Free Meixner states

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PLAN.

1. Fock space construction.

2. Free Meixner states: definition.

3. First property: orthogonal polynomials; martingales.

4. Second property: free cumulants.

5. Realization of the process.

6. Analogy: quadratic exponential families.

7. Examples.

Fock space construction.

Initial data:

Number $t > 0$.

$B_1, \ldots, B_d$ symmetric $d \times d$ matrices.

C diagonal $d^2 \times d^2$ matrix.

FOCK SPACE CONSTRUCTION.

Initial data:

Number $t > 0$.

$B_1, \dots, B_d$ symmetric $d \times d$ matrices.

C diagonal $d^2 \times d^2$ matrix.

Such that

$$t(I \otimes I) + C \geq 0$$

and

$$(B_i \otimes I)C = C(B_i \otimes I).$$

Let $\mathcal{H} = \mathbb{C}^d$ with an orthonormal basis $e_1, e_2, \ldots, e_d$.

$B_i =$ operator on $\mathcal{H}$, $C =$ operator on $\mathcal{H} \otimes \mathcal{H}$.

Let $\mathcal{H} = \mathbb{C}^d$ with an orthonormal basis $e_1, e_2, \ldots, e_d$.

$B_i =$ operator on $\mathcal{H}$, $C =$ operator on $\mathcal{H} \otimes \mathcal{H}$.

$$\begin{aligned}\mathcal{F}_{\mathsf{alg}}(\mathcal{H}) &= \bigoplus_{i=0}^{\infty} \mathcal{H}^{\otimes i} = \mathbb{C}\Omega \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus \ldots \\ &= \text{ vector space of non-commutative polynomials in } e_1, e_2, \ldots, e_d.\end{aligned}$$

Inner product

$$\langle \eta, \zeta \rangle_C = \langle \eta, K_C \zeta \rangle \,,$$

$\langle \cdot, \cdot \rangle =$ the usual tensor inner product.

K_C is the non-negative kernel: on $\mathcal{H}^{\otimes 4}$,

$$K_C = t^4 \Big(I^{\otimes 2} \otimes (I^{\otimes 2} + C/t) \Big) \Big(I \otimes (I^{\otimes 2} + C/t) \otimes I \Big) \Big((I^{\otimes 2} + C/t) \otimes I^{\otimes 2} \Big).$$

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Factor out vectors of length zero, complete, get the Fock space $\mathcal{F}_C^{(t)}(\mathcal{H})$.

Operators

$$
\begin{aligned}
a_i^+ \quad & \left(e_{u(1)} \otimes e_{u(2)} \otimes \ldots \otimes e_{u(k)}\right) = e_i \otimes e_{u(1)} \otimes e_{u(2)} \otimes \ldots \otimes e_{u(k)}, \\
a_i^- \quad & \left(e_{u(1)} \otimes e_{u(2)} \otimes \ldots \otimes e_{u(k)}\right) = \delta_{i,u(1)} e_{u(2)} \otimes \ldots \otimes e_{u(k)},
\end{aligned}
$$

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\begin{aligned}
B_i \quad & = B_i \otimes I^{\otimes(k-1)} \text{ on } \mathcal{H}^{\otimes k}, \\
a_i^- C \quad & = a_i^- (C \otimes I^{\otimes(k-2)}) \text{ on } \mathcal{H}^{\otimes k}.
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Define

$$X_i = a_i^+ + t a_i^- + B_i + a_i^- C.$$

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Define

$$X_i = a_i^+ + t a_i^- + B_i + a_i^- C.$$

Lemma. Each X_i factors through to $\mathcal{F}_C^{(t)}(\mathcal{H})$. Each X_i is symmetric and bounded, so self-adjoint.

Non-commutative polynomials.

$$\mathbb{R}\langle \mathbf{x} \rangle = \mathbb{R}\langle x_1, x_2, \ldots, x_d \rangle$$

Involution

$$(x_1\, x_2\, x_1\, x_1\, x_3)^* = x_3\, x_1\, x_1\, x_2\, x_1.$$

NON-COMMUTATIVE POLYNOMIALS.

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φ a state on $\mathbb{R}\langle \mathbf{x} \rangle$: linear functional, $\varphi[1] = 1$,

$$\varphi[P^*] = \varphi[P],$$

$$\varphi[P^* P] \geq 0.$$

Example of a state: joint distribution $\varphi^{(t)}_{C,\{B_i\}}$ of $\{X_1, X_2, \ldots, X_d\}$ is a state on $\mathbb{R}\langle x_1, \ldots, x_d \rangle$

$$\varphi^{(t)}_{C,\{B_i\}}\left[P(x_1, x_2, \ldots, x_d)\right] = \langle \Omega, P(X_1, X_2, \ldots, X_d)\Omega \rangle \, .$$

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Definition. A state φ is a free Meixner state if

$$\varphi = \varphi^{(t)}_{C,\{B_i\}}$$

for some $t, C, \{B_i\}$.

