Heisenberg modules and arithmetic properties of noncommutative tori

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Noncommutative tori

Let $\theta \in \mathbb{R}$ be an irrational number

$$\begin{aligned} A_\theta &= C^*\langle U, V \mid U, V \text{ unitaries}, \; UV = e^{2\pi\imath\theta} VU \rangle \\ &= C(\mathbb{T}^2_\theta) \end{aligned}$$

$$\begin{aligned} A_\theta \supset \mathcal{A}_\theta &= \text{Smooth elements for the } \mathbb{T}^2 \text{ action} \\ &= C^\infty(\mathbb{T}^2_\theta) \end{aligned}$$

Noncommutative tori

The Lie algebra $L = \mathbb{R}^2$ acts on $\mathcal{A}_\theta$ by derivations. A basis for this action is given by the derivations:

$$\delta_1(U) = 2\pi\imath U; \qquad \delta_1(V) = 0$$
$$\delta_2(U) = 0; \qquad \delta_2(V) = 2\pi\imath V.$$

Complexified noncommutative tori

A complex parameter

$$\tau \in \mathbb{C} \setminus \mathbb{R}$$

induces a complex structure on $L = \mathbb{R}^2$.
The corresponding complex structure on $\mathcal{A}_\theta$ is given by the derivation:

$$\delta_\tau = \tau\delta_1 + \delta_2 .$$

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Complexified noncommutative tori

The complexified noncommutative torus $\mathbb{T}^2_{\theta,\tau}$ can be consider as:

- The noncommutative torus $\mathbb{T}^2_\theta$ together with the complex structure determined by τ.
- The elliptic curve $X_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ together with the noncommutative structure determined by θ.
- The quotient $\mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z} \oplus \theta\mathbb{Z})$

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Holomorphic vector bundles on $\mathbb{T}^2_{\theta,\tau}$

A holomorphic vector bundle on $\mathbb{T}^2_{\theta,\tau}$ is given by a pair $(E, \bar{\nabla})$ where

- E is a right $\mathcal{A}_\theta$-module (finite type, projective).
- $\bar{\nabla} : E \to E$ is a $\mathbb{C}$-linear operator satisfying

$$\bar{\nabla}(ea) = \bar{\nabla}(e)a + e\delta_\tau(a), \qquad e \in E, a \in \mathcal{A}_\theta$$

Holomorphic vector bundles on $\mathbb{T}^2_{\theta,\tau}$

Theorem (Polishchuk, Schwarz)

The category of holomorphic vector bundles on $\mathbb{T}^2_{\theta,\tau}$

$$\mathcal{H}ol(\mathbb{T}^2_{\theta,\tau})$$

is an abelian category.

Holomorphic vector bundles on $\mathbb{T}^2_{\theta,\tau}$

There is an equivalence of categories

$$\mathcal{H}ol(\mathbb{T}^2_{\theta,\tau}) \simeq \mathcal{C}^\theta$$

where

$$\mathcal{C}^\theta \subset \mathcal{D}^b(X_\tau)$$

is the heart of a t-structure on $\mathcal{D}^b(X_\tau)$ depending on θ.

- Can $\mathbb{T}^2_{\theta,\tau}$ be considered as a noncommutative projective algebraic variety?
- What is the ring of homogeneous coordinates for this noncommutative projective algebraic variety?

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Holomorphic sections

Let $(E, \bar{\nabla})$ be a holomorphic vector bundle on $\mathbb{T}^2_{\theta,\tau}$

$$\bar{\Gamma}(E, \bar{\nabla}) \quad := \quad Ker(\bar{\nabla})$$

Rings of holomorphic sections

Let $(E, \bar{\nabla})$ be a holomorphic vector bundle on $\mathbb{T}^2_{\theta,\tau}$ and assume E is a $\mathcal{A}_\theta$-$\mathcal{A}_\theta$-bimodule.
Let

$$E^{\otimes_\theta n} := \underbrace{E \otimes_{\mathcal{A}_\theta} E \otimes_{\mathcal{A}_\theta} \cdots \otimes_{\mathcal{A}_\theta} E}_{n}$$

and

$$B(E, \bar{\nabla}) := \bigoplus_{n \geqslant 0} \bar{\Gamma}(E^{\otimes_\theta n}, \bar{\nabla})$$

Real multiplication noncommutative tori

Theorem (Rieffel)

Let $\theta \in \mathbb{R}$ be an irrational number. Then there are nontrivial $\mathcal{A}_\theta$-$\mathcal{A}_\theta$-bimodules if and only if θ is a quadratic irrationality.

