Type III factors distinguish type III E_0-semigroups

by

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Definitions & Notation

$H :=$ Separable infinite dim. Hilbert space.

A semigroup $\alpha = \{\alpha_t\}_{t>0}$ of unital $*$-endomorphisms of $\mathbb{B}(H)$ s.t. $t \mapsto \alpha_t(x)$ is continuous in the strong operator topology for $\forall x \in \mathbb{B}(H)$ is said to be an E_0-semigroup.

A continuous family of unitaries $\{U_t\}_{t>0}$ in $\mathbb{B}(H)$ is an α-cocycle
$\overset{\text{def}}{\Leftrightarrow}$ $U_{s+t} = U_s \alpha_s(U_t)$, $\forall s, t > 0$.

α acting on $\mathbb{B}(H)$ and β acting on $\mathbb{B}(K)$ are cocycle conjugate
$\overset{\text{def}}{\Leftrightarrow}$ $\exists V : H \to K$ unitary, $\exists \{U_t\}$ α-cocycle s.t.

$$\mathrm{Ad}(V^*) \cdot \beta_t \cdot \mathrm{Ad}(V) = \mathrm{Ad}(U_t) \cdot \alpha_t.$$

Goal
Classification up to cocycle conjugacy.

For an E_0-semigroup α, set

$\mathcal{E}_\alpha(t) := \{V \in \mathbb{B}(H);\ Vx = \alpha_t(x)V,\ \forall x \in \mathbb{B}(H)\}$.

$\mathcal{E}_\alpha(t)$ has a Hilbert space structure

$$\langle V, W \rangle 1_{\mathbb{B}(H)} := W^* V.$$

$\mathcal{E}_\alpha(s+t)$ is identified with $\mathcal{E}_\alpha(\alpha) \otimes \mathcal{E}_\alpha(\beta)$ by

$$\mathcal{E}_\alpha(s+t) \ni VW \leftrightarrow V \otimes W \in \mathcal{E}_\alpha(s) \otimes \mathcal{E}_\alpha(t).$$

$\{\mathcal{E}_\alpha(t)\}_{t>0}$ is (continuous tensor) <u>product system</u> (of Hilbert spaces).

$\{\mathcal{E}_\alpha(t)\}_{t>0}$ is a complete invariant for cocycle conjugacy (Arveson).

A C_0-semigroup $V = \{V_t\}_{t>0}$ of isometries acting on H is said to be a unit if $V_t \in \mathcal{E}_\alpha(t)$ for $\forall t > 0$.

Let $\mathcal{U}_\alpha$ denote the set of units.

- α is of type I $\overset{\text{def}}{\Leftrightarrow}$ α has enough units, i.e.

$$\text{span}\{V^1_{t_1}V^2_{t_2}\cdots V^n_{t_n};\ t_1 + \cdots t_n = t,\ V^i \in \mathcal{U}_\alpha\}$$

is dense in $\mathcal{E}_\alpha(t)$ for all $t > 0$.

- α is of type II $\overset{\text{def}}{\Leftrightarrow}$ $\mathcal{U}_\alpha \neq \emptyset$ but not of type I.

- α is of type III $\overset{\text{def}}{\Leftrightarrow}$ $\mathcal{U}_\alpha = \emptyset$.

Example

CCR(=CAR) flows.

$G = L^2((0,\infty), K)$,
$H = e^G$: Symmetric Fock space of G

$$H = \bigoplus_{n=0}^{\infty} G^{\otimes_s n}.$$

$W(f)$: The Weyl operator for $f \in G$,

$S = \{S_t\}_{t>0}$: Shift of $L^2((0,\infty), K)$,

$$(S_t f)(x) = \begin{cases} 0 & (x < t) \\ f(x-t) & (t \le x) \end{cases}$$

CCR flow of index $n = \dim K$ is defined by

$$\alpha_t(W(f)) = W(S_t f).$$

Every type I E_0-semigroup of index n is cocycle conjugate to the CCR flows of index n.

