Pullbacks of comodule algebras and finite free distributive lattices

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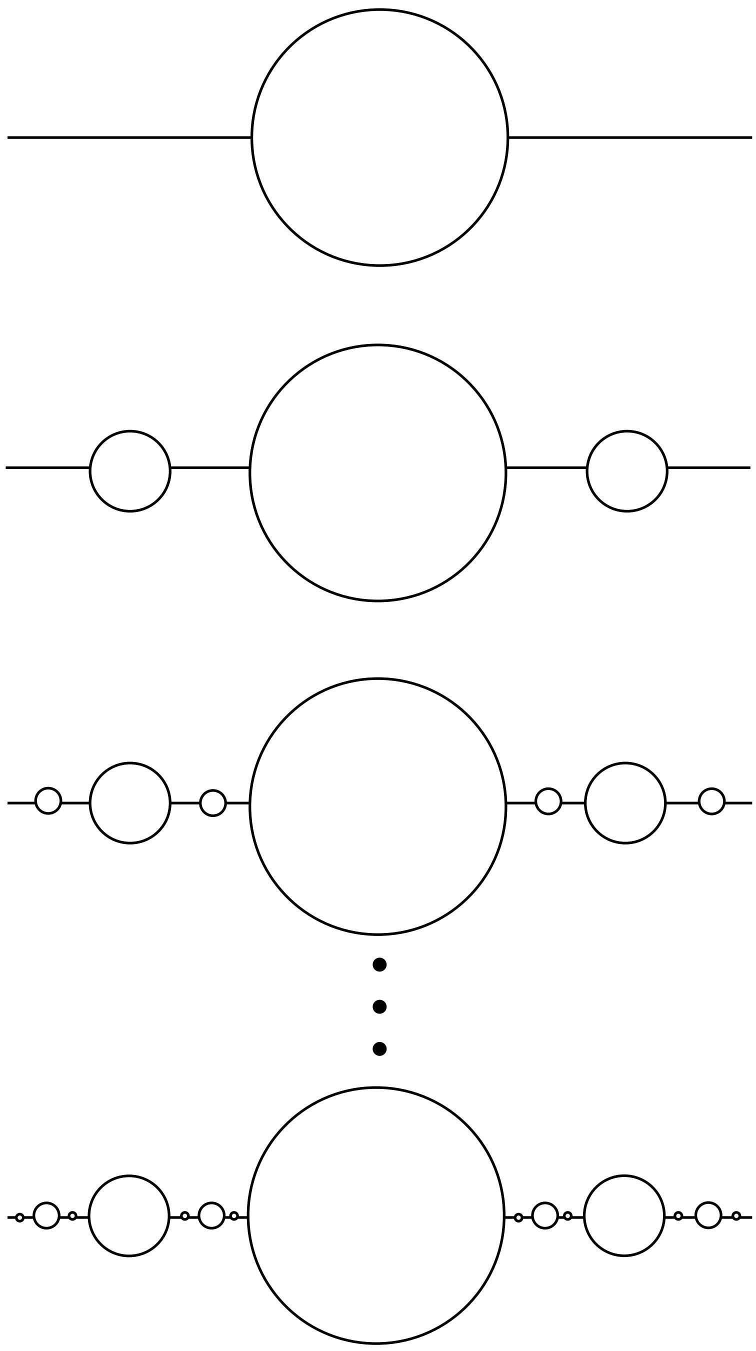
Bubble space and piecewise triviality



The Peter-Weyl subalgebra

Let H be the C^*-algebra of a compact quantum group and $PW(H)$ its dense Hopf $*$-subalgebra spanned by the matrix coefficients of the irreducible unitary corepresentations. Let A be a unital C^*-algebra and let $\delta : A \to A \otimes_{\min} H$ be a coaction. We define the Peter-Weyl subalgebra of A as:

$$PW_H(A) := \{\, a \in A \,|\, \delta(a) \in A \underset{\mathrm{alg}}{\otimes} PW(H) \,\}.$$

The Peter-Weyl subalgebra $PW_H(A)$ is:

- a $PW(H)$-comodule algebra,
- a dense $*$-subalgebra.