In this case we call $\mathbb{T}^2_\theta$ a real multiplication noncommutative torus.

Real multiplication noncommutative tori

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Heisenberg groups

Consider a split central extension:

$$1 \rightarrow \mathbb{C}_1^* \rightarrow G \rightarrow K \rightarrow 0$$

where K is a locally compact abelian group and

$$\mathbb{C}_1^* = \{z \in \mathbb{C} | \, |z| = 1\}$$

Assume that the exact sequence splits thus $G = \mathbb{C}_1^* \times K$ and the group structure is given by:

$$(\lambda, x)(\mu, y) \quad = \quad (\lambda\mu\psi(x, y), x + y)$$

where $\psi : K \times K \to \mathbb{C}_1^*$ is a two-cocycle in K with values in $\mathbb{C}_1^*$.

Heisenberg groups

The cocycle ψ induces a skew multiplicative pairing

$$\begin{aligned} e: \; K \times K & \longrightarrow \mathbb{C}_1^* \\ (x, y) & \longmapsto \frac{\psi(x,y)}{\psi(y,x)}. \end{aligned}$$

Definition

If the morphism from K to its Pontrjagin dual $\widehat{K}$ given by e is a isomorphism then G is a *Heisenberg group*.

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Definition

A subgroup H of K is called *isotropic* if there exist a section of G over H:

$$\begin{aligned} \sigma : K &\longrightarrow G \\ x &\longmapsto (\alpha(x), x). \end{aligned}$$

Heisenberg groups

Theorem (Stone, Von Neumann, Makey)

Let G be a Heisenberg group. Then G has a unique irreducible unitary representation in which $\mathbb{C}_1^*$ acts by multiples of the identity.

Theorem (Stone, Von Neumann, Makey)

Let G be a Heisenberg group. Given a maximal isotropic subgroup $H \subset K$ let $\mathcal{H}$ be the space of measurable functions on K satisfying

1. $f(x+h) = \alpha(h)\psi(h,x)^{-1} f(x)$ *for all* $h \in H$.
2. $\int_{K/H} |f(x)|^2 dx < \infty$.

Then G acts on $\mathcal{H}$ by

$$U_{(\lambda,y)} f(x) = \lambda\psi(x,y) f(x+y).$$

and $\mathcal{H}$ is an irreducible unitary representation of G.

Heisenberg groups

Theorem

If

$$1 \to \mathbb{C}_1^* \to G_i \to K_i \to 0 \qquad i = 1, 2$$

are two Heisenberg groups with Heisenberg representations $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *then*

$$1 \to \mathbb{C}_1^* \to G_1 \times G_2 / \{(\lambda, \lambda^{-1}) | \lambda \in \mathbb{C}_1^*\} \to K_1 \times K_2 \to 0$$

is a Heisenberg group and its Heisenberg representation is $\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2$.

Real Heisenberg groups

Let $K = \mathbb{R}^2$ and let $\varepsilon \in \mathbb{R}^*_+$.
$G = K \times \mathbb{C}^*_1$ is a Heisenberg group with

$$\begin{aligned} \psi(x, y) &= exp\left(\frac{\pi \imath}{\varepsilon}(x_1 y_2 - y_1 x_2)\right) \\ e(x, y) &= exp\left(\frac{2\pi \imath}{\varepsilon}(x_1 y_2 - y_1 x_2)\right). \end{aligned}$$

where $x = (x_1, x_2),\ y = (y_1, y_2)\ \in K$.

