Upper and lower bounds for normal derivatives of spectral clusters of Dirichlet Laplacian

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Outline of Talk:

1. Problem setting and results

2. Examples

3. Rellich-type estimates and Perturbation estimates

4. Sketch proof of upper bounds

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I. Problem setting: Compact manifold (M, g) with boundary $Y = \partial M$.

- Dirichlet eigenvalue problem: $\Delta_M u + \lambda^2 u = 0,\ on\, M;\ u(x) = 0,\ on\, Y = \partial M$

- Discrete spectrum $0 \le \lambda_1^2 \le \lambda_2^2 \le \cdots \le \lambda_j^2 \le \cdots \to \infty$

- L^2-normalized eigenfunctions $\{e_0(x), e_1(x), \cdots, e_j(x) \cdots\}$

- Define the spectral projection operator(**Spectral Cluster**)

$$\begin{aligned}
\chi_\lambda^s f &= \sum_{\lambda_j \in [\lambda, \lambda+s)} e_j(f) = \int_M [\sum_{\lambda_j \in [\lambda, \lambda+s)} e_j(x) e_j(y)] f(y) dy \\
e_j(f) &= e_j(x) \int_M e_j(y) f(y) dy \quad \text{p}rojection\ onto\ eigenspace\ of\ \lambda_j.
\end{aligned}$$

Question: Do there exist constants $C_1, C_2 > 0$, depending on M but not on λ, such that

$$C_1 \lambda ||\chi_\lambda^s f||_{L^2(M)} \le ||\partial_n \chi_\lambda^s f||_{L^2(Y)} \le C_2 \lambda ||\chi_\lambda^s f||_{L^2(M)}\ ? \qquad (*)$$

• By heat kernel techniques, Ozawa [Osaka J. Math, 1993] showed the averaged version of (*) holds:

$$\sum_{j=1}^{\infty} e^{-t\lambda_j^2}[\partial_n e_j(y)]^2 \sim C_1(y)t^{-n/2-1}+C_2(y)t^{-n/2-1/2}+\cdots+C_k(y)t^{-n/2-3/2+k/2}+\cdots$$

$$\sum_{\lambda_j\le\lambda}[\partial_n e_j(y)]^2 = D_1\lambda^{n+2}+o(\lambda^{n+2}), \quad for\ each\ y\in Y.$$

Conjecture 1:

$$\sum_{\lambda_j\le\lambda}[\partial_n e_j(y)]^2 = D_1\lambda^{n+2}+D_2H(y)\lambda^{n+1}+o(\lambda^{n+1}), \quad for\ each\ y\in Y.$$

Conjecture 2: Do there exist constants $C_1, C_2 > 0$, depending on M but not on λ, such that

$$C_1\lambda_j \le ||\partial_n e_j||_{L^2(Y)} \le C_2\lambda_j$$

Single eigenfunction case:

- Hassell-Tao[Math.Res.Lett., 2002] answered for single eigenfunction:

Theorem (Hassell-Tao, 2002): Let (M, g) be a smooth compact Riemannian manifold with boundary.

Upper Bound: There exists $C > 0$ independent of j, such that

$$||\partial_n e_j||_{L^2(Y)} \leq C\lambda_j$$

Lower bound: Suppose M can be embedded in the interior of a compact manifold with boundary, N, of the same dimension, such that every geodesic in M eventually meets the boundary of N. There exists $C > 0$ independent of j, such that

$$||\partial_n e_j||_{L^2(Y)} \geq C\lambda_j$$

- In particular, both results hold for bounded domain of Euclidean space.

Spectral clusters case:

Theorem(X. Xu): Let (M, g) be a smooth compact Riemannian manifold with boundary.

Upper Bound:[2004] For any constant $s > 0$, there exists $C > 0$ independent of λ, such that

$$\sup_{y \in Y} \sum_{\lambda_j \in [\lambda, \lambda+s)} [\partial_n e_j(y)]^2 \leq C_s \lambda^{n+1}$$

$$||\partial_n \chi_\lambda^s f||_{L^2(Y)} \leq C\lambda ||\chi_\lambda^s f||_{L^2(M)}$$

Lower bound:[2007] Suppose M can be embedded in the interior of a compact manifold with boundary, N, of the same dimension, such that every geodesic in M eventually meets the boundary of N. For small constant $s > 0$, there exists $C > 0$ independent of λ, such that

$$||\partial_n \chi_\lambda^s f||_{L^2(Y)} \geq C\lambda ||\chi_\lambda^s f||_{L^2(M)}$$

In particular, both results hold for bounded domain of Euclidean space.

II. Examples:

• *Example 1 – the disc.* Let $M = \{x \in R^2 || x| < a\}$ for some $a > 0$. One has an equality

$$\int_{S^1} [\partial_n e_j(y)]^2 dy = \frac{2\lambda_j}{a}.$$

This follows from Rellichs identity, which also gives the same result in all dimensions.

