form estimate holds.

c) For $n = 2$, $n = 3$ if $r > 2$ and $\frac{1}{q} < \frac{n}{4}$, or if $1 < r \leq 2$, null form estimate holds.

New necessary conditions

For $n = 3$. *Low frequency interactions* $(+-)$ *:*

$$\alpha_1 + \alpha_2 \geq \frac{3}{q} + \frac{1}{r} - 2. \tag{15}$$

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Theorem (S.Lee-K. Rogers-V)

For $n = 3$. Suppose (1) holds and that (2)– (15) and (16) hold with strict inequalities.

Then, the null form estimate holds.

Some ideas of the proof

1) Littlewood-Paley decomposition of the waves :

$$\phi = \sum_{j=-\infty}^{\infty} \phi_j, \quad \psi = \sum_{k=-\infty}^{\infty} \psi_k$$

ϕ_j, ψ_k are waves having frequency supports on region $|\xi| \sim 2^j$, $|\xi| \sim 2^k$.

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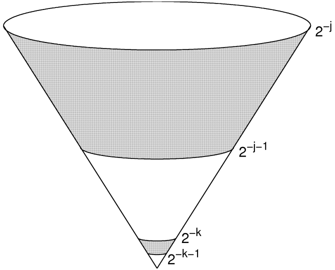

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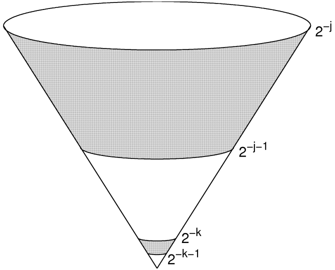


Then, to prove the estimate, it is sufficient to show that

$$\|D_0^{\beta} D_+^{\beta_+} D_-^{\beta_-}(\phi_j \psi_k)\|_{q,r} \leq C 2^{-\epsilon|j-k|}(2^{\alpha_1 j}\|\phi_j(0)\|_{L^2})(2^{\alpha_2 k}\|\psi_k(0)\|_{L^2}).$$

Some ideas of the proof

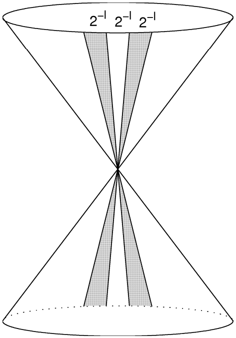
2) For each $l \geq 0$, decompose dyadically the double light cone into projective sectors Γ, Γ' with angle 2^{-l}

$$\phi_0 = \sum_{\Gamma} \phi_{0,\Gamma}, \qquad \psi_k = \sum_{\Gamma'} \psi_{k,\Gamma'}$$

We denote by $\angle(\Gamma, \Gamma')$ the angle between the sectors. Then, we have a Whitney type decomposition

$$\phi_0 \psi_k = \sum_{l \geq 0} \sum_{\Gamma, \Gamma'; \angle(\Gamma, \Gamma') \sim 2^{-l}} \phi_{0,\Gamma} \psi_{k,\Gamma'}.$$

Some ideas of the proof



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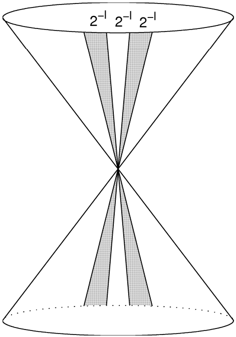
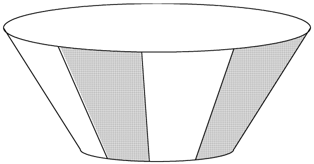


It is enough to show that if $\angle(\Gamma, \Gamma') \sim 2^{-l}$, then for $\epsilon > 0$ and $k, l \geq 0$,

$$\|D_0^{\beta} D_+^{\beta_+} D_-^{\beta_-} (\phi_{0,\Gamma} \psi_{k,\Gamma'})\|_{L_t^q L_x^r} \leq C 2^{-\epsilon(k+l)} 2^{\alpha_2 k} \|\phi_{0,\Gamma}(0)\|_{L^2} \|\psi_{k,\Gamma'}(0)\|_{L^2}.$$

Some ideas of the proof



Some ideas of the proof

Use the bilinear restriction theorem for the cone.

