Automorphisms of the type II Arveson system of Warren's noise

Boris Tsirelson

Tel Aviv University //www.tau.ac.il/~tsirel/

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Arveson system = product system of Hilbert spaces:

$$H_{s+t} = H_s \otimes H_t \quad \text{for } s,t \in (0,\infty)$$ (associative, measurable; separable).

Classical part = maximal type I subsystem:

Fock spaces; generated by units.

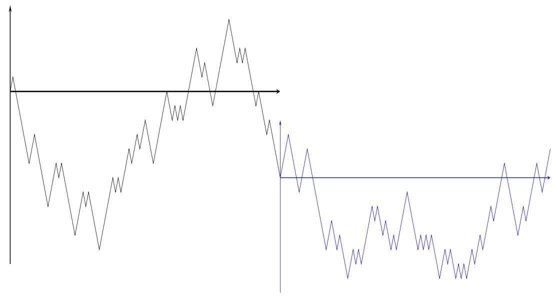
Unit: $u_{s+t} = u_s \otimes u_t\,, \quad u_t \in H_t\,, \quad \|u_t\| = 1$ (measurable; normalized).

A simple example of type II. The classical part:

$\Omega_t^{\text{white}} = \{$Brownian paths on $[0,t]\}$;

$\Omega_{s+t}^{\text{white}} = \Omega_s^{\text{white}} \times \Omega_t^{\text{white}}$; (assoc., meas.)

$H_t^{\text{white}} = L_2(\Omega_t^{\text{white}})$.

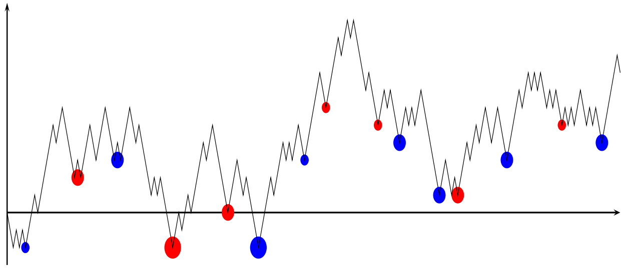


A nonclassical extension (J. Warren, 1999):

$\Omega_t = \{$Brownian paths on $[0,t]$ with independent random signs attached to local minima$\}$;

$\Omega_{s+t} = \Omega_s \times \Omega_t$; (assoc., meas.)

$H_t = L_2(\Omega_t) \supset H_t^{\text{white}}$.



(countable, dense)

Automorphism: $\theta_{s+t} = \theta_s \otimes \theta_t$, $\theta_t : H_t \to H_t$ unitary (meas.)

If (u_t) is a unit then $(\theta_t u_t)_t$ is a unit.

For the type I system, (H_t^{white}), over the Brownian motion (B_t):

Units: $u_t^z(\omega) = \exp(zB_t(\omega)) \underbrace{\exp(-(\operatorname{Re} z)^2 t)}_{\text{normalization}} \underbrace{\exp(\mathrm{i}at)}_{\text{trivial}}$.

$u^z = (u_t^z)_t$, $z \in \mathbb{C}$

Automorphisms act on units: $u^z \mapsto u^{Mz}$, $M : \mathbb{C} \to \mathbb{C}$ a motion (shift, rotat.)

Imaginary shift, $z \mapsto z + \mathrm{i}\lambda$: $\theta_t = \exp(\mathrm{i}\lambda B_t)$, multiplication operator.

Real shift, $z \mapsto z + \lambda$: drift, $B_t \mapsto B_t - 2\lambda t$. (nonsingular; Jacobian)

Rotation, $z \mapsto \mathrm{e}^{\mathrm{i}\lambda} z$: Wiener chaos spaces, $\mathrm{d}B_t \mapsto \mathrm{e}^{\mathrm{i}\lambda}\mathrm{d}B_t$.

Especially, $z \mapsto -z$ (rotation by π): $B_t \mapsto -B_t$.

Extends or not to the type II system, (H_t), over Warren's noise?

Imaginary shift: yes. } Transitive on units — evident

Real shift: yes. } Transitive on units — sharp minima

$z \mapsto -z$ (rotation by π): no. (Evident?) maxima are not minima

THEOREM. For every $\lambda \in (0, 2\pi)$ the automorphism of the classical part (H_t^{white}) of the Arveson system (H_t), corresponding to the rotation $z \mapsto e^{i\lambda} z$, cannot be extended to an automorphism of the whole system.