The operation PW_H commutes with taking equivariant pullbacks. Also,

$$A^{coH} = PW_H(A)^{coPW(H)}.$$

One-surjective pullbacks of principal comodule algebras

Main result: *Let H be a Hopf algebra with bijective antipode, and let*

$$
\begin{array}{ccccc}
 & & P & & \\
 & \swarrow & & \searrow & \\
P_1 & & & & P_2 \\
 & \underset{\pi_1}{\searrow} & & \underset{\pi_2}{\swarrow} & \\
 & & P_{12} & &
\end{array}
$$

be a one-surjective pullback diagram of H-comodule algebras. Then P is principal, if P_1 and P_2 are principal.

Corollary: *Let $\mathcal{P}$ be a flabby sheaf of H-comodule algebras over a topological space X. If $\{U_i\}_i$ is a finite open covering such that all $\mathcal{P}(U_i)$'s are principal, then $\mathcal{P}(U)$ is principal for any open subset $U \subseteq X$.*

Compact principal bundles

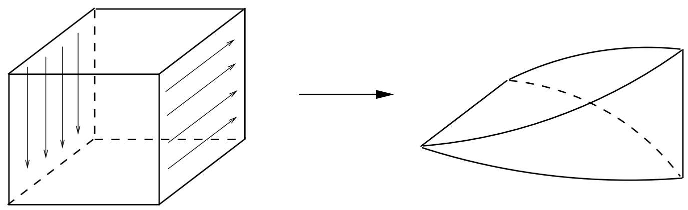
Freeness is equivalent to the injectivity of the map

$$X \times G \ni (x, g) \longmapsto (x, xg) \in X \underset{X/G}{\times} X.$$

Combining the local triviality of G-principal bundles when G is a Lie group with the pullback theorem for principal comodule algebras, one can show:

Corollary: *Let G be a compact group and X a compact Hausdorff G-bundle. Then X is a principal bundle if and only if $PW_{C(G)}(C(X))$ is a principal comodule algebra.*

Noncommutative join construction



$$P_1 := \{f \in C([0,1], \bar{H}) \otimes H \mid f(0) \in \Delta(H)\},$$
$$P_2 := \{f \in C([0,1], \bar{H}) \otimes H \mid f(1) \in \mathbb{C} \otimes H\}.$$

The P_i's are H-comodule algebras via

$$\Delta_{P_i} = \mathrm{id}_{C([0,1],\bar{H})} \otimes \Delta,$$

and the subalgebras of H-invariants are

$$B_1 := \{f \in C([0,1], \bar{H}) \mid f(0) \in \mathbb{C}\},$$
$$B_2 := \{f \in C([0,1], \bar{H}) \mid f(1) \in \mathbb{C}\}.$$

Heegaard quantum 3-spheres

$$
\begin{array}{ccccc}
 & & C(S^3_{\theta,p,q}) & & \\
 & \swarrow & & \searrow & \\
\mathcal{T} \rtimes_\theta \mathbb{Z} & & & & \mathcal{T} \rtimes_{-\theta} \mathbb{Z} \\
 & \searrow_{\pi_1} & & \swarrow_{\pi_2} & \\
 & & C(S^1) \rtimes_\theta \mathbb{Z} & &
\end{array}
$$

Finite coverings

A finite family $\{\pi_i : P \to P_i\}_{i\in\{1,\ldots,N\}}$ of surjective algebra homomorphisms is called a **weak covering** if $\cap_{i=1,\ldots,N} \ker \pi_i = \{0\}$.

Denote by Ker_N^π the lattice of ideals generated by the $\ker \pi_i$'s with $\cap$ and $+$ as the join and meet operations, respectively. A weak covering is called a **covering** if the lattice Ker_N^π is distributive.

An ordered family $(\pi_i : P \to P_i)_{i\in\{1,\ldots,N\}}$ is called an **ordered covering** if the set $\{\pi_i : P \to P_i\}_{i\in\{1,\ldots,N\}}$ is a covering.