Free Meixner states come in families:

$$\left\{\varphi_{C,\{B_i\}}^{(t)} : t \in [t_0, +\infty)\right\}$$

for fixed $C, \{B_i\}$: $t_0 = -\min C_{ij}$. Usually write

$$\varphi^{(t)} = \varphi_{C,\{B_i\}}^{(t)}.$$

Will interpret t as a convolution parameter.

“Noncommutative dynamics.”

Free Meixner states: first property.

Monic orthogonal polynomials

$$P_i(\mathbf{x}) = x_i + \dots,$$
$$P_{ij}(\mathbf{x}) = x_i x_j + \dots$$

$$\{P_{\vec{u}}(\mathbf{x})\} = \Big\{1, P_i(\mathbf{x}), P_{ij}(\mathbf{x}), P_{ijk}(\mathbf{x}), \dots\Big\}$$

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Gram-Schmidt: can make orthogonal

$$\langle P_{\vec{u}}, P_{\vec{v}} \rangle = \varphi\left[P_{\vec{u}}^* P_{\vec{v}}\right] = 0$$

unless $\vec{u} = \vec{v}$.

Proposition. A state φ is a free Meixner state if and only if its monic orthogonal polynomials have a generating function

$$\sum_{\vec{u}} P_{\vec{u}}(\mathbf{x}) z_{\vec{u}} = 1 + \sum_i P_i(\mathbf{x}) z_i + \sum_{i,j} P_{ij}(\mathbf{x}) z_i z_j + \dots$$
$$= F(\mathbf{z}) \left(1 - \sum_i x_i G_i(\mathbf{z}) \right)^{-1}$$

for some $F(\mathbf{z}), G_i(\mathbf{z})$.

Proposition. More precisely, for φ a free Meixner state

$$\begin{aligned}\sum_{\vec{u}} P_{\vec{u}}(\mathbf{x}) z_{\vec{u}} &= 1 + \sum_i P_i(\mathbf{x}) z_i + \sum_{i,j} P_{ij}(\mathbf{x}) z_i z_j + \dots \\ &= \left(1 - \sum_i x_i U_i(\mathbf{z}) + R(\mathbf{U}(\mathbf{z}))\right)^{-1}\end{aligned}$$

for some $U_i(\mathbf{z})$ and $R(\mathbf{z}) =$ free cumulant generating function of φ.

Proposition. Even more precisely, for φ a free Meixner state

$$\begin{aligned}\sum_{\vec{u}} P_{\vec{u}}(\mathbf{x}) z_{\vec{u}} &= 1 + \sum_i P_i(\mathbf{x}) z_i + \sum_{i,j} P_{ij}(\mathbf{x}) z_i z_j + \dots \\ &= \left(1 - \sum_i x_i (\mathbf{D}R)^{\langle -1 \rangle}(\mathbf{z}) + R((\mathbf{D}R)^{\langle -1 \rangle}(\mathbf{z}))\right)^{-1}\end{aligned}$$

for $R(\mathbf{z}) =$ free cumulant generating function of φ, $\mathbf{D}$ a "left non-commutative gradient", and $A^{\langle -1 \rangle}$ the inverse under composition.

Example:

$$D_1(z_1 z_2 z_1^2) = z_2 z_1^2$$
$$D_1(z_2 z_1 z_2 z_1^2) = 0.$$

Why interesting?

In a von Neumann algebra $(\mathcal{A}, \tau)$, let $\{Y_t : t \geq 0\}$ be a process with freely independent increments. Let

$$\mathcal{A}_t = W^*(Y_s : 0 \leq s \leq t)$$

and $R_t(z) =$ free cumulant generating function of Y_t. Then (Biane 1998) for any function U,

$$\tau\left[\left.\frac{1}{1 - Y_t U(z) + R_t(U(z))}\right| \mathcal{A}_s\right] = \frac{1}{1 - Y_s U(z) + R_s(U(z))}.$$

That is,

$$\frac{1}{1 - Y_t U(z) + R_t(U(z))}$$

is a martingale.

True in many variables, at least if τ is a trace:

$$\left(1 - \sum_i X_i(t) U_i(\mathbf{z}) + R_t(\mathbf{U}(\mathbf{z}))\right)^{-1}$$

is a martingale.