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Prove a bilinear restriction theorem for two waves Fourier supported at different frequences :

Theorem

$\widehat{f_1}$ supported in $\{\xi \in \mathbf{R}^n : \ |\xi| \sim 1, \ 0 < \xi \cdot e_1 < 1/4\}$ and $\widehat{f_2}$ supported in $\{\xi \in \mathbf{R}^n : \ |\xi| \sim 2^k, \ 0 < \xi \cdot e_2 < 1/4\}$.
Then, for $\epsilon > 0$ and $1 < q, r \le 2$ satisfying $1/q < \min(1, \frac{n+1}{4})$, $1/q < \frac{n+1}{2}(1 - \frac{1}{r})$, there is a constant C (independent of k, ϕ and ψ) such that

$$\|\phi\psi\|_{L^q_t L^r_x} (or \ \|\phi\bar{\psi}\|_{L^q_t L^r_x}) \le C 2^{(\frac{1}{q} - \frac{1}{2} + \epsilon)k} \|\phi(0)\|_{L^2} \|\psi(0)\|_{L^2}.$$

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Tao proved it for the case $q = r$.

Instead of the conditions :

$$\alpha_1 + \alpha_2 \geq \frac{1}{q},$$

$$\alpha_1 + \alpha_2 \geq \frac{3}{q} - n(1 - \frac{1}{r}).$$

Tao (Tataru) had to assume something stronger :

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Critical :

$$q = r = \frac{n+2}{n}, \qquad \alpha_1 + \alpha_2 = \frac{1}{q}.$$

This turned out to be related to the bilinear restriction theorem for paraboloids in $\mathbf{R} \times \mathbf{R}^{n-1}$.

Some ideas of the proof

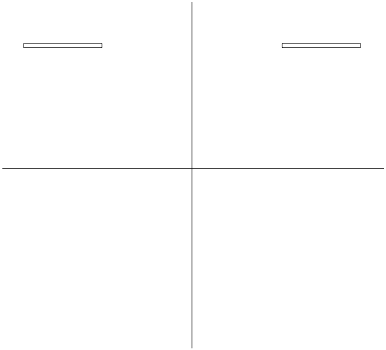
$$S_i = \{\xi \in \mathbb{R}^n : b \leq \xi_n \leq b + a, |\xi'/\xi_n + (-1)^i e'_{n-1}| \leq 1/2\}, \; i = 1, 2.$$

Theorem (++)

Let $\widehat{\phi(0)}$, $\widehat{\psi(0)}$ supported in S_1 and S_2 respectively. If $0 < a \ll 1, 1 \leq b \leq 2$ and $n \geq 2$, then for q, r satisfying $1/q < \min(1, n/4)$, $2/q < n(1 - 1/r)$, and for any $\epsilon > 0$, there is a constant C, independent of a and b, such that

$$\|\phi\psi\|_{L^q_t L^r_x} \leq C a^{1-\frac{1}{r}-\epsilon} \|\phi(0)\|_2 \|\psi(0)\|_2.$$

Some ideas of the proof



Some ideas of the proof

ϵ–removal lemma for mixed norms :

$I_R = [-R/2, R/2]$ and $Q'_R \subset \mathbb{R}^{n-1}$ It is enough to show that for and $\epsilon > 0$,

$$\|\phi_1\phi_2\|_{L^q_t(I_R)L^r_{x',x_n}(Q'_R\times\mathbb{R})} \leq Ca^{1-1/r}R^{\epsilon}\|f_1\|_2\|f_2\|_2$$

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The case $q = \infty$, $r = 1$ is trivial (Plancherel). Hence, by interpolation, it is enough to consider the case $q = 1$, $r = \frac{n}{n-2}$.

Some ideas of the proof

$$\theta(\xi') = \sqrt{1 + |\xi'|^2} - 1, \quad \xi' = (\xi_1, \xi_2, \ldots, \xi_{n-1}).$$

$$\phi_i(x,t) = \int_{S_i} e^{i(x'\xi' + x_n\xi_n + t\xi_n\theta(\xi'/\xi_n))} f_i(\xi) d\xi|, \quad i = 1, 2$$

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$$\phi_i(x,t) = \int_b^{b+a} \phi_i^s(x',t) e^{isx_n}\, ds.$$

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For fixed s, we decompose in wave packets, corresponding to tubes τ of dimension $(R^{1/2})^{n-1} \times R$.

$$\phi_i^s(x',t) = \sum_{\tau \in \mathcal{T}_i^s} \phi_{i,\tau}^s(x',t)$$

Some ideas of the proof

$$\|\phi_i^s(\cdot,t)\|_{L^2} \sim \Big(\sum_{\tau\in\mathcal{T}_i^s} \|\phi_{i,\tau}^s(\cdot,t)\|_{L^2}^2\Big)^{1/2}.$$

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$$\|\phi_i^s(\cdot,t)\|_{L^2} \sim \Big(\sum_{\tau\in\mathcal{T}_i^s} \|\phi_{i,\tau}^s(\cdot,t)\|_{L^2}^2\Big)^{1/2}.$$