Examples of type III

• Powers (87): Quasi-free representation of CAR

• Tsirelson (2003): Off white noise (uncountably many).

$X(t)$: Gaussian generalized process
$E(X(t)X(s)) = C(s-t)$,
C : Positive definite distribution
$\widehat{C} = e^{\rho(\lambda)} d\lambda$: Fourier transformation
$e^{\rho(\lambda)}$: Spectral density function

$$\int_{\mathbb{R}^2} \frac{|\rho(x) - \rho(y)|^2}{|x-y|^2} dx dy < \infty$$

$e^{\rho(\lambda)} \sim \log^{\beta} |\lambda|$ for large λ, $\beta < 0$.

Main result (M.I.-Srinivasan)

There exist uncountably many mutually non-cocycle conjugate E_0-semigroups of type III satisfying:

- They have the same Tsirelson invariant as the CCR flows

$$e^{\rho(\lambda)} \sim 1, \quad (|\lambda| \to \infty)$$

- They are distinguished by the type of "local observable algebras".

Invariant

For E_0-semigroup α and finite interval $I = (s, t) \subset (0, \infty)$,

$$\mathcal{A}^\alpha(I) := \alpha_s(\mathbb{B}(H)) \cap \alpha_t(\mathbb{B}(H))'$$

For a bounded open set $U \subset (0, \infty)$,

$$\mathcal{A}^\alpha(U) := \bigvee_{I \subset U} \mathcal{A}^\alpha(I)$$

- $\{\mathcal{A}^\alpha(U)\}_U$ is a cocycle conjugacy invariant

- U finite union of intervals
$\Rightarrow \mathcal{A}^\alpha(U)$ is a type I factor

- α is type I $\Rightarrow \mathcal{A}^\alpha(U)$ is a type I factor

- $\mathcal{U}_\alpha \neq \emptyset \Rightarrow \mathcal{A}^\alpha(U)$ has a type I summand

Generalized CCR flows

Lemma

G : Real Hilbert space.

$\{S_t\}_{t\geq 0}$, $\{T_t\}_{t\geq 0}$: C_0-semigroups satisfying

(C1) $T_t^* S_t = I$ for $\forall t > 0$.

(C2) $T_t - S_t$ is Hilbert-Schmidt for $\forall t > 0$.

H : Symmetric Fock space of $G^{\mathbb{C}} = G \otimes \mathbb{C}$.

$W(f + ig)$, $f, g \in G$: Weyl operator.

$\Rightarrow \exists_1$ E_0-semigroup α satisfying

$$\alpha_t(W(f + ig)) = W(S_t f + iT_t g).$$

Def. Call α the generalized CCR flow associated with $(\{S_t\}, \{T_t\})$.

Th.(Bhat-Srinivasan, M.I.-Srinivasan)

GCCR flows are either of type I or type III

• When S_t is an isometry for all $t > 0$, one may identify G with $L^2((0,\infty),K)$ and $\{S_t\}_{t\geq 0}$ with the shift up to cocycle conjugacy.

• Tsirelson's examples of type III E_0-semigroups are obtained in this way with $\dim K = 1$.

Perturbation Problem

$\{S_t\}_{t>0}$: Shift semigroup of $L^2(0,\infty)$.
Classify C_0-semigroups $\{T_t\}_{t>0}$ acting on $L^2(0,\infty)$ satisfying

(C1) $T_t^* S_t = I$ for $\forall t > 0$.
(C2) $T_t - S_t$ is Hilbert-Schmidt for $\forall t > 0$.

Notation

For $f, g \in L^2(0, \infty)$,
(f, g) =Dual paring (not inner product),

$$\int_0^\infty f(x)g(x)dx.$$

$f \otimes g$: Rank one operator $(f \otimes g)h = (g, h)f$.

$\mathbb{H}_r$: The right half plane.

$e_z(x) := e^{-xz}$ for $z \in \mathbb{H}_r$.