Corollary. For free Meixner states, orthogonal polynomials are martingales for the corresponding processes.

Free cumulant functional.

For any functional φ, have a unique free cumulant functional R_φ on $\mathbb{R}\langle x_1, x_2, \ldots, x_d \rangle$. Free cumulant generating function:

$$R(\mathbf{z}) = \sum_{\vec{u}} R\left[x_{\vec{u}}\right] z_{\vec{u}} = \sum_{i} R\left[x_i\right] z_i + \sum_{i,j} R\left[x_i x_j\right] z_i z_j + \ldots .$$

Properties (Voiculescu, Speicher):

1) φ is a free product $\varphi = \varphi_1 * \varphi_2 * \ldots * \varphi_d$ on $\mathbb{R}\langle x_1, x_2 \ldots, x_d \rangle$

$$\Leftrightarrow$$

$$x_1, x_2, \ldots, x_d \text{ freely independent}$$

$$\Leftrightarrow$$

$$R\left[x_{u(1)} x_{u(2)} \ldots x_{u(k)}\right] = 0$$

unless all $u(1) = u(2) = \ldots u(k)$.

2) φ is a free convolution $\varphi = \mu \boxplus \nu$

$$\Leftrightarrow$$

$$R_\varphi = R_\mu + R_\nu.$$

Operator model for the free cumulant functional.

Recall

$$\varphi\left[P(x_1, \ldots, x_d)\right] = \left\langle \Omega, P(X_1, \ldots, X_d)\Omega \right\rangle .$$

Theorem. For a free Meixner state, $R\left[x_i\right] = 0$, and

$$R\left[x_i P(x_1, \ldots, x_d) x_j\right] = t \left\langle e_i, P(S_1, \ldots, S_d) e_j \right\rangle ,$$

where

$$S_i = a_i^+ + B_i + a_i^- C = X_i - t a_i^- .$$

Free Meixner states: second property.

Corollary. φ is a free Meixner state if and only if its free cumulant generating function R satisfies the PDEs

$$D_i D_j R(\mathbf{z}) = \delta_{ij} t + \sum_k B_{ij}^k (D_i R)(\mathbf{z}) + \frac{1}{t} C_{ij} (D_i R)(\mathbf{z})\ (D_j R)(\mathbf{z}).$$

Free Meixner states: third property.

Recursion relation for their orthogonal polynomials.

Since

$$R_{\varphi}^{(t)}\left[x_i P(x_1, \ldots, x_d) x_j\right] = t\left\langle e_i, P(S_1, \ldots, S_d) e_j\right\rangle,$$

$$\varphi^{(t)} \boxplus \varphi^{(s)} = \varphi^{(t+s)},$$

so

$$\left\{\varphi^{(t)} : t \in [t_0, +\infty)\right\}$$

form a free convolution semigroup.

If $t \in [0, +\infty)$, φ freely infinitely divisible.

Always can continue $\varphi^{(t_0)}$ to larger t: free phenomenon.

PROCESS.

If φ freely infinitely divisible, can realize all $\varphi^{(t)}$ as distributions of a single process: can put all $X_i(t)$ on the same Hilbert space. An idea of Śniady.

Let

$$H = \mathcal{H} \otimes L^\infty([0,1], dx) \subset \mathcal{H} \otimes L^2([0,1], dx).$$

Inner product

$$\langle \eta \otimes f, \zeta \otimes g \rangle = \langle \eta, \zeta \rangle \int_0^1 f(x) g(x) \, dx = \langle \eta, \zeta \rangle \langle fg \rangle.$$

Its algebraic Fock space

$$\mathcal{F}_{\mathsf{alg}}(H).$$

Inner product

$$\begin{aligned}
&\langle (\eta_1 \otimes f_1) \otimes \ldots \otimes (\eta_l \otimes f_l), (\zeta_1 \otimes g_1) \otimes \ldots \otimes (\zeta_n \otimes g_n) \rangle_C \\
&\quad = \delta_{ln} \sum_{\substack{\Lambda \subset \{1,\ldots,n-1\} \\ \pi(\Lambda)=(V_1,V_2,\ldots,V_k)}} \left\langle \eta_1 \otimes \ldots \otimes \eta_n, C^{\Lambda^c} (\zeta_1 \otimes \ldots \otimes \zeta_n) \right\rangle \\
&\qquad \times \prod_{j=1}^{k} \left(\int_{\mathbb{R}} \left[\prod_{i \in V_j} f_i(x) g_i(x) \right] dx \right),
\end{aligned}$$

where Λ^c is the complement $\{1, \ldots, n-1\} \setminus \Lambda$, and

$$C^{\Lambda^c} = \prod_{i \in \Lambda^c} I^{\otimes(i-1)} \otimes C \otimes I^{\otimes(n-i-1)}.$$