• *Example 2 the cylinder.* Let $M = [0, 1] \times S^1$, the product of an interval with a unit circle . Then the normalized eigenfunctions are

$$u = \sqrt{1/\pi}\, \sin(m\pi x) e^{in\theta/2},\ m, n\ integers; \quad \lambda^2 = (m\pi)^2 + (n/2)^2$$

The L^2 norm of its normal derivative is $||\partial_n u||_2 = 2m\pi$.

Upper bound holds but no lower bound holds if one holds m fixed and sends n to infinity.

• *Example 3 the upper hemisphere.* Let M be the hemisphere

$$M = \{x \in R^3 | \, |x| = 1, < x, (0,0,1) >\geq 0\}.$$

In this case, the eigenfunctions are given by those spherical harmonics which are odd under reflection in the $(x1, x2)$ plane, namely, spherical harmonics

$$u = Y_{lm} = e^{im\phi} P_{lm}(cos\theta), \quad \lambda^2 = l(l+1),$$

where $-l \leq m \leq l$ and $l - m$ is odd, using spherical polar coordinates.

Take $m = l - 1$, then eigenfunction is

$$u = c_l e^{i(l-1)\phi} (\sin\theta)^{l-1} \cos\theta, \quad \partial_n u = c_l e^{i(l-1)\phi}$$

where c_l is the normalization factor. Direct computation gives

$$c_l^{-2} = \frac{1}{2l+1} \frac{(2l-2)(2l-4)\cdots 2}{(2l-1)(2l-3)\cdots 3} = \frac{1}{2l+1} 2^{2(l-1)} \frac{(l-1)!(l-1)!}{(2l-1)!}$$

which has asymptotic

$$c_l^{-2} \sim l^{-3/2}.$$

While

$$||\partial_n u||_2 = \sqrt{4\pi} c_l \sim l^{3/4} \sim \lambda^{3/4}.$$

III. Rellich-type estimates and Perturbation estimates

Lemma 1.(Rellich-type estimates) Let $u = \chi_\lambda^s f$ be the spectral projection of f. Then for any differential operator A,

$$\int_Y \partial_n u A u d\sigma = \int_M < u, [-\Delta, A]u > dg + \int_M < (-\Delta - \lambda^2)u, Au > dg$$
$$- \int_M < u, A(-\Delta - \lambda^2)u > dg.$$

Proof: Follow from $[-\Delta, A] = [-\Delta - \lambda^2, A]$ and

$$\int_Y \partial_n u A u d\sigma = \int_M < (-\Delta - \lambda^2)u, Au > dg - \int_M < u, (-\Delta - \lambda^2)Au > dg.$$

which follows from Green's formula and $u \equiv 0$ on Y.

Lemma 2.(Perturbation estimates) Let $u = \chi_\lambda^s f$ be the spectral projection of f, one has

$$||(-\Delta - \lambda^2)u||_2 \leq 3s\lambda||u||_2.$$

Proof: Direct computation:

$$\begin{aligned} ||(-\Delta - \lambda^2)u||_2^2 &= \int_M < (-\Delta - \lambda^2)u, (-\Delta - \lambda^2)u > dg \\ &= \sum_{\lambda_j \in [\lambda, \lambda+s)} (\lambda_j^2 - \lambda^2)^2 e_j^2(f) \\ &< \sum_{\lambda_j \in [\lambda, \lambda+s)} (2s\lambda + s^2)^2 e_j^2(f) \\ &< 9s^2\lambda^2||u||_2^2 \end{aligned}$$

IV. Sketch proof of upper bound

Choose geodesic coordinates (r, y) on $[0, \delta] \times Y$. Pick $A = \chi(r)\partial_r$, where $\chi \in C_c^\infty(R)$, $\chi(0) = 1$ and $\chi(r) = 0$ for $r > \delta$. From Lemma 1,

$$\int_Y \partial_n u A u d\sigma = \int_M < u, [-\Delta, A]u > dg + \int_M < (-\Delta - \lambda^2)u, Au > dg$$
$$- \int_M < u, A(-\Delta - \lambda^2)u > dg.$$

There are first order (vector-valued) differential operators B_1, B_2 with smooth coefficients.

$$||\partial_n u||^2_{L^2(Y)} = \int_Y \partial_n u A u d\sigma$$
$$|\int_M < u, [-\Delta, A]u > dg| = |\int_M < B_1 u, B_2 u > dg| \leq C \int_M < \nabla u, \nabla u > dg \leq 2\lambda^2 ||u||_2^2$$
$$|\int_M < (-\Delta - \lambda^2)u, Au > dg| \leq ||(-\Delta - \lambda^2)u||_2 ||Au||_2 \leq Cs\lambda^2 ||u||_2^2$$
$$|\int_M < u, A(-\Delta - \lambda^2)u > dg| \leq ||A(-\Delta - \lambda^2)u||_2 ||u||_2 \leq Cs\lambda^2 ||u||_2^2$$

Remark: Choose $A = Q^* Q \partial_r$ near the boundary, with Q an elliptic differential operator of order k in the y variables, one has the upper bound

$$||u||_{H^k(Y)} \leq C\lambda^k ||u||_2$$

for any integer k, and hence (by interpolation) any real k.