B. Tsirelson, arXiv:math.OA/0612303v1

The proof combines two arguments, one 'commutative' and one 'noncommutative'. The simplest case $\lambda = \pi$ $(z \mapsto -z)$ needs only the 'commutative' argument: maxima are not minima.

$$H_t = L_2(\Omega_t) \supset L_2(\Omega_t^{\text{white}}) = H_t^{\text{white}};$$

$$\Omega_t \ni \omega = (\omega^{\text{white}}, \eta), \quad \eta : \text{LocMin}(\omega^{\text{white}}) \to \{-1, +1\};$$

$f \in H_t$, general form: $\quad f(\omega) = f(\omega^{\text{white}}, \eta) = \sum_S f_S(\omega^{\text{white}}) \eta_S;$ (enumerate)

finite $S \subset \text{LocMin}(\omega^{\text{white}});$ $\quad \eta_S = \prod_{s \in S} \eta_s;$ $\quad$ and $\|f\|^2 = \sum_S \|f_S\|^2.$

$$\sum_S (\dots) = \sum_{n=0}^{\infty} \sum_{|S|=n} (\dots);$$

$n = 0$: H_t^{white}, the classical part; $\quad$ invariant under automorphisms $\quad$ (any n)

$n = 1$: $H_t^{(1)}$, the first superchaos; $\quad$ also i.u.a. $\quad$ (chaos = Wiener chaos space)

$f \in H_t^{(1)}$, general form: $f(\omega) = \sum_s f_s(\omega^{\text{white}}) \eta_s$, $s \in \text{LocMin}(\omega^{\text{white}});$

$H_t^{(1)} = L_2\big((0,t) \times \Omega_t^{\text{white}}$, a σ-finite measure $\big);$ $\quad$ (not locally finite)

$f_s(\omega^{\text{white}}) = f(s, \omega^{\text{white}}),$ $\quad$ only $s \in \text{LocMin}(\omega^{\text{white}})$ matter.

Commutative algebra $\mathrm{BoB}\big((0,t)\times\Omega_t^{\mathrm{white}}\big)$ of all bounded Borel functions acts on $H_t^{(1)}$. Special observables...

Automorphisms act on observables, $A \mapsto \theta_t^{-1} A \theta_t$.

Assume that some automorphism $\theta^{(\pi)}$ extends the rotation by π $(z \mapsto -z)$.

Striving to a contradiction we seek an observable A such that

$(*)$ $\quad (\theta_1^{(\pi)})^{-1} A \theta_1 = -A;$ (minima $\leftrightarrow$ maxima)

$(**)$ $\quad A = +1$ always. (minima only)

$$\text{Try} \quad A(s,\omega^{\mathrm{white}}) = \begin{cases} +1 & \text{if } s \text{ is a local minimum of } \omega^{\mathrm{white}}, \\ -1 & \text{if } s \text{ is a local maximum of } \omega^{\mathrm{white}}, \\ 0 & \text{otherwise;} \end{cases}$$

$(**)$ holds; $\quad (*)$ — ? $\quad$ Approximation by simpler observables is needed.

LEMMA. The action of $\mathrm{BoB}(0,t)$ on $H_t^{(1)}$ commutes with $\theta_t|_{H_t^{(1)}}$ for every automorphism θ.

HINT. The indicator of $(0, 0.5)$ leads to the orthogonal projection onto the θ-invariant subspace $H_{0.5}^{(1)} \otimes H_{0.5}^{\mathrm{white}} \subset H_1^{(1)}$.

$\theta^{(\pi)}$ acts on $\mathrm{BoB}(\Omega_t^{\mathrm{white}})$ by $\omega \mapsto -\omega$, and on $\mathrm{BoB}(0,t)$ trivially $(\forall\theta)$.

If $A' \in \mathrm{BoB}(0,1)$, $A'' \in \mathrm{BoB}(\Omega_1^{\mathrm{white}})$ and $A \in \mathrm{BoB}\big((0,1)\times\Omega_1^{\mathrm{white}}\big)$ are (say)

$$A'(\cdot) = 1 \text{ on } (0,0.5), \quad A'(\cdot) = 0 \text{ on } (0.5,1)\,,$$
$$A''(\omega^{\mathrm{white}}) = \operatorname{sgn}\big(\omega^{\mathrm{white}}(1) - \omega^{\mathrm{white}}(0.5)\big)\,,$$
$$A(s,\omega^{\mathrm{white}}) = A'(s)A''(\omega^{\mathrm{white}})\,,$$

then $\theta_1^{(\pi)}$ transforms A into $-A$. Hint: $\theta_1^{(\pi)} = \theta_{0.5}^{(\pi)} \otimes \theta_{0.5}^{(\pi)}$.

$(*)$ $\theta_1^{(\pi)}$ transforms $A_{n,\delta}$ into $-A_{n,\delta}$; here $(\delta \in (0,0.5),\ n = 1,2,\dots)$

$A_{n,\delta}(s,\omega^{\mathrm{white}}) = \sum_{k=1}^n A'_{n,k}(s)A''_{n,k,\delta}(\omega^{\mathrm{white}})$,

$A'_{n,k}$ is the indicator of $[\frac{k-1}{2n}, \frac{k}{2n})$,

$A''_{n,k,\delta}(\omega^{\mathrm{white}}) = \operatorname{sgn}\big(\omega^{\mathrm{white}}(\frac{k}{2n}+\delta) - \omega^{\mathrm{white}}(\frac{k}{2n})\big)$.

