Constrained Liftings

(joint work with *Rolf Gohm*)

Santanu Dey

Institut für Mathematik u. Informatik
E-M-A-Universität Greifswald
17487 Germany

1. Introduction

- **Row-contraction:** $\underline{T} = (T_1, \cdots, T_d)$, $T_i \in B(\mathcal{H})$, $\sum_{i=1}^{d} T_i T_i^* \leq \mathbf{1}$ (or $\underline{T}\,\underline{T}^* \leq \mathbf{1}$).

  Coisometric if $\underline{T}\,\underline{T}^* = \mathbf{1}$.

- Minimal isometric dilation (mid) (G. Popescu) for row-contractions.

- Constrained dilations for row-constractions satisfying polynomial relations. (B.V.R.Bhat, Popescu, Zacharias, S.D.) (Arveson, Drury - commuting dilation)

- **Polynomial relations:** $\{p_\eta(\underline{z})\}_{\eta \in J}$ a finite set of polynomials in d n.c. variables (indexed by J).
  $\underline{T}$ is said to be J-constrained if

$$p_\eta(\underline{T}) = 0 \qquad \forall \eta \in J.$$

- Given $\underline{R}$ on $\mathcal{L}$, there exists maximal subspace $\mathcal{L}^J$ s.t.

  $$R_i = \begin{pmatrix} R_i^J & 0 \\ * & R_i^N \end{pmatrix} \quad \text{where} \quad R_i^J = P_{\mathcal{L}^J} R_i|_{\mathcal{L}^J}$$

  and $p_\eta(\underline{R}^J) = 0$ for all $\eta \in J$.
  $\underline{R}^J$:= **maximal** J**-constrained piece**.

- 1. $p_{i,j}(\underline{z}) = z_i z_j - z_j z_i$ for $(i,j) \in \{1, 2, \cdots, d\}^2$.

  2. $p_{i,j}(\underline{z}) = z_i z_j - q_{ji} z_j z_i$ for $(i,j) \in \{1, 2, \cdots, d\}^2$ where $|q_{ij}| = 1$ and $q_{ij} = q_{ji}^{-1}$.

  3. $p_{i,j}(\underline{z}) = z_i z_j - a_{ij} z_i z_j$ for $(i,j) \in \{1, 2, \cdots, d\}^2$ where $A = (a_{ij})_{d\times d}$ is a $0-1$-matrix.

  If case 1, we write J^s instead of J.
  If combination of all three, we write J'.

- $\underline{E} = (E_1, \ldots, E_d)$ on $\mathcal{H}_E = \mathcal{H}_C \oplus \mathcal{H}_A$.
  if $\underline{E}\ \underline{E}^* \leq \mathbf{1}$, $\quad E_i = \begin{pmatrix} C_i & 0 \\ B_i & A_i \end{pmatrix}$
  then $\underline{E}$ is a **lifting** of $\underline{C}$ by $\underline{A}$.

2. Minimal constrained dilation

- Denote by $\underline{V}$ the mid of $\underline{T}$.

- A result from previous work:

  Theorem 1 *Let* $\underline{T}$ *be a* J'*-constrained row contraction. Then* $\underline{V}^{J'}$ *is the minimal* J'*-constrained dilation of* $\underline{T}$.

- **Question:** Is $\underline{V}^{J'}$ the minimal constrained dilation of $\underline{T}^{J'}$?

  Theorem 2 *Suppose* $\dim \mathcal{H} < \infty, T_i \in B(\mathcal{H})$ *and* $\underline{T}\ \underline{T}^* = 1$*. Then:*
  (1) the maximal commuting subspace of $\underline{V}$ *is contained in* $\mathcal{H}$.
  (2) $\underline{V}^{J^s}$ *is the standard commuting dilation of* $\underline{T}^{J^s}$.