Example:

$$
\begin{aligned}
&\Big\langle (\eta_1 \otimes f_1) \otimes (\eta_2 \otimes f_2) \otimes (\eta_3 \otimes f_3), (\zeta_1 \otimes g_1) \otimes (\zeta_2 \otimes g_2) \otimes (\zeta_3 \otimes g_3) \Big\rangle_C \\
&= \Big\langle \eta_1 \otimes \eta_2 \otimes \eta_3, (C \otimes I)(I \otimes C)(\zeta_1 \otimes \zeta_2 \otimes \zeta_3) \Big\rangle \langle f_1 g_1 f_2 g_2 f_3 g_3 \rangle \\
&\quad + \Big\langle \eta_1 \otimes \eta_2 \otimes \eta_3, (C \otimes I)(\zeta_1 \otimes \zeta_2 \otimes \zeta_3) \Big\rangle \langle f_1 g_1 \rangle \langle f_2 g_2 f_3 g_3 \rangle \\
&\quad + \Big\langle \eta_1 \otimes \eta_2 \otimes \eta_3, (I \otimes C)(\zeta_1 \otimes \zeta_2 \otimes \zeta_3) \Big\rangle \langle f_1 g_1 f_2 g_2 \rangle \langle f_3 g_3 \rangle \\
&\quad + \Big\langle \eta_1 \otimes \eta_2 \otimes \eta_3, \zeta_1 \otimes \zeta_2 \otimes \zeta_3 \Big\rangle \langle f_1 g_1 \rangle \langle f_2 g_2 \rangle \langle f_3 g_3 \rangle
\end{aligned}
$$

Complete $\mathcal{F}_{\mathsf{alg}}(H)$, get $\mathcal{F}_C(H)$.

Operators:

$$a_i^{+(t)}\Big((\eta_1\otimes f_1)\otimes\ldots\otimes(\eta_n\otimes f_n)\Big)$$
$$= (e_i\otimes \mathbf{1}_{[0,t)})\otimes(\eta_1\otimes f_1)\otimes\ldots\otimes(\eta_n\otimes f_n),$$
$$a_i^{-(t)}\Big((\eta_1\otimes f_1)\otimes\ldots\otimes(\eta_n\otimes f_n)\Big)$$
$$= \langle e_i,\eta_1\rangle\left(\int_0^t f_1(x)\,dx\right)(\eta_2\otimes f_2)\otimes\ldots\otimes(\eta_n\otimes f_n),$$
$$B_i^{(t)}\Big((\eta_1\otimes f_1)\otimes\ldots\otimes(\eta_n\otimes f_n)\Big)$$
$$= (B_i\eta_1\otimes f_1\mathbf{1}_{[0,t)})\otimes(\eta_2\otimes f_2)\otimes\ldots\otimes(\eta_n\otimes f_n),$$
$$\tilde{a}_i^{(t)}\Big((\eta_1\otimes f_1)\otimes\ldots\otimes(\eta_n\otimes f_n)\Big)$$
$$= \Big((a_i^- C(\eta_1\otimes\eta_2))\otimes(f_1\mathbf{1}_{[0,t)}f_2)\Big)\otimes(\eta_3\otimes f_3)\otimes\ldots\otimes(\eta_n\otimes f_n),$$

where $\mathbf{1}_{[0,t)}$ is the indicator function of the interval $[0,t)$. Let

$$X_i(t) = a_i^{+(t)} + B_i^{(t)} + a_i^{-(t)} + \tilde{a}_i^{(t)}.$$

Each $X_i(t)$ self-adjoint on $\mathcal{F}_C(H)$.

If all $f_i = g_i = \mathbf{1}_{[0,t)}$, then

$$\begin{aligned}\langle(\eta_1 \otimes f_1) \otimes \ldots \otimes (\eta_n \otimes f_n), (\zeta_1 \otimes g_1) \otimes \ldots \otimes (\zeta_n \otimes g_n)\rangle_C \\ = t^n \langle \eta_1 \otimes \ldots \otimes \eta_n, \zeta_1 \otimes \ldots \otimes \zeta_n \rangle_{C/t} .\end{aligned}$$

Moreover, each $X_i(t)$ restricted to

$$\mathcal{F}_C(\mathcal{H} \otimes \mathsf{Span}\left(\mathbf{1}_{[0,t)}\right)) \cong \mathcal{F}_C^{(t)}(\mathcal{H})$$

is exactly X_i.