$\mathcal{L}[f]$: Laplace transformation of f

$$\mathcal{L}[f](z) = \int_0^\infty f(x)e^{-xz}dx.$$

$\mathcal{G}(A)$: Graph of an operator A.

Infinitesimal theory

• A:=Generator of $\{S_t\}_{t>0}$.
$Af = -f'$, $f(0) = 0$
$A^* f = f'$, without boundary condition

• B:=Generator of $\{T_t\}_{t>0}$.
(C1) $\Leftrightarrow \langle S_t f, T_t g\rangle = \langle f, g\rangle \Leftrightarrow B \subset -A^*$
B is a differential operator $Bf = -f'$.
$D(B)$ is perturbed!

• $\exists$ non-zero vector in $\mathcal{G}(-A^*) \cap \mathcal{G}(B)^\perp$.

• For $p \in D(A^*)$, set $A_p f := -f'$

$$D(A_p) := \{f \in D(A^*);\ (f,p) + (f',p') = 0\}.$$

A_p is densely defined iff $p' \notin D(A)$.

• $\mathcal{O} := \{p \in D(A^*);\ p' \notin D(A)\}$.
$\exists p \in \mathcal{O}$ s.t. $B \subset A_p$ (in fact $B = A_p$).

Th. (M.I.)

(1) For $p \in \mathcal{O}$ and $z \in \mathbb{H}_r$, set

$$\begin{aligned}\mathcal{M}[p](z) &= (p, e_z) + (p', e_z') \\ &= \int_0^\infty \Big(p(x) - zp'(x)\Big) e^{-xz} dx.\end{aligned}$$

$\Rightarrow$ z is in the resolvent set of A_p if and only if $\mathcal{M}[p](z) \neq 0$ and

$$(zI - A_p)^{-1} = (zI - A)^{-1} + e_z \otimes \xi_{p,z},$$

$$\xi_{p,z} = -\frac{zp + p' + (1 - z^2) \int_0^\infty e^{-zt} S_t^* p dt}{\mathcal{M}[p](z)}.$$

(2) $\{T_t\}$: C_0-semigroup satisfying (C1)

B: generator of $\{T_t\}$

$\Rightarrow$ $B = A_p$ for some $p \in \mathcal{O}$.

$H^2(\mathbb{H}_r)$:= Hardy space for the right half plane.

$\mathcal{HD}$:= The set of analytic functions $M(z)$ on $\mathbb{H}_r$ s.t.
$M(z)/(1+z) \in H^2(\mathbb{H}_r)$ and $M(z) \notin H^2(\mathbb{H}_r)$.

Lemma

The map $\mathcal{O} \ni p \mapsto \mathcal{M}[p] \in \mathcal{HD}$ is a bijection.

- When $M = \mathcal{M}[p]$, we abuse notation
$A_M := A_p$, $\xi_{M,z} := \xi_{p,z}$.

For $M = 1$, $\xi_{M,z} = 0$ and $A_M = A$.

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$$\mathcal{L}[\xi_{M,z}](w) = \frac{M(z) - M(w)}{(z-w)M(z)}.$$

- $|M(i\lambda)|^2$ corresponds to spectral density function of Tsirelson's off white noise.
Our class of functions is slightly larger than the class he considered.

Def.

$\mathcal{HD}_b :=$ The set of $M \in \mathcal{HD}$ such that
A_M generates a C_0-semigroup
((C1) is automatically satisfied).

For $1 \leq p \leq \infty$,
$\mathcal{HD}_p :=$ The set of $M \in \mathcal{HD}_b$ such that
$e^{tA_M} - S_t$ belongs to the Schatten class C_p
for all $t > 0$.

Problem Characterize $\mathcal{HD}_2$.

It is a much harder problem to characterize $\mathcal{HD}_b$ (or $\mathcal{HD}_p$ with $p \neq 2$).

Global theory

For $M \in \mathcal{HD}_2$, set $K_t = e^{tA_M} - S_t \in C_2$.

