The Gauge Group of a Strongly Spatial E_0-semigroup

Daniel Markiewicz
joint work with Robert T. Powers

U. of Toronto, Fields Inst., Ben Gurion U.

Fields Institute, July 2007

CP-semigroups and E_0-semigroups

Definition

A CP-semigroup is a family $\{\alpha_t : t \geq 0\}$ of normal completely positive maps of $B(H)$ such that:

- $\alpha_t \circ \alpha_s = \alpha_{t+s}$, for all $t, s \geq 0$
- $\alpha_0(X) = X$, for all $X \in B(H)$;
- the map $t \mapsto \langle \alpha_t(X)\xi, \eta \rangle$ is continuous, for all $\xi, \eta \in H$, $X \in B(H)$

We call it an E-semigroup if α_t is a *-endomorphism for all t. If $\alpha_t(I) = I$ it is called an E_0-semigroup .

Definition

A unit for an E_0-semigroup α on $B(H)$ is a strongly continuous one-parameter semigroup of operators T_t such that $T_0 = I$ and

$$\alpha_t(X)T_t = T_t X, \qquad \forall t \geq 0, \forall X \in B(H).$$

Cocycles and the Gauge group

Definition

An cocycle is a strongly continuous family of operators $(C_t)_{t\geq 0}$ in $B(H)$ such that

$$C_{t+s} = C_t \alpha_t(C_s) \quad \text{for} \quad t, s \geq 0.$$

We say that the cocycle C_t is local if $C_t \in \alpha_t(B(H))'$ for all $t > 0$.

Definition

The gauge group of an E_0-semigroup α is given by

$$G(\alpha) = \{ \text{ unitary local cocycles of } \alpha \}$$

Cocycle Conjugacy

Definition

Two E_0-semigroups α, β are cocycle equivalent if there exists a unitary α-cocycle $(U_t)_{t\geq 0}$ such that

$$\beta_t(X) = U_t\alpha_t(X)U_t^*.$$

They are cocycle conjugate ($\alpha \sim \beta$) if there exists a conjugacy θ such that α and $\theta^{-1} \circ \beta \circ \theta$ are cocycle equivalent.

Remark

If E is the product system of an E_0-semigroup α, then $G(\alpha) \simeq Aut(E)$.

Action of the Gauge group on the Units

Remark

If U_t is a local unitary cocycle and T_t is a unit, then U_tT_t is a unit.

Theorem (Arveson, Memoirs AMS 1989)

If α is an E_0-semigroup of type I_n, then for every $(a, \psi) \in \mathbb{C} \times \mathbb{C}^n$ there exists a unique unit $T_t(a, \psi)$. Furthermore, if U_t is a unitary local cocycle, there exists a unique triple $(\lambda, \xi, W) \in \mathbb{R} \times \mathbb{C}^n \times U(n)$ such that

$$U_tT_t(a, \psi) = T_t(a + i\lambda - \|\xi\|^2/2 - \langle W\psi, \xi\rangle, \xi + W\psi), \qquad \forall a, \psi.$$

Corollary

Let α be an E_0-semigroup of type I. The action of $G(\alpha)$ on the set of normalized units is transitive.

Full tuples of units

Definition

Given two CP-semigroups α and β, we say that β is a subordinate of α, denoted $\beta \leq \alpha$, if $\alpha_t - \beta_t$ is completely positive for all $t \geq 0$.

Definition

Let α be an E_0-semigroup of index n. We will say that a k-tuple of units ($k \geq n+1$) is full if together its elements generate the subordinate E-semigroup of type I_n.

Definition

Two full n-tuples of units $(U_t^{(0)}, U_t^{(1)}, \ldots, U_t^{(n)})$ and $(V_t^{(0)}, V_t^{(1)}, \ldots, U_t^{(n)})$ share the same covariances if

$$(U_t^{(j)})^* U_t^{(k)} = (V_t^{(j)})^* V_t^{(k)}, \qquad \forall j,k,t \geq 0$$

N-fold transitivity

Definition

Let α be an E_0-semigroup of index n. We will say that $G(\alpha)$ is (n+1)-fold transitive if given any two full $n+1$-tuples $((U_t^{(0)}, U_t^{(1)}, \ldots, U_t^{(n)})$ and $(V_t^{(0)}, V_t^{(1)}, \ldots, U_t^{(n)})$ sharing the same covariances, there exists a local unitary cocycle θ_t such that $\theta_t U_t^{(k)} = V_t^{(k)}$, for all $k, t \geq 0$

Remark

If an E_0-semigroup is of type I_n, then its gauge group is (n+1)-fold transitive.

Lack of N-fold transitivity

Theorem (M.-Powers)

There exists an E_0-semigroup of type II_1 whose gauge group is not 2-fold transitive.

Theorem (Tsirelson)

There exists an E_0-semigroup whose gauge group is not transitive.

Corollary ($II_1 \neq II_0 \otimes I_1$)

There exists a type II_1 E_0-semigroup which is not the tensor product of an E_0-semigroup of type II_0 and another of type I_1.