To obtain the usual Heisenberg representation consider the maximal isotropic subgroup

$$H = \{x = (x_1, x_2) \in K \mid x_2 = 0\}$$

Then

$$\mathcal{H} \simeq L^2(\mathbb{R})$$

And

$$\begin{aligned} U_{(1,(y_1,0))} f(x) &= f(x + y_1). \\ U_{(1,(0,y_2))} f(x) &= e\left(\frac{2\pi\imath}{\varepsilon} x y_2\right) f(x). \end{aligned}$$

Real Heisenberg groups

$f \in \mathcal{H}$ is smooth if

$$\delta U_{X_1} \delta U_{X_2} \cdots \delta U_{X_n}(f)$$

is well defined for any n and any $X_1, X_2, \ldots X_n \in Lie(G)$ where

$$\delta U_X(f) = \lim_{t \to 0} \frac{U_{\exp(tX)} f - f}{t}$$

Real Heisenberg groups

Choose a basis $\{A, B, C\}$ for the Lie algebra $Lie(G)$ such that

$\exp(tA) = (1,(t,0))$, $\exp(tB) = (1,(0,t))$, $\exp(tC) = (e^{2\pi\imath t},(0,0))$

Given $\tau \in \mathbb{C} \setminus \mathbb{R}$ let

$$\begin{aligned} W_\tau &= \langle \delta U_A - \tau \delta U_B \rangle \\ W_{\bar{\tau}} &= \langle \delta U_A - \bar{\tau} \delta U_B \rangle. \end{aligned}$$

Theorem

In any Heisenberg representation of G there exists an element f_τ, unique up to a scalar, such that $\delta U_X(f_\tau)$ is defined and equal to 0 for all $X \in W_\tau$.

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Finite Heisenberg groups

Let c be a positive integer and let $K = (\mathbb{Z}/c\mathbb{Z})^2$.

$$\begin{aligned} \psi(([n_1],[n_2]),([m_1],[m_2])) &= exp\left(\frac{2\pi\imath}{2c}(n_1 m_2 - m_1 n_2)\right) \\ e([n_1],[n_2]),([m_1],[m_2])) &= exp\left(\frac{2\pi\imath}{c}(n_1 m_2 - m_1 n_2)\right) \end{aligned}$$

where $([n_1],[n_2]),([m_1],[m_2]) \in K$.

Finite Heisenberg groups

If we choose as maximal isotropic subgroup $H = \{([n_1], [n_2]) \in K \mid [n_2] = 0\}$ we may realize the Heisenberg representation as the action of G on $C(\mathbb{Z}/c\mathbb{Z})$ given by

$$
\begin{aligned}
U_{(1,([m_1],0))}\phi([n]) &= \phi([n + m_2]). \\
U_{(1,(0,[m_2]))}\phi([n]) &= exp\left(\frac{2\pi\imath}{c} n m_2\right) f([n]).
\end{aligned}
$$

Heisenberg group schemes

A *Heisenberg group scheme* is a central extensions of the form

$$1 \to \mathbb{G}_m \to \mathcal{G} \to \mathcal{K} \to 0$$

where $\mathcal{K}$ is a finite abelian group scheme aver a base field k. These groups arise in a natural way by considering ample line bundles on abelian varieties over the base field k.

Heisenberg modules

Let $\theta \in \mathbb{R}$ be a quadratic irrationality and let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

be a matrix fixing θ.

Holomorphic sections of Heisenberg modules

Let

$$\varepsilon = \frac{c\theta + d}{c}$$

and consider the following operators on the Schwartz space $\mathcal{S}(\mathbb{R})$:

$$\begin{aligned}
(\check{U}f)(x) &= f(x-\varepsilon) \\
(\check{V}f)(x) &= exp(2\pi\imath x)f(x) \\
(\hat{U}f)(x) &= f\left(x - \frac{1}{c}\right) \\
(\hat{V}f)(x) &= exp\left(\frac{2\pi\imath x}{c\varepsilon}\right) f(x).
\end{aligned}$$

The Schwartz space $\mathcal{S}(\mathbb{R})$ is identified with the set of smooth elements of the real Heisenberg representation.

Heisenberg modules

Consider also the following operators on $C(\mathbb{Z}/c\mathbb{Z})$

$$\begin{aligned}
(\check{u}\phi)([n]) &= \phi([n-1]) \\
(\check{v}\phi)([n]) &= exp\left(-\frac{2\pi\imath dn}{c}\right)\phi([n]) \\
(\hat{u}\phi)([n]) &= \phi([n-a]) \\
(\hat{v}\phi)([n]) &= exp\left(-\frac{2\pi\imath n}{c}\right)\phi([n]).
\end{aligned}$$