- A lifting $\underline{E}$ of $\underline{C}$ is **subisometric** if the mids of $\underline{E}$ and $\underline{C}$ are unitarily equivalent.

Corollary 3 *Let* $\underline{E}$ *on a f.d. Hilbert space be a coisometric lifting of* $\underline{C}$ *by a* $*$*-stable* $\underline{A}$ *(i.e.,* $\lim_{n\to\infty} \sum_{|\alpha|=n} \underline{A}_\alpha \underline{A}_\alpha^* = 0$*).*
Then $\underline{E}^{J^s} = \underline{C}^{J^s}$.

Theorem 4 *Suppose* $\underline{T}\ \underline{T}^* = \mathbf{1}$ *and* $\underline{T}^N$ *is* $*$*-stable. Then* $\underline{V}^{J'}$ *is the minimal* J'*-constrained dilation of a* $\underline{T}^{J'}$.

3. Characteristic function

- $\underline{E}$ is contractive lifting of $\underline{C}$, then there exist a contraction $\gamma : \mathcal{D}_{*,A} \to \mathcal{D}_C$ between the defect spaces.

- $\mathcal{H}_A^1 := \{h \in \mathcal{H}_A : \sum_{|\alpha|=n} \|A_\alpha^* h\|^2 = \|h\|^2$ for all $n \in \mathbb{N}\}$. If $\mathcal{H}_A^1 = 0$, then $\underline{A}$ is called **completely non-coisometric (cnc)**.

- $\underline{E}$ is **reduced lifting** of $\underline{C}$ if $\underline{A}$ is cnc and γ is injective.

- $\Gamma := \mathbb{C} \oplus \mathbb{C}^d \oplus (\mathbb{C}^d)^{\otimes 2} \oplus \cdots \oplus (\mathbb{C}^d)^{\otimes n} \oplus \cdots$

  $$L_i x := e_i \otimes x \quad \text{for} \quad x \in \Gamma, i = 1, \cdots, d,$$

  where $\{e_1, \cdots, e_d\}$ std basis of $\mathbb{C}^d$. Popescu's construction of mid on $\mathcal{H} \oplus (\Gamma \otimes \mathcal{D}_T)$ is

  $$\begin{aligned} V_i(h \oplus \sum_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes d_\alpha) &= T_i h \oplus [e_0 \otimes (D_T)_i h \\ &\qquad + e_i \otimes \sum_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes d_\alpha]. \end{aligned}$$

- a dilation $\underline{S}^T$ of $\underline{T}$ obtained by a compression of the Popescu's realization of mid to $\mathcal{H} \oplus (\Gamma_J \otimes \mathcal{D}_T)$ is called **pseudo J-constrained dilation.**

- $\mathbb{W} \underline{S}^E = (\underline{S}^C \oplus \underline{Z}) \mathbb{W}$
  for some unitary $\mathbb{W}$. The **constrained characteristic function** is defined as

  $$M_{J,C,E} := P_{\Gamma_J \otimes \mathcal{D}_C} \mathbb{W}|_{\Gamma_J \otimes \mathcal{D}_E}.$$

- Denote by $\mathbf{K}_J$ the space on which $\underline{S}^E$ are defined. $\Delta_{\theta_{J,C,E}} := (\mathbf{1} - M^*_{J,C,E} M_{J,C,E})^{\frac{1}{2}}$. For reduced liftings

$$\mathbb{W}\mathbf{K}_J := \mathcal{H}_C \oplus (\Gamma_J \otimes \mathcal{D}_C) \oplus \overline{\Delta_{\theta_{J,C,E}}(\Gamma_J \otimes \mathcal{D}_E)}, \tag{1}$$

$$\mathbb{W}\mathcal{H}_E := \mathbb{W}\mathbf{K}_J \ominus (M_{J,C,E} \oplus \Delta_{\theta_{J,C,E}})(\Gamma_J \otimes \mathcal{D}_E). \tag{2}$$

Theorem 5 *For a constrained row contraction,*
its **constrained reduced liftings** *are unitarily equivalent* <u>*if and only if*</u> *its* **constrained characteristic functions** *coincide.*

4. Invariants of Hilbert modules

- Assume γ to be isometry.