- Semigroup relation

$\Rightarrow \exists$ kernel function $k(x, y)$ such that

$$K_t f(x) = \begin{cases} \int_0^\infty k(x-t, y) f(y) dy & (x < t) \\ 0 & (x \geq t) \end{cases}$$

$$||K_t||^2_{\text{H.S.}} = \int_0^t \int_0^\infty |k(x, y)|^2 dy dx < \infty, \quad \forall t > 0.$$

- For $z \in \mathbb{H}_r$ with sufficiently large Re z,

$$\begin{aligned} (zI - A_M)^{-1} &= \int_0^\infty e^{-tz} e^{tA_M} dt \\ &= (zI - A)^{-1} + \int_0^\infty e^{-tz} K_t dt. \end{aligned}$$

- Compare this with

$$(zI - A_M)^{-1} = (zI - A^{-1}) + e_z \otimes \xi_{M,z}.$$

$$\int_0^\infty e^{-tz} K_t dt = e_z \otimes \xi_{M,z}$$

implies

$$\xi_{M,z}(y) = \int_0^\infty k(r,y) e^{-rz} dr.$$

$$\int_0^\infty \int_0^\infty k(r,s) e^{-(rz+sw)} dr ds = \frac{M(z) - M(w)}{(z-w)M(z)}.$$

• The Paley-Wiener theorem characterizes $M \in \mathcal{HD}_2$!

<u>Th.</u> (M.I.)
$M \in \mathcal{HD}_2 \Leftrightarrow \exists a > 0$ s.t.

$$\sup_{x \geq a} \frac{1}{x} \int_{-\infty}^{\infty} \Big(\frac{\mathcal{P}[|M(i\cdot)|^2](x+iy)}{|M(x+iy)|^2} - 1 \Big) dy < \infty,$$

where $\mathcal{P}[f](x+iy)$ is the Poisson integral of f,

$$\mathcal{P}[f](x+iy) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x f(\lambda)}{x^2 + (y-\lambda)^2} d\lambda, \quad x > 0.$$

Cor

Let $M \in \mathcal{HD}$.

If there exist positive constants a, n, and C s.t.

$$\frac{1}{|M(z)|} \leq C(1+|z|)^n, \quad \forall \text{Re } z \geq a,$$

$$\int_{-\infty}^{\infty} \Big(\frac{\mathcal{P}[|M(i\cdot)|^2](x+iy)}{|M(x+iy)|^2} - 1 \Big) dy < \infty, \quad \exists x > a,$$

then $M \in \mathcal{HD}_2$.

Th.(M.I.)

$M(z) \in \mathcal{HD}$,

$B(z)$: Blachke component of $M(z)$,

$S(z)$: Singular inner component of $M(z)$,

$F(z)$: Outer component of $M(z)$.

Then,

$M \in \mathcal{HD}_2 \Leftrightarrow$ each component is in $\mathcal{HD}_2$.

Moreover,

- $B(z) \in \mathcal{HD}_2$ iff

$$B(z) = \Big(\frac{z-1}{z+1}\Big)^k \prod_n \frac{|1-\beta_n^2|}{1-\beta_n^2} \cdot \frac{z-\beta_n}{z+\overline{\beta_n}},$$

$$\sum_n \text{Re}\, \beta_n < +\infty.$$

- $S(z) \in \mathcal{HD}_2$ iff

$$S(z) = \exp\Big[-\int_{-\infty}^{\infty} \frac{\lambda z + i}{\lambda + iz} d\mu(\lambda)\Big],$$

$$\int_{-\infty}^{\infty} (1+\lambda^2) d\mu(\lambda) < +\infty.$$

- $F \in \mathcal{HD}_2$ if and only if

$$\sup_{y\in\mathbb{R}} \Big(\log \mathcal{P}[e^\rho](x+iy) - \mathcal{P}[\rho](x+iy)\Big) = O(1),$$

$$\int_{-\infty}^{+\infty} \Big(\log \mathcal{P}[e^\rho](x+iy) - \mathcal{P}[\rho](x+iy)\Big) dy = O(x),$$

as x tends to $+\infty$, where $\rho(\lambda) = \log |M(i\lambda)|^2$.