Heisenberg modules

$$E_g = \mathcal{S}(\mathbb{R}) \otimes C(\mathbb{Z}/c\mathbb{Z})$$

becomes a $\mathcal{A}_\theta$-$\mathcal{A}_\theta$-bimodule by defining:

$$\begin{aligned}
(f \otimes \phi)U &= (\check{U} \otimes \check{u})(f \otimes \phi) \\
(f \otimes \phi)V &= (\check{V} \otimes \check{v})(f \otimes \phi) \\
U(f \otimes \phi) &= (\hat{U} \otimes \hat{u})(f \otimes \phi) \\
V(f \otimes \phi) &= (\hat{V} \otimes \hat{v})(f \otimes \phi)
\end{aligned}$$

where $f \in \mathcal{S}(\mathbb{R})$ and $\phi \in C(\mathbb{Z}/c\mathbb{Z})$.

Heisenberg modules

There is a natural identification:

$$E_g \otimes_{\mathcal{A}_\theta} E_g \quad \simeq \quad E_{g^2}.$$

$$\begin{aligned} E_g \hat{\otimes} E_g &= [\mathcal{S}(\mathbb{R}) \otimes C(\mathbb{Z}/c\mathbb{Z})] \,\hat{\otimes}\, [\mathcal{S}(\mathbb{R}) \otimes C(\mathbb{Z}/c\mathbb{Z})] \\ &= \left[\mathcal{S}(\mathbb{R}) \hat{\otimes} \mathcal{S}(\mathbb{R})\right] \otimes [C(\mathbb{Z}/c\mathbb{Z}) \otimes C(\mathbb{Z}/c\mathbb{Z})] \\ &= [\mathcal{S}(\mathbb{R} \times \mathbb{R})] \otimes [C(\mathbb{Z}/c\mathbb{Z} \times \mathbb{Z}/c\mathbb{Z})] \,. \end{aligned}$$

Heisenberg modules

To pass from $E_g \hat{\otimes} E_g$ to $E_g \otimes_{\mathcal{A}_\theta} E_g$ we have to quotient $E_g \hat{\otimes} E_g$ by the space spanned by the relations:

$$
\begin{aligned}
[(f \otimes \phi)U] \,\hat{\otimes}\, [g \otimes \omega] &= [f \otimes \phi] \,\hat{\otimes}\, [U(g \otimes \omega)] \\
[(f \otimes \phi)V] \,\hat{\otimes}\, [g \otimes \omega] &= [f \otimes \phi] \,\hat{\otimes}\, [V(g \otimes \omega)]
\end{aligned}
$$

where $f, g \in \mathcal{S}(\mathbb{R})$ and $\phi, \omega \in C(\mathbb{Z}/c\mathbb{Z})$. At the level of the Heisenberg representations involved this amounts to restrict to the subspaces of $\mathcal{S}(\mathbb{R} \times \mathbb{R})$ and $C(\mathbb{Z}/c\mathbb{Z} \times \mathbb{Z}/c\mathbb{Z})$ which are invariant under the action of the subgroups of $\mathrm{Heis}(\mathbb{R}^4)$ and $\mathrm{Heis}((\mathbb{Z}/c\mathbb{Z})^4)$ generated by the elements giving these relations.

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Holomorphic sections of Heisenberg modules

The corresponding space of invariant elements in $\mathcal{S}(\mathbb{R}\times\mathbb{R})$ is canonically isomorphic to the space $\mathcal{S}(\mathbb{R})$ of smooth elements of the Heisenberg representation of $\mathrm{Heis}(\mathbb{R}^2)$ with $c\varepsilon^2/(a+d)$ playng the role of ε. In $C((\mathbb{Z}/c\mathbb{Z})\times(\mathbb{Z}/c\mathbb{Z}))$ the corresponding invariant subspace is canonically isomorphic to the Heisenber representation $C(\mathbb{Z}/c(a+d)\mathbb{Z})$ of $\mathrm{Heis}((\mathbb{Z}/c(a+d)\mathbb{Z})^2)$. Thus we get:

$$\begin{aligned} E_g \otimes_{\mathcal{A}_\theta} E_g &\simeq \mathcal{S}(\mathbb{R})\otimes C(\mathbb{Z}/c(a+d)\mathbb{Z}) \\ &= E_{g^2} \end{aligned}$$

The compatibility of the module structures in this isomorphism is implied by the the compatibility of the Heisenberg representations involved.