So $\{\mathbf{X}(t)\} =$ process with free increments.

ANALOGY: QUADRATIC EXPONENTIAL FAMILIES.

Let $\mu =$ probability measure.

Denote

$$\mathcal{L}_\mu(z) = \log \int_{-\infty}^{\infty} e^{xz}\, d\mu(x)$$

Analogy: quadratic exponential families.

Let $\mu =$ probability measure.

Denote

$$\mathcal{L}_\mu(z) = \log \int_{-\infty}^{\infty} e^{xz}\, d\mu(x)$$

For a function $U(z)$, let

$$d\mu^{(z)}(x) = e^{xU(z)-\mathcal{L}(U(z))}\, d\mu(x).$$

Exponential family generated by μ.

Cf. Gibbs measures.

A family of polynomials $\{P_n(x)\}$ is a Sheffer family if

$$\sum_{n=0}^{\infty} \frac{1}{n!} P_n(x) z^n = e^{xU(z) - \mathcal{L}(U(z))}.$$

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An exponential family is quadratic if

$$\mathcal{L}''(z) = 1 + b\mathcal{L}'(z) + c\left(\mathcal{L}'(z)\right)^2.$$

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Theorem. (Meixner 1934, Morris 1982) μ generates a quadratic exponential family $\Leftrightarrow$ μ-orthogonal polynomials are Sheffer.

Many other characterizations.

Multi-dimensional version: Feinsilver 1986, Pommeret 1996.

Partially classified: Letac 1989, Casalis 1991, 1996.

Compare

$$\mathcal{L}''(z) = 1 + b\mathcal{L}'(z) + c\left(\mathcal{L}'(z)\right)^2.$$

$$\Leftrightarrow$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} P_n(x) z^n = e^{xU(z) - \mathcal{L}(U(z))}.$$

with

$$(D^2 R)(z) = \delta_{ij} + b(DR)(\mathbf{z}) + c\Big((DR)(z)\Big)^2.$$

$$\Leftrightarrow$$

$$\sum_{n=0}^{\infty} P_n(x) z^n = \Big(1 - xU(z) + R(U(z))\Big)^{-1}$$

ONE DIMENSIONAL FREE MEIXNER DISTRIBUTIONS.

Probability measures on $\mathbb{R}$.

Orthogonal polynomials $\{P_n(x)\}$.

Have a generating function

$$\sum_{n=0}^{\infty} P_n(x) z^n = F(z) \frac{1}{1 - xG(z)}.$$

Proposition. The R-transform R_μ satisfies

$$\frac{R_\mu}{z} = t + bR_\mu + (c/t)R_\mu^2.$$

for $b \in \mathbb{R}$, $c \geq -t$.

Complete description:

$$\frac{1}{2\pi t}\frac{\sqrt{4(t+c)-(x-b)^2}}{1+(b/t)x+(c/t^2)x^2}\,dx + \text{ zero, one, or two atoms.}$$

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Particular cases.

Free probability: free normal, free Poisson, free binomial.

Random matrices: semicircular, Marchenko-Pastur, Jacobi ensemble.

Kesten measures.

Other appearances:

Szegö (1922), Bernstein (1930), Boas & Buck (1956), Carlin & McGregor (1957), Geronimus (1961), Allaway (1972), Askey & Ismail (1983), Cohen & Trenholme (1984), Kato (1986), Freeman (1998), Saitoh & Yoshida (2001), M.A. (2003), Kubo, Kuo & Namli (2006), Belinschi & Nica (2007), . . .

Multi-dimensional examples.

Concentrate on tracial states.

1. Free semicircular system. Data $C = 0$, $B_i = 0$.

Each X_i has the semicircular distribution, freely independent.

$$\sum_{\vec{u}} P_{\vec{u}}(\mathbf{x}) z_{\vec{u}} = \left(1 - \sum_i x_i z_i + \sum_i z_i^2\right)^{-1}.$$

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$$\begin{aligned}
P_i(\mathbf{x}) &= x_i + \dots &&= U_1(x_i), \\
P_{ij}(\mathbf{x}) &= x_i x_j + \dots &&= U_1(x_i) U_1(x_j), \\
P_{ii}(\mathbf{x}) &= x_i^2 + \dots &&= U_2(x_i), \\
P_{1,2,1,1,1}(\mathbf{x}) &= x_1 x_2 x_1^3 + \dots &&= U_1(x_1) U_1(x_2) U_3(x_1),
\end{aligned}$$

where $U_i =$ Chebyshev polynomials.