- From equation (1) and (2) we get

  $\|(M_0^J)^* v\| = \|P_{\mathbb{W}\mathcal{H}_E}(v \oplus 0)\|,$ for $v \in \Gamma \otimes \mathcal{D}_C$

  and

  $$\|v\|^2 = \|P_{\mathbb{W}\mathcal{H}_E}(v \oplus 0)\|^2 + \|M^*_{J,C,E} v\|^2.$$

  (where $M_0^J : \mathcal{H}_A \to \Gamma_J(\mathbb{C}^d) \otimes \mathcal{D}_C$

  $$M_0^J h := \sum_\alpha e_\alpha \otimes \gamma D_{*,A} A^*_\alpha h)$$

- comparing above two equations we get

$$M_0^J (M_0^J)^* + M_{J,C,E} M^*_{J,C,E} = \mathbf{1}_{\Gamma_J(\mathbb{C}^d) \otimes \mathcal{D}_C}. \tag{3}$$

- **Hilbert module** (Popescu, Kribs; and before in commuting case by Arveson):

  $\mathbb{C}\mathbb{F}_n^+$ = free semigroup algebra with n generators.
  To $\underline{E} = (E_1, \cdots, E_d)$ on $\mathcal{H}_E$, a Hilbert (left) module over $\mathbb{C}\mathbb{F}_n^+$ is associated, by

  $$g.h := g(E_1, \cdots, E_d)h, \text{ for } g \in \mathbb{C}\mathbb{F}_n^+, h \in \mathcal{H}_E,$$

  $$\|g_1 h_1 + \cdots + g_n h_n\|^2 \leq \|h_1\|^2 + \cdots + \|h_n\|^2.$$

- Assume rank $D_C < \infty$

- Following well-known invariants are defined as:

  $$curv A := \lim_{n\to\infty} \frac{\mathrm{tr}[K^*(A)(P_{\leq n} \otimes \mathbf{1}_{\mathcal{D}_{*,A}})K(A)]}{d^n},$$

  $$\chi(A) := \lim_{n\to\infty} \frac{\mathrm{rk}[K^*(A)(P_{\leq n} \otimes \mathbf{1}_{\mathcal{D}_{*,A}})K(A)]}{1 + d + \ldots + d^{n-1}},$$

  where $K(A)$ is the Poisson kernel of $\underline{A}$.

Theorem 6

$$curv A = \textit{rk} D_C - \lim_{n\to\infty} \frac{\textit{tr}[M_{C,E} M_{C,E}^* (P_{\leq n} \otimes \mathbf{1})]}{d^n},$$

$$\chi(A) = \lim_{n\to\infty} \frac{\textit{rk}[(\mathbf{1} - M_{C,E} M_{C,E}^*)(P_{\leq n} \otimes \mathbf{1})]}{1 + d + \ldots + d^{n-1}}.$$

- In commuting case

$$curv A = \int_{\partial \mathbb{B}_d} \lim_{r\to 1} \mathrm{tr}[\mathbf{1} - \theta_{J^s}(r\zeta)\theta_{J^s}^*(r\zeta)] d\sigma(\zeta),$$

$$= \mathrm{rk} D_C - (d-1)! \lim_{n\to\infty} \frac{\mathrm{tr}[\theta_{J^s}\theta_{J^s}^* (Q_{\leq n} \otimes \mathbf{1})]}{d^n},$$

$$\chi(A) = d! \lim_{n\to\infty} \frac{\mathrm{rk}[(\mathbf{1} - \theta_{J^s}\theta_{J^s}^*)(Q_{\leq n} \otimes \mathbf{1})]}{n^d}.$$