There are isomorphisms:

$$\underbrace{E_g \otimes_{\mathcal{A}_\theta} \cdots \otimes_{\mathcal{A}_\theta} E_g}_{n} \simeq E_{g^n}.$$

Heisenberg modules

$$g^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, \qquad n > 0.$$

Given a Heisenberg $\mathcal{A}_\theta$-$\mathcal{A}_\theta$-bimodule E_g we may define a holomorphic structure on E_g by:

$$\nabla_0 = \tau\nabla_1 + \nabla_2$$

where

$$\begin{aligned} (\nabla_1 f \otimes \phi)(x,[n]) &= 2\pi\imath\left(\frac{x}{\varepsilon}\right)(f \otimes \phi)(x,[n]) \\ (\nabla_2 f \otimes \phi)(x,[n]) &= \frac{d}{dx}(f \otimes \phi)(x,[n]). \end{aligned}$$

Holomorphic sections of Heisenberg modules

The operators ∇_1 and ∇_2 correspond to the action of the Lie algebra of $\mathrm{Heis}(\mathbb{R}^2)$ on the left factor of $\mathcal{S}(\mathbb{R}) \otimes C(\mathbb{Z}/c\mathbb{Z})$ given by the derivations δU_A and δU_B.

Let

$$\begin{aligned} R_g &= \{f_\tau \otimes \phi \in E_g \mid \phi \in C(\mathbb{Z}/c\mathbb{Z})\} \\ &= \left\{ exp\left(\frac{2\pi\imath}{2\varepsilon}\tau x^2\right) \otimes \phi \in E_g \mid \phi \in C(\mathbb{Z}/c\mathbb{Z}) \right\} \\ &\simeq \bar{\Gamma}(E_g, \bar{\nabla}_0) \end{aligned}$$

Homogeneous coordinate ring

We denote by $f_{\tau,n}$ the corresponding element on the left factor of E_{g^n} and let

$$R_{g^n} = \left\{ f_{\tau,n} \otimes \phi \in E_{g^n} \,|\, \phi \in C(\mathbb{Z}/c_n\mathbb{Z}) \right\}.$$

Definition

The homogeneous coordinate ring for the complexified noncommutative torus $\mathbb{T}^2_{\theta,\tau}$ is the graded ring:

$$\begin{aligned} B_g(\theta,\tau) &= \bigoplus_{n\geq 0} R_{g^n} \\ &\simeq B(E_g, \bar{\nabla}_0) \end{aligned}$$

Holomorphic sections of Heisenberg modules

It is possible to realize each space R_{g^n} as the space of sections of a line bundle over X_τ. For this we consider the matrix coefficients obtained by pairing $f_{\tau,n}$ with functionals in the distribution completion of the Heisenberg representation which are invariant under the action of elements in $\mathrm{Heis}(\mathbb{R}^2)$ corresponding to a lattice in $\mathbb{R}^2$ associated to g^n.

Homogeneous coordinate rings for $\mathbb{T}^2_{\theta,\tau}$

This fact has the following consequence:

Theorem

Let $\theta \in \mathbb{R}$ be a quadratic irrationality fixed by a matrix $g \in SL_2(\mathbb{Z})$. Let k be the minimal field of definition of the elliptic curve X_τ. Then the algebra $B_g(\theta, \tau)$ admits a rational presentation over a finite algebraic extension of k.

If $n \mid m$ then $c_n \mid c_m$, explicitly

$$c_m = \eta_n^m c_n$$

where

$$\eta_n^m = \Sigma_{i=0}^{l} a_n^{l-i} d_{(i-1)n}, m = ln$$

Thus there are morphisms

$$\eta_n^m : \mathbb{Z}/c_m\mathbb{Z} \to \mathbb{Z}/c_n\mathbb{Z}$$

An algebraic endomotive from $\mathbb{T}^2_{\theta,\tau}$

The data

$$\{\mathbb{Z}/c_m\mathbb{Z}\}_{m\in\mathbb{N}}\ , \qquad S = \mathbb{N}$$

Defines an algebraic endomotive over $\mathbb{Q}$.

An algebraic endomotive from $\mathbb{T}^2_{\theta,\tau}$

We consider η_n^m as an isogeny on the elliptic curve whose period lattice is spanned by τ. Via $n \mapsto (c\theta + d)^n$ we can view the semigroup S as the semigroup of positive units of $k = \mathbb{Q}(\theta)$.