V. Sketch proof of lower bound

1. Simple proof for Euclidean bounded domains $M \subset R^n$

Choose $A = \sum_{i=1}^{n} x_i \partial_i$. One has $[-\Delta, A] = -2\Delta$, from Lemma 1,

$$\int_Y \partial_n u A u d\sigma = \int_M < u, [-\Delta, A]u > dg + \int_M < (-\Delta - \lambda^2)u, Au > dg$$
$$- \int_M < u, A(-\Delta - \lambda^2)u > dg.$$

One has

$$\int_Y \partial_n u A u d\sigma = \int_Y < n, x > (\partial_n u)^2 d\sigma \leq C||\partial_n u||^2_{L^2(Y)}$$
$$\int_M < u, [-\Delta, A]u > dg = \int_M < u, -2\Delta u > dg = 2\int_M < \nabla u, \nabla u > dg \geq 2\lambda^2 ||u||_2^2$$
$$|\int_M < (-\Delta - \lambda^2)u, Au > dg| \leq ||(-\Delta - \lambda^2)u||_2 ||Au||_2 \leq Cs\lambda^2 ||u||_2^2$$
$$|\int_M < u, A(-\Delta - \lambda^2)u > dg| \leq ||A(-\Delta - \lambda^2)u||_2 ||u||_2 \leq Cs\lambda^2 ||u||_2^2$$

Our next task is to get an identity for a first order classical pseudodifferential operator A similar to Lemma 1. Since A is now pseudodifferential and therefore non-local, it is important to take into account the fact that u is defined only on M. Let $\bar{u}$ denote the extension by zero to N.

Lemma 1'.(Rellich-type estimates) Let $u = \chi_\lambda^s f$ be the spectral projection of f, and let A be a first order classical pseudodifferential operator satisfying the transmission condition. Then

$$\begin{aligned}\int_M \langle \bar{u}, [H, A]\bar{u}\rangle \, dg \quad &+ \quad \int_M < (-\Delta - \lambda^2)u, Au > dg - \int_M < u, A(-\Delta - \lambda^2)u > dg \\ &= \quad 2\Im \int_Y \frac{\partial u}{\partial \nu} A\bar{u} \, d\sigma - \int_Y (\frac{\partial u}{\partial \nu})^2 c \, d\sigma,\end{aligned}$$

where $c(y) = \lim_{\rho\to\infty} \rho^{-1} a(0, y, \rho, 0)$.

2. Estimates for spectral clusters near the boundary

Choose geodesic coordinates (r, y) on $[0, \delta] \times Y$. The metric g can be written as

$$g = dr^2 + h_{ij} dy_i dy_j$$

Denote Y_r as the the submanifold $r \times Y$ under the geodesic coordinates with $r < \delta$.

Lemma 3. There exists $C > 0$, independent of λ, for $u = \chi_\lambda^s f$, such that

$$\int_{Y_r} u^2 d\sigma \leq C\lambda^2 r^2, \quad \forall\, r \in [0, \delta/2];$$

$$\int_{[0,r]\times Y} u^2 dr d\sigma \leq C\lambda^2 r^3, \quad \forall\, r \in [0, \delta/2].$$

3. Existence of A with positive commutator $[-\Delta_M, A]$

Goal: Construct a first order pseudodifferential operator A with positive commutator $[-\Delta_M, A]$.

Lemma 4.(Lemma 4.1 in [Hassell-Tao,2002]) Given any geodesic γ in S^*N, there is a first order, classical, self- adjoint pseudodifferential operator Q satisfying the transmission condition, and properly supported on N, such that the principal symbol $\sigma(i[H,Q])$ of $i[H,Q]$ is nonnegative on T^*M, and

$$\sigma(i[H,Q]) \geq \sigma(H) = |\xi|^2, \ \textit{on a conic neighbourhood } U_\gamma \textit{ of } \gamma \cap T^*M.$$

For each geodesic γ in S^*N, there is a conic neighbourhood U_γ. By compactness of S^*M, a finite number of the U_γ cover S^*M. Define $A = \sum Q_\gamma$. From Lemma 4, one has

$$\sigma(i[H,A]) \geq |\xi|^2, \quad \textit{on } T^*M.$$

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