Examples of $M(z) \in \mathcal{HD}_2$

- off white noise

$$\int_{\mathbb{R}^2} \frac{|\rho(x)-\rho(y)|^2}{|x-y|^2} dxdy < \infty$$

$$M(z) = \exp\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\lambda z + i}{\lambda + iz} \cdot \frac{\rho(\lambda)d\lambda}{1+\lambda^2}\}$$

- $M(z) = \Big(\log(a+z)\Big)^{\beta}$, $a \geq 1$, $\beta > 0$.

- $\varphi \in L^1_{\text{loc}}[0,\infty) \cap L^2((0,\infty),(1-e^{-x})dx)$,

$$M(z) = 1 - \int_0^{\infty} \varphi(x) e^{-xz} dx,$$

Example of $M(z) \in \mathcal{HD}_b \setminus \mathcal{HD}_2$

- $M(z) = (1+z)^{\beta}$, $-1/2 < \beta < 1/2$, $\beta \neq 0$.

- $M(z) = 1 - re^{-az}$, $a > 0$.

Applications

$$\varphi \in L^1_{\mathrm{loc}}[0,\infty) \cap L^2((0,\infty),(1-e^{-x})dx)_{\mathbb{R}},$$

$$M(z) := 1 - \int_0^\infty \varphi(x) e^{-xz} dx,$$

α^φ : GCCR flow for $T_t = e^{tA_M}$.

Th. (M.I.-Srinivasan)
α^φ is of type III $\Leftrightarrow$ $\varphi \notin L^2(0,\infty)$.

α^φ has the same Tsirelson invariant as the CCR flow of index 1.

Lemma (M.I.-Srinivasan)
$\mathcal{T}_M$: Toeplitz operator

$$\mathcal{T}_M f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{L}[f](i\lambda) M(i\lambda) e^{i\lambda x} d\lambda$$

Then

$$\mathcal{A}^{\alpha^\varphi}(U) = \{W(\mathcal{T}_M f + i(\mathcal{T}_M^*)^{-1} g);\ f,g \in L^2(U)_{\mathbb{R}}\}''$$

Th. (M.I.-Srinivasan)
$\mathcal{A}^{\alpha^\varphi}(U)$ is either a type I or type III factor.

When φ is non-increasing on a neighborhood of 0, $\mathcal{A}^{\alpha^\varphi}(U)$ is type III $\Leftrightarrow$

$$\int_0^\infty ||1_U - 1_{x+U}||_{L^1} |\varphi(x)|^2 dx = \infty.$$

Functions $\varphi(x) = x^{\beta-1}e^{-x}$, $0 < \beta \leq 1/2$, give mutually non-cocycle conjugate type III GCCR flows.

Conjecture

(1) α^{φ_1} and α^{φ_2} are cocycle conjugate
$\Leftrightarrow \varphi_1 - \varphi_2 \in L^2(0, \infty)$.
($\Leftarrow$ is true)

(2) For general GCCR flows,
the asymptotic behavior of $||S_t - T_t||_{H.S.}$
for $t \to +0$ is a cocycle conjugacy invariant.

(2)' $||S_t - T_t||^2_{H.S.} = O(t) \Leftrightarrow$ type I
(true for $M = 1 - \mathcal{L}[\varphi]$)

cf.

- For $\varphi \in L^2(0, t)$, $||S_t - T_t||^2_{H.S.} \sim t||\varphi||^2_2$

- For $\varphi(x) \sim x^{\beta-1}L(x)$, $(x \to +0)$,
$0 < \beta < 1/2$, $L(x)$ slowly varying,

$$||S_t - T_t||^2_{H.S.} \sim \frac{t^{2\beta}L(t)^2}{2\beta(1-2\beta)}$$

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