Powers Approach (New York Journal of Math. 2003)

Definition

Let K be a separable Hilbert space, and let $H = K \otimes L^2(0, \infty)$. Let U_t denote the right translation of H by $t \geq 0$. We will say that a CP semigroup α of $B(H)$ is a CP-flow if U_t is a unit of α.

Basic definitions

- δ = generator of CP-flow α
- $-d$ = generator of U_t
- π_0 = boundary representation of $\mathcal{D}(\delta)$ on $K \simeq \mathcal{D}(d^*)/\mathcal{D}(d)$
- $\Lambda : B(K) \to B(H)$ given by $(\Lambda(A)f)(x) = e^{-x}Af(x)$
- $\Lambda = \Lambda(I)$
- Boundary weight map $\rho \mapsto \omega(\rho)$ from $B(H)_*$ to boundary weights on $[(I - \Lambda^{1/2})B(H)(I - \Lambda^{1/2})]_*$.

Strongly Spatial CP-flows

Definition

A CP-semigroup α is strongly spatial if there exists β cocycle conjugate to α whose boundary representation is σ-weakly continuous and unital.

Theorem (Powers)

Suppose $\pi : B(H) \to B(K)$ is a σ-weakly continuous CP contraction. Then the map

$$\omega(\rho) = \hat{\pi}(\rho) + \hat{\pi}(\hat{\Lambda}(\hat{\pi}(\rho))) + \hat{\pi}(\hat{\Lambda}(\hat{\pi}(\hat{\Lambda}(\hat{\pi}(\rho))))) + \dots$$

converges as a weight and it is the boundary weight of the minimal CP flow which is derived from π. Furthermore it is the unique such CP-flow if $(\pi \circ \Lambda)^n(I) \to 0$ weakly.

Framework for the examples

- $K = \otimes_{j=1}^{\infty} L^2(0,\infty)$ with reference vector $v_1 \otimes v_2 \otimes \cdots$ where
$$v_j(x) = \lambda_j e^{-\frac{1}{2}\lambda_j^2 x}$$
where $\lambda_j > 0$ for all j and
$$\sum_{j=1}^{\infty} \frac{|\lambda_j - \lambda_{j+1}|^2}{\lambda_j^2 + \lambda_{j+1}^2} < \infty, \qquad \sum_{j=1}^{\infty} \lambda_j^{-2} < \infty$$
(example $\lambda_j = j$ but not 2^j).
- $H = K \otimes L^2(0,\infty)$
- $\pi : B(H) \to B(K)$ given by $\pi(X) = SXS^*$ where
$$S(f_1 \otimes f_2 \otimes \cdots \otimes h) = h \otimes f_1 \otimes f_2 \otimes \cdots$$
- $\Delta := \lim(\pi \circ \Lambda)^n(I) \neq 0.$

Existence

Theorem (M.-Powers)

Suppose α is a CP-flow derived from π as given above, with boundary weight ω. The ω is of the form

$$\omega(\rho) = \omega^1(\rho) + \rho(\Delta)\xi$$

where ω^1 is the boundary weight of the minimal CP-flow derived from π and ξ is a positive boundary weight with $\xi(I - \Lambda) \leq 1$. The CP flow α is unital if and only if $\xi(I - \Lambda) = 1$. Furthermore, there exist unital CP-flows derived from π which are strongly spatial but with $\xi \neq 0$.

Theorem (M.-Powers)

If α is the Bhat dilation of a strongly spatial CP-flow as above with $\xi \neq 0$, then it has index one but its gauge group is not 2-fold transitive.

Corners

Definition (Powers)

Suppose that α and β are E_0-semigroups. We say that γ is a corner from α to β if θ_t given by

$$\theta_t \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \alpha_t(A) & \gamma_t(B) \\ \gamma_t^*(C) & \beta_t(D) \end{pmatrix}$$

is a CP semigroup. We say γ is hypermaximal if for every CP semigroup θ_t' subordinate to θ_t of the form

$$\theta_t' \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \alpha_t'(A) & \gamma_t(B) \\ \gamma_t^*(C) & \beta_t'(D) \end{pmatrix},$$

we have that $\alpha_t' = \alpha_t$ and $\beta_t' = \beta_t$.

Some main ideas of the proof

Remark

Unitary Local Cocycles for α $\Leftrightarrow$ Hypermaximal corners from α to α

Outline

- In our case, look for local unitary cocycles leaving right shift invariant (=flow corners)
- Observe that if γ is a flow corner corresponding to a rotation, then $\theta_t = \begin{pmatrix} \alpha'_t & \gamma_t \\ \gamma^*_t & \beta'_t \end{pmatrix}$ has to be derived from $\begin{pmatrix} \pi & z\pi \\ \bar{z}\pi & \pi \end{pmatrix}$ for some $|z| \leq 1$.
- Describe possible boundary weight maps of possible θ_t; in this case a dychotomy appears: $z = 1$ versus $z \neq 1$.
- Only $z = 1$ possible in the hypermaximal case.