2. Free products. Data $C_{ij} = c_i \delta_{ij}$, $B_i = b_i E_{ii}$.

Here $E_{ii} =$ projection onto e_i.

$$S_i = a_i^+ + b_i E_{ii} + a_i^- C$$

acts only on $\mathrm{Span}\left(e_i^{\otimes k} | k \geq 0\right)$. So

$$R\left[x_{u(1)} \ldots x_{u(k)}\right] = 0$$

unless all $u(j)$ equal. This means

$(X_1, X_2, \ldots, X_d) \sim$ free product of 1-dim free Meixner states.

Proposition. Let $C_{ij} = c_i \delta_{ij}$, B_i arbitrary, φ tracial. Then $\varphi =$ rotation of a free product state.

Particular cases:

Multi-dimensional semicircular distribution $=$ free product of semicircular distributions.

Multi-dimensional free Poisson distribution $=$ rotation of a free product of free Poisson distributions.

3. Multinomial. $C_{ij} \equiv -1$. (Recall $I \otimes I + C \geq 0$.)

Choose vectors $\{f_1, \ldots, f_d\}$ with

$$\langle f_i, f_i \rangle = p_i(1 - p_i),$$
$$\left\langle f_i, f_j \right\rangle = -p_i p_j,$$

where

$$p_i > 0, \qquad i = 1, 2, \ldots, d, \qquad p_1 + p_2 + \ldots + p_d = 1.$$

Let

$$B_i(f_i) = (1 - 2p_i) f_i,$$
$$B_i(f_j) = -p_i f_j - p_j f_i,$$

and define

$$X_i = a_i^{+} + B_i + a_i^{-} + a_i^{-} C + p_i.$$

Proposition. $X_i =$ orthogonal projection onto $\mathrm{Span}\,(f_i + p_i\Omega)$.

Orthogonal projections adding up to the identity.

They commute, φ factors through to a state on $\mathbb{R}[x_1, \ldots, x_d]$.

Can be identified with the measure

$$\varphi = p_1\delta_{e_1} + p_2\delta_{e_2} + \ldots p_d\delta_{e_d},$$

the multinomial distribution.

Note that the multinomial distribution is also classical Meixner.

Note: have

$$\left\{\varphi^{(t)} : t \geq 1\right\}$$

free multinomials.

No longer on commuting variables for $t > 1$. But:

Lemma. Some $\varphi^{(t)}$ a trace $\Leftrightarrow$ all $\varphi^{(t)}$ traces.

4. Generalization.

Lemma. If $\varphi_{C,\{B_i\}}$ is tracial, then for all i, j,

$$B_i e_j = B_j e_i$$

and

$$[B_i, B_j] = B_i B_j - B_j B_i = C_{ji} E_{ij} - C_{ij} E_{ji}.$$

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Let $C_{ij} \equiv c$. Analogs of “Simple quadratic exponential families.” Classified by Casalis. In a very different way.

Proposition. In this case, the necessary conditions

$$B_i e_j = B_j e_i$$

and

$$[B_i, B_j] = B_i B_j - B_j B_i = cE_{ij} - cE_{ji}.$$

are also sufficient for φ to be a trace.

Tracial on words of length $3, 4 \Rightarrow$ Trace.

OPERATOR ALGEBRAS.

1. Free products = free products.

2. $C_{ij} \equiv c$, $T_i = 0$. Ricard:

$$W^*(X_1, \ldots, X_d) = \begin{cases} L(\mathbb{F}_d) \\ L(\mathbb{F}_d) \oplus \mathcal{B}(\ell^2) \end{cases}$$

depending on d, c.

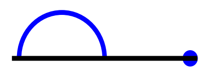
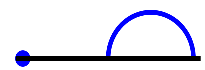
PROOFS: FREE CUMULANTS.

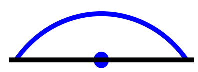
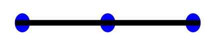
$$\varphi\left[x_{\vec{u}}\right] = \text{ moments of } \varphi.$$

$R_\varphi =$ free cumulant functional, another functional on $\mathbb{R}\langle \mathbf{x} \rangle$.