Let $\{B\}$ be a collection of $R^{1-\delta}$ cubes which partition $Q_R' \times I_R$. Then, there are relations $\sim_1^s$, $\sim_2^s$ between B and $\tau \in \mathcal{T}_i^s$,

$$\sum_B \| \sum_{\tau\sim_i^s B} f_{i,\tau}^s\|_2^2 \leq CR^{\epsilon}\|f_i^s\|_2^2$$

$$\| \sum_{\tau\not\sim_1^s B \text{ or } \tau'\not\sim_2^u B} \phi_{1,\tau}^s \phi_{2,\tau'}^u \|_{L^2(B)} \leq CR^{-(n-2)/4+c\delta}\|f_1^s\|_2\|f_2^u\|_2$$

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$$\Big\| \int_b^{b+a}\int_b^{b+a} \sum_{\tau \not\sim_1^s B \text{ or } \tau' \not\sim_2^{s'} B} \phi_{1,\tau}^s \phi_{s,\tau}^{s'} e^{i(s+s')x_n}\,ds\,ds' \Big\|_{L_t^1(I)L_{x',x_n}^{\frac{n}{n-2}}(B'\times\mathbb{R})}$$

$$\leq C a^{2/n} R^{c\delta} \|f_1\|_2 \|f_2\|_2.$$

Some ideas of the proof

$$\Big\| \int_b^{b+a} \int_b^{b+a} \sum_{\tau \not\sim_1^s B \text{ or } \tau' \not\sim_2^{s'} B} [\phi_{1,\tau}^s \phi_{s,\tau}^s] e^{i(s+s')x_n} ds ds' \Big\|_{L^2_{x',t}(B) L^2_{x_n}}$$

$$\leq C \int_b^{b+a} \Big\| \chi_{[b,b+a]}(s'-s) \Big(\sum_{\tau \not\sim_1^s B \text{ or } \tau' \not\sim_2^{s'-s} B} \phi_{1,\tau}^s \phi_{2,\tau}^{s'-s} \Big) \Big\|_{L^2_{x',t}(B) L^2_{s'}} ds$$

$$\leq C R^{-(n-2)/4 + C\delta} \int_b^{b+a} \big\| \chi_{[b,b+a]}(s'-s) \| f_1^s \|_2 \| f_2^{s-s'} \|_2 \big\|_{L^2_{s'}} ds$$

$$\leq C R^{-(n-2)/4 + C\delta} \int_b^{b+a} \| f_1^s \|_2 \Big(\int_b^{b+a} \| f_2^{s-s'} \|_2^2 ds' \Big)^{1/2} ds$$

$$\leq C R^{-(n-2)/4 + C\delta} a^{1/2} \| f \|_2 \| g \|_2$$

Some ideas of the proof

By Hölder's inequality in t,

$$\| \iint \sum_{\tau \not\sim_1^s B \text{ or } \tau' \not\sim_2^{s'} B} (\cdot) \|_{L^1_t(I) L^2_{x', x_n}(B' \times \mathbb{R})} \leq C a^{1/2} R^{-(n-4)/4 + c\delta} \|f\|_2 \|g\|_2.$$

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Finally, we interpolate

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We will take $\widehat{\phi(0)}$ supported in $B((1, 0, \ldots, 0), 2^{-m})$ and $\widehat{\psi(0)}$ supported in $B((-1, 2^{-m+3}, 0, \ldots, 0), 2^{-m})$.

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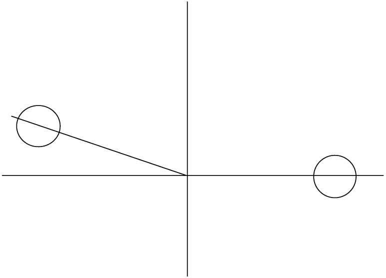

The counterexamples :

1) **About the multiplier :** $D_0^{\beta} D_+^{\beta_+} D_-^{\beta_-}$

If $(\xi, \tau) \in supp\,\widehat{(\phi\psi)} = supp\,(\widehat{\phi} * \widehat{\psi})$, then, $|\xi| \sim 2^{-m}$ and $|\tau| \sim 1$. Hence $|\xi| + |\tau| \sim 1$ and $|\tau| - |\xi| \sim 1$.

$$\|D_0^{\beta} D_+^{\beta_+} D_-^{\beta_-}(\phi_j \psi_k)\|_{q,r} \sim 2^{-m\beta_0} \|\phi_j \psi_k\|_{q,r}.$$

The counterexamples :

2) **About** ϕ : Take $\widehat{\phi(0)}$ the characteristic function of $B(e_1, 2^{-m})$.

$$\phi(x,t) = \int_B e^{i(x\cdot\xi + t|\xi|)}\, d\xi.$$

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If $x = (x_1, x')$ and $|x'| \le \frac{1}{10}2^m$, $|x_1 + t| \le \frac{1}{10}2^m$ and $|t| \le \frac{1}{10}2^{2m}$, then $|\phi(x,t)| \ge c2^{-nm}$.