Lemma. For every $f \in H_{0.5}^{(1)} \otimes H_{0.5}^{\mathrm{white}}$, ($\langle A\rangle_f$ means $\langle Af,f\rangle$)

$(**)$ $\liminf_{n\to\infty}\langle A_{n,\delta}\rangle_f \to \|f\|^2$ as $\delta \to 0+$.

Hint. The natural measure on $(0,1)\times\Omega_1^{\mathrm{white}}$ is σ-finite, but the spectral measure of any state f is finite.

Contradiction between $(*)$ and $(**)$, $\|\theta_1^{(\pi)} f\|^2 = -\|f\|^2$,
proves the theorem for $\lambda = \pi$.

The worst case: $\lambda = 2\pi/3$.

The 'commutative' argument — as before; $A_{n,\delta}$, $(**)$.

However, $(*)$ fails; $\theta_1^{(2\pi/3)}$ transforms $A_{n,\delta}$ into an observable beyond the commutative algebra (surely not $-A_{n,\delta}$). Instead,

$$(*) \qquad \langle A_{n,\delta} \rangle_f + \langle A_{n,\delta} \rangle_g + \langle A_{n,\delta} \rangle_h \le (3 - \varepsilon) \| f \|^2 ,$$

where $g = \theta_1^{(2\pi/3)} f$, $h = (\theta_1^{(2\pi/3)})^{-1} f$, and ε is a positive absolute constant.

Define Q on $L_2(\Omega_1^{\text{white}})$ as multiplication by $\omega^{\text{white}}(1)$, then $\theta_1^{(\pi/2)}$ transforms Q into P such that $[P, Q] = -2\mathrm{i}$ (CCR; Weyl relations, in fact), and $\theta_1^{(2\pi/3)}$ transforms Q into $-\frac{1}{2} Q + \frac{\sqrt{3}}{2} P$.

Theorem. Let selfadjoint operators P, Q, R be such that $P + Q + R = 0$ and $[P, Q] = [Q, R] = [R, P] = -\mathrm{i}$. Then $2 < \| \operatorname{sgn} P + \operatorname{sgn} Q + \operatorname{sgn} R \| < 3$.

Enough to prove for the irreducible representation of CCR. This is done in:

B. Tsirelson, arXiv:quant-ph/0611147v1: "How often is the coordinate of a harmonic oscillator positive?"

Why $\|\operatorname{sgn} P + \operatorname{sgn} Q + \operatorname{sgn} R\| > 2\,$?

Numerics (finite dimension, number states); approximately 2.1.

Clearly, $\|\operatorname{sgn} P + \operatorname{sgn} Q + \operatorname{sgn} R\| \le 3$; why $\|\operatorname{sgn} P + \operatorname{sgn} Q + \operatorname{sgn} R\| < 3\,$?

In fact, $\|\operatorname{sgn}(P+c) + \operatorname{sgn}(Q+c) + \operatorname{sgn}(R+c)\| < 3$ for every $c \in \mathbb{R}$.

Easy part: 3 is not an eigenvalue of $\operatorname{sgn} P + \operatorname{sgn} Q + \operatorname{sgn} R$,

since 2 is not an eigenvalue of $\operatorname{sgn} P + \operatorname{sgn} Q$

by a classical theorem of F. and M. Riesz on Fourier transform.

Nevertheless, $\|\operatorname{sgn} P + \operatorname{sgn} Q\| = 2$.

Hard part: the spectrum of $\operatorname{sgn} P + \operatorname{sgn} Q + \operatorname{sgn} R$ is discrete, except for two accumulation points, ± 1. Moreover,

$$\operatorname{trace}\big((\operatorname{sgn} P + \operatorname{sgn} Q + \operatorname{sgn} R)^2 - \mathbf{1}\big)^2 < \infty\,.$$

Why $\operatorname{trace}\big((\operatorname{sgn} P + \operatorname{sgn} Q + \operatorname{sgn} R)^2 - \mathbf{1}\big)^2 < \infty$?

Weyl transform of an operator is a function of p, q. (not always)

Tomographic definition: $\operatorname{Weyl}\big(f(aP + bQ)\big)(p, q) = f(ap + bq)$,

extended by linearity. (some continuity)

$$\operatorname{trace}(A^2) = \frac{1}{2\pi} \iint |\operatorname{Weyl}(A)|^2 \, \mathrm{d}p\mathrm{d}q \, .$$

LEMMA. $\operatorname{Weyl}\big((\operatorname{sgn} P) \circ (\operatorname{sgn} Q)\big)(p, q) = \frac{2}{\pi} \operatorname{Si}(2pq)$.

Here $A \circ B = (AB + BA)/2$, and $\operatorname{Si}(x) = \int_0^x \frac{\sin u}{u} \, \mathrm{d}u$.

Maybe, the spectrum is always discrete outside the interval

$$\Big[\liminf_{p^2+q^2\to\infty} \operatorname{Weyl}(A)(p, q), \limsup_{p^2+q^2\to\infty} \operatorname{Weyl}(A)(p, q) \Big] \quad ?$$