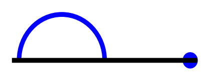
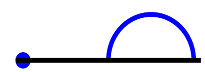
$$R_\varphi\left[x_{\vec{u}}\right] = \varphi\left[x_{\vec{u}}\right] - \sum_{\substack{\pi \in NC(n) \\ \pi \neq \hat{1}}} \prod_{B \in \pi} R_\varphi\left[\prod_{i \in B} x_{u(i)}\right].$$

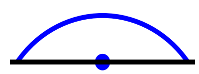
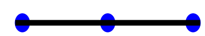
$$
\begin{aligned}
R\left[x_i\right] &= \varphi\left[x_i\right], \\
R\left[x_i x_j\right] &= \varphi\left[x_i x_j\right] - R\left[x_i\right] R\left[x_j\right], \\
R\left[x_i x_j x_k\right] &= \varphi\left[x_i x_j x_k\right] - R\left[x_i x_j\right] R\left[x_k\right] - R\left[x_i\right] R\left[x_j x_k\right] \\
&\quad - R\left[x_i x_k\right] R\left[x_j\right] - R\left[x_i\right] R\left[x_j\right] R\left[x_k\right]
\end{aligned}
$$

$= \varphi\left[x_i x_j x_k\right] -$  $-$ 

$-$  $-$  .

$$
\begin{aligned}
R\left[x_i\right] &= \varphi\left[x_i\right], \\
R\left[x_i x_j\right] &= \varphi\left[x_i x_j\right] - R\left[x_i\right] R\left[x_j\right], \\
R\left[x_i x_j x_k\right] &= \varphi\left[x_i x_j x_k\right] - R\left[x_i x_j\right] R\left[x_k\right] - R\left[x_i\right] R\left[x_j x_k\right] \\
&\qquad - R\left[x_i x_k\right] R\left[x_j\right] - R\left[x_i\right] R\left[x_j\right] R\left[x_k\right]
\end{aligned}
$$

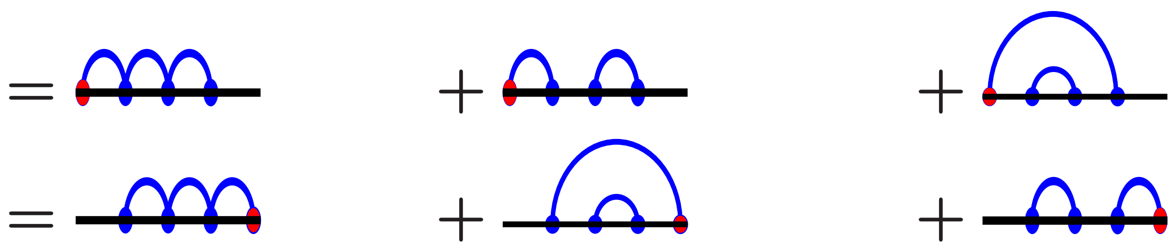
$= \varphi\left[x_i x_j x_k\right] -$  $-$ 

$-$  $-$  .

For simplicity, assume $\varphi\left[x_i\right] = 0$, $\varphi\left[x_i x_j\right] = \delta_{ij}$: mean zero, identity covariance.

Proof. φ a trace $\Leftrightarrow$ R_φ a trace.

$$\begin{aligned}\varphi\left[x_1x_2x_3x_4\right] &= R\left[x_1x_2x_3x_4\right] + R\left[x_1x_2\right]R\left[x_3x_4\right] + R\left[x_1x_4\right]R\left[x_2x_3\right] \\ &= R\left[x_2x_3x_4x_1\right] + R\left[x_2x_1\right]R\left[x_3x_4\right] + R\left[x_2x_3\right]R\left[x_4x_1\right].\end{aligned}$$



Proof of Lemma.

Recall $S_i = a_i^+ + B_i + ca_i^-$.

$$\begin{aligned} R[x_1x_2x_3x_4x_5] &= \langle e_1, S_2S_3S_4e_5\rangle \\ &= \langle e_1, B_2B_3B_4e_5\rangle + c\,\langle e_1, e_5\rangle\,\langle e_2, B_3e_4\rangle \\ &\quad + c\,\langle e_1, B_4e_5\rangle\,\langle e_2, e_3\rangle + c\,\langle e_1, B_2e_5\rangle\,\langle e_3, e_4\rangle\,. \end{aligned}$$

Proof of Lemma.

Recall $S_i = a_i^+ + B_i + c a_i^-$.

$$
\begin{aligned}
R\left[x_1 x_2 x_3 x_4 x_5\right] &= \langle e_1, S_2 S_3 S_4 e_5 \rangle \\
&= \langle e_1, B_2 B_3 B_4 e_5 \rangle + c \langle e_1, e_5 \rangle \langle e_2, B_3 e_4 \rangle \\
&\quad + c \langle e_1, B_4 e_5 \rangle \langle e_2, e_3 \rangle + c \langle e_1, B_2 e_5 \rangle \langle e_3, e_4 \rangle .
\end{aligned}
$$