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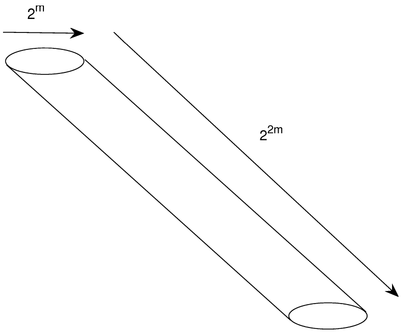

The counterexamples :

Proof : Note that

$$|\xi| = \sqrt{\xi_1^2 + |\xi'|^2} = |\xi|\sqrt{1 + \frac{|\xi'|^2}{|\xi_1|^2}} \sim \xi_1 \left[1 + O\left(\frac{|\xi'|^2}{|\xi_1|^2}\right)\right]$$

$$\sim \xi_1 + \frac{|\xi'|^2}{|\xi_1|} \sim \xi_1 + O(2^{-2m}).$$

Hence,

$$|x\cdot\xi + t|\xi|| \le |x'\cdot\xi'| + |x_1\xi_1 + t\xi_1 + tO(2^{-2m})| \le |x'\cdot\xi'| + |(x_1+t)\xi_1| + |tO(2^{-2m})|$$

3) **About** ψ : Take $B' = B((-1, 2^{-m+3}, 0, \ldots, 0), 2^{-m})$.

$$\widehat{\psi_i(0)}(\xi) = e^{i\xi_i \cdot x_j} \chi_{B'}$$

for some convenient x_i, and

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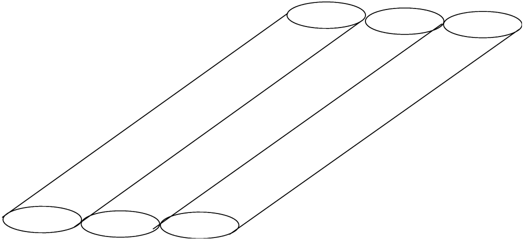

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This is the projection on the ξ-variable of the region

$$1/2 \leq \xi_1 \leq 3/4, \quad \tau = |\xi|, \quad |\tau + \xi_1 - 2| \leq 2^{-k}$$

The counterexamples :

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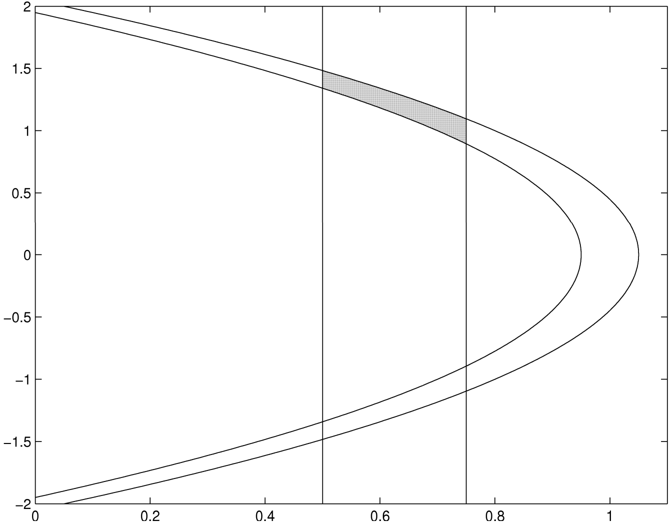

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Then $\psi(x,t) \sim 2^{-k}$ in the region $|x'| \leq \frac{1}{10}$, $|x_1 - t| \leq \frac{1}{10}$, $|t| \leq \frac{1}{10}2^k$.

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$D_0 \sim D_+ \sim 2^k, \qquad D_- \sim 1$

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Interaction between high and low frequency :

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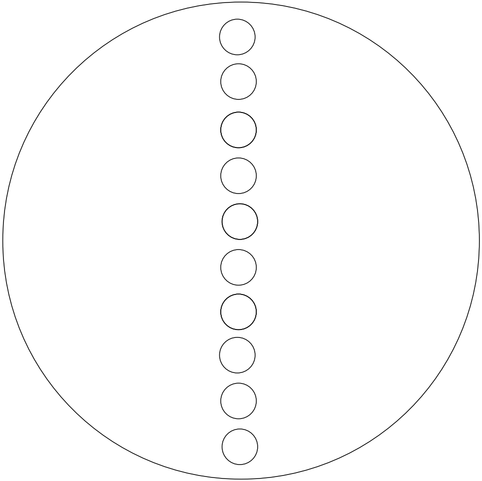

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