Finite projective space with Alexandrov topology

Let $\mathbb{P}^{N-1}(\mathbb{Z}/2)$ be the 2-element field $(N-1)$-projective space

$$\mathbb{P}^{N-1}(\mathbb{Z}/2) := \{0,1\}^N \setminus \{(0,\ldots,0)\}$$

whose topology subbasis is its covering by affine spaces, i.e., this topology is generated by the subsets

$$A_i := \{(z_1,\ldots,z_N) \in \mathbb{P}^{N-1}(\mathbb{Z}/2) \mid z_i \neq 0\}.$$

The category of compact Hausdorff spaces X with a given ordered closed covering $(C_i \subseteq X)_{i\in\{1,\ldots,N\}}$ is equivalent to the opposite category of flabby sheaves of commutative unital C^*-algebras over $\mathbb{P}^{N-1}(\mathbb{Z}/2)$.

Finite free distributive lattices

By Koichi YAMAMOTO

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1.—Introduction.—The problem to determine the order $f(n)$ of the free distributive lattice $FD(n)$ generated by n symbols $\gamma_1, \cdots, \gamma_n$ was first proposed by Dedekind, but very little is known about this number [1, p. 146]. Only the first six values of $f(n)$ are computed, and enumerations of further $f(n)$ appear to lie beyond the scope of any reasonable methods known today. It might, however, be pointed out that Morgan Ward, who found $f(6)$ by the help of computing machines, stated [2] an asymptotic relation

$$\log_2 \log_2 f(n) \sim n$$

and that the present author proved in a previous note [3] that

$$f(n) \equiv 0 \pmod 2 \qquad \text{if} \qquad n \equiv 0 \pmod 2).$$

An inspection of numerical results $f(n)$, $n \leqq 6$ suggests strongly the following asymptotic equivalence

$$(*) \qquad \log_2 f(n) \sim \sqrt{\frac{2}{\pi}}\, 2^n n^{-\frac{1}{2}}.$$

The author cannot prove or disprove this interesting relation, but he proves in the present paper that

$$\sqrt{\frac{2}{\pi}}\, 2^n n^{-\frac{1}{2}}(1+O(n^{-1})) < \log_2 f(n) < \sqrt{\frac{2}{\pi}}\, 2^n n^{-\frac{1}{2}} \log_2 \sqrt{\frac{n\pi}{2}}\,(1+O(n^{-1}))$$

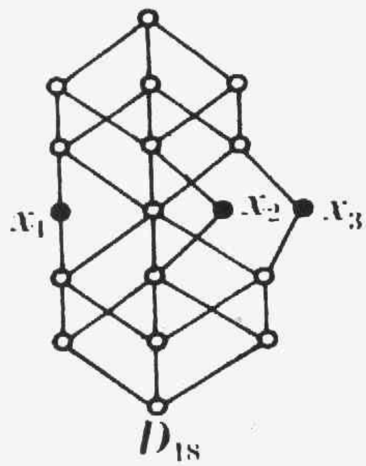

FIGURE 8

Covering Lemma

If $(\Lambda, \vee, \wedge)$ is a lattice generated by $\lambda_1, \ldots, \lambda_N$, then we can define maps

$$\lambda \stackrel{L^\Lambda}{\longmapsto} \{\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, N\} \mid \lambda_{i_1} \wedge \ldots \wedge \lambda_{i_k} \leq \lambda\}$$

$$\alpha \stackrel{R^\Lambda}{\longmapsto} \bigvee_{\{i_1,\ldots,i_k\} \in \alpha} \left(\lambda_{i_1} \wedge \ldots \wedge \lambda_{i_k}\right) .$$

Lemma: Let $\mathbf{C}_N$ be the category of ordered N-coverings of algebras, and $\mathbf{F}_N$ be the category of flabby sheaves of algebras over $\mathbb{P}^{N-1}(\mathbb{Z}/2)$. Then the following assignments

$$(\pi_i : P \to P_i)_i \stackrel{F}{\longmapsto} \Big(\mathcal{P} : U \mapsto P/R^{\mathsf{Ker}^\pi_N}(L^{\Gamma_N}(U))\Big)_U$$

$$\mathcal{P} \stackrel{G}{\longmapsto} \left(\mathcal{P}(\mathbb{P}^{N-1}(\mathbb{Z}/2)) \to \mathcal{P}(A_i)\right)_i$$

are functors establishing an equivalence of the categories $\mathbf{C}_N$ and $\mathbf{F}_N$.