But

$$
\begin{aligned}
\langle e_1, B_2 B_3 B_4 e_5 \rangle &= \langle e_2, B_1 B_3 B_4 e_5 \rangle \\
&= \langle e_2, [B_1, B_3] B_4 e_5 \rangle + \langle e_2, B_3 [B_1, B_4] e_5 \rangle + \langle e_2, B_3 B_4 B_5 e_1 \rangle \\
&= \langle e_2, B_3 B_4 B_5 e_1 \rangle \\
&\quad + c\Big(\langle e_2, e_1 \rangle \langle e_3, B_4 e_5 \rangle + \langle e_2, B_3 e_1 \rangle \langle e_4, e_5 \rangle \Big) \\
&\quad - c\Big(\langle e_2, e_3 \rangle \langle e_1, B_4 e_5 \rangle + \langle e_2, B_3 e_4 \rangle \langle e_1, e_5 \rangle \Big)
\end{aligned}
$$

Recall $S_i = a_i^+ + B_i + c a_i^-$.

$$\begin{aligned} R[x_1 x_2 x_3 x_4 x_5] &= \langle e_1, S_2 S_3 S_4 e_5 \rangle \\ &= \langle e_1, B_2 B_3 B_4 e_5 \rangle + c \langle e_1, e_5 \rangle \langle e_2, B_3 e_4 \rangle \\ &\quad + c \langle e_1, B_4 e_5 \rangle \langle e_2, e_3 \rangle + c \langle e_1, B_2 e_5 \rangle \langle e_3, e_4 \rangle . \end{aligned}$$

But

$$\begin{aligned} \langle e_1, B_2 B_3 B_4 e_5 \rangle &= \langle e_2, B_1 B_3 B_4 e_5 \rangle \\ &= \langle e_2, [B_1, B_3] B_4 e_5 \rangle + \langle e_2, B_3 [B_1, B_4] e_5 \rangle + \langle e_2, B_3 B_4 B_5 e_1 \rangle \\ &= \langle e_2, B_3 B_4 B_5 e_1 \rangle \\ &\quad + c\Big(\langle e_2, e_1 \rangle \langle e_3, B_4 e_5 \rangle + \langle e_2, B_3 e_1 \rangle \langle e_4, e_5 \rangle \Big) \\ &\quad - c\Big(\langle e_2, e_3 \rangle \langle e_1, B_4 e_5 \rangle + \langle e_2, B_3 e_4 \rangle \langle e_1, e_5 \rangle \Big) \end{aligned}$$

and

$$c \langle e_1, B_2 e_5 \rangle \langle e_3, e_4 \rangle = c \langle e_2, B_5 e_1 \rangle \langle e_3, e_4 \rangle ,$$

so

$$
\begin{aligned}
&R\left[x_1 x_2 x_3 x_4 x_5\right] \\
&\quad = \langle e_1, B_2 B_3 B_4 e_5 \rangle + c \langle e_1, e_5 \rangle \langle e_2, B_3 e_4 \rangle \\
&\qquad + c \langle e_1, B_4 e_5 \rangle \langle e_2, e_3 \rangle + c \langle e_1, B_2 e_5 \rangle \langle e_3, e_4 \rangle \\
&\quad = \langle e_2, B_3 B_4 B_5 e_1 \rangle + c \langle e_2, e_1 \rangle \langle e_3, B_4 e_5 \rangle \\
&\qquad + \langle e_2, B_3 e_1 \rangle \langle e_4, e_5 \rangle + c \langle e_2, B_5 e_1 \rangle \langle e_3, e_4 \rangle \\
&\quad = \langle e_2, S_3 S_4 S_5 e_1 \rangle = R\left[x_2 x_3 x_4 x_5 x_1\right].
\end{aligned}
$$

so

$$
\begin{aligned}
R\left[x_1 x_2 x_3 x_4 x_5\right] \\
&= \langle e_1, B_2 B_3 B_4 e_5 \rangle + c\,\langle e_1, e_5 \rangle \langle e_2, B_3 e_4 \rangle \\
&\quad + c\,\langle e_1, B_4 e_5 \rangle \langle e_2, e_3 \rangle + c\,\langle e_1, B_2 e_5 \rangle \langle e_3, e_4 \rangle \\
&= \langle e_2, B_3 B_4 B_5 e_1 \rangle + c\,\langle e_2, e_1 \rangle \langle e_3, B_4 e_5 \rangle \\
&\quad + \langle e_2, B_3 e_1 \rangle \langle e_4, e_5 \rangle + c\,\langle e_2, B_5 e_1 \rangle \langle e_3, e_4 \rangle \\
&= \langle e_2, S_3 S_4 S_5 e_1 \rangle = R\left[x_2 x_3 x_4 x_5 x_1\right].
\end{aligned}
$$

The same calculation works in general.