

A spectral triple for harmonic oscillator Moyal space

(work in progress)

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(joint with Harald Grosse, [arxiv:0709.0095](https://arxiv.org/abs/0709.0095))

Introduction

An interesting quantum field theoretical model

action functional for real-valued field ϕ on $\mathbb{R}^4$:

$$S[\phi] = \int d^4x \left(\frac{1}{2} \phi \star (-\Delta + \Omega^2 \tilde{x}^2 + \mu^2) \phi + \frac{\lambda}{4} \phi \star \phi \star \phi \star \phi \right)(x)$$

Moyal product $\star$ defined by Θ and $\tilde{x} := 2\Theta^{-1} \cdot x$

- **renormalisable as formal power series** in λ
[H.Grosse+R.W.; 2004]
means: well-defined **perturbative** quantum field theory
- **β -function vanishes to all orders** in λ for $\Omega = 1$
[M.Disertori, R.Gurau, J.Magnen + V.Rivasseau; 2006]
means: model is believed to exist **non-perturbatively**

Noncommutative spin manifolds = spectral triples

- suggested 1996 by A. Connes (**axioms of spectral triples**) using insight from the **standard model** of particle physics
- commutative spectral triples are indeed compact spin manifolds [**distinguished lecture by A. Connes**]
- equivalence classes of spectral triples describe **Yang-Mills theory** (inner automorphisms) and **gravity** (outer automorphisms)
- **spectral action principle** [A. Chamseddine, A. Connes; 1996] provides action functional = **starting point for QFT**
some divergent one-loop Feynman graphs are renormalisable by construction

Spectral triples and Yang-Mills theory for Moyal space

- Moyal spaces are non-compact spectral triples [V.Gayral, J.M.Gracia-Bondía, B.Iochum, T.Schücker, J.Várilly; 2003]
- spectral action for Moyal space [V.Gayral, B.Iochum; 2004] is Yang-Mills with Moyal product
problem: spacial regularisation required

for renormalisation we need oscillator potential. . .

Is there a spectral triple associated with a Dirac operator with oscillator spectrum?

- answer is no if all noncommutative dimensions coincide
- examples (Podleś quantum sphere, standard model) require different metric and KO dimensions

answer is yes for independent metric and KO dimensions

Axioms for non-compact spectral triples

(adapted from Gayral et al; subject to alteration . . .)

Definition (non-compact spectral triple $(\mathcal{A}, \mathcal{B}, \mathcal{H}, \mathcal{D}, J, \chi, \mathbf{c})$)

- a **non-unital algebra** $\mathcal{A}$ acting faithfully on a **Hilbert space** $\mathcal{H}$ (via a representation π)
- a preferred **unitisation** $\mathcal{B}$ of $\mathcal{A}$ acting on $\mathcal{H}$, too
- a densely defined **selfadjoint unbounded operator** $\mathcal{D}$ on $\mathcal{H}$ s.t. $[\mathcal{D}, \pi(b)]$ extends to a bounded operator $\forall b \in \mathcal{B}$
- in the even case: a **selfadjoint operator** χ on $\mathcal{H}$ satisfying $\chi^2 = 1$, $\chi\pi(b) = \pi(b)\chi \quad \forall b \in \mathcal{B}$, $\mathcal{D}\chi = -\chi\mathcal{D}$
- in the real case: an **antiunitary operator** J on $\mathcal{H}$ which satisfies conditions 4 and 5 below

subject to the following conditions 0–6.

The conditions

0 Compactness.

The operator $\pi(a)(\mathcal{D} - \lambda)^{-1}$ is compact for all $a \in \mathcal{A}$ and all λ not contained in the spectrum of $\mathcal{D}$.

1 Regularity and dimension spectrum.

Both $\pi(b)$ and $[\mathcal{D}, \pi(b)]$ belong for any $b \in \mathcal{B}$ to

$\bigcap_{n=1}^{\infty} \text{dom}(\delta^n)$, with $\delta T := [\langle \mathcal{D} \rangle, T]$ and $\langle \mathcal{D} \rangle := (\mathcal{D}^2 + 1)^{\frac{1}{2}}$.

For any element α of the algebra $\Psi_0(\mathcal{A})$ generated by $\delta^n \pi(a)$ and $\delta^n [\mathcal{D}, \pi(a)]$, with $a \in \mathcal{A}$, the function

$\zeta_{\alpha}(z) := \text{Tr}(\alpha \langle \mathcal{D} \rangle^{-z})$ extends holomorphically to $\mathbb{C} \setminus \text{Sd}$ for some discrete set $\text{Sd} \subset \mathbb{C}$ (dimension spectrum).

2 Metric dimension.

For the metric dimension $d := \sup\{r \in \mathbb{R} \cap \text{Sd}\}$, the Dixmier trace $\text{Tr}_\omega(\pi(a)\langle \mathcal{D} \rangle^{-d})$ is finite for any $a \in \mathcal{A}$ and positive for positive elements of $\mathcal{A}$.

3 Finiteness.

The algebras $\mathcal{A}$ and $\mathcal{B}$ are pre- C^* -algebras.

The space $\mathcal{H}^\infty := \bigcap_{k=0}^{\infty} \mathcal{H}^k$, with $\mathcal{H}^k := \text{dom}(\mathcal{D}^k)$ completed with norm $\|\xi\|_k^2 := \|\xi\|^2 + \|\mathcal{D}^k \xi\|^2$, is a finitely generated projective $\mathcal{A}$ -module $p\mathcal{A}^m$, for some projector $p = p^2 = p^* \in M_m(\mathcal{B})$.

The scalar product on $\mathcal{H}^\infty$ is recovered from the hermitian structure $\langle \cdot, \cdot \rangle_{\mathcal{A}} : \mathcal{H}^\infty \times \mathcal{H}^\infty \rightarrow \mathcal{A}$ and the Dixmier trace:

$$(\xi, \eta) = \text{Tr}_\omega\left(\langle \xi, \eta \rangle_{\mathcal{A}} \langle \mathcal{D} \rangle^{-d}\right), \quad \xi, \eta \in \mathcal{H}^\infty$$

4 Reality.

J defines a **real structure of KO-dimension** $k \in \mathbb{Z}_8$, i.e.

$$J^2 = \varepsilon, \quad J\mathcal{D} = \varepsilon'\mathcal{D}J, \quad J\chi = \varepsilon''\chi J \quad (\text{even case})$$

$k \bmod 8$	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

The action π of $\mathcal{B}$ on $\mathcal{H}$ satisfies $[\pi(b_1), \pi^o(b_2)] = 0$ for all $b_1, b_2 \in \mathcal{B}$, where $\pi^o(b_2) = J\pi(b_2^*)J^{-1}$.

5 First order.

$[[\mathcal{D}, \pi(b_1)], \pi^o(b_2)] = 0$ for all $b_1, b_2 \in \mathcal{B}$.

6 Orientability.

$\exists$ **Hochschild d -cycle $\mathbf{c}$** on $\mathcal{B}$ with values in $\mathcal{B} \otimes \mathcal{B}^o$,

The representation $\pi(\mathbf{c})$ defined by

$$\begin{aligned} & \pi((b_{-1} \otimes b_0) \otimes b_1 \otimes \cdots \otimes b_d) \\ & := J\pi(b_{-1}^*)J^{-1}\pi(b_0)[\mathcal{D}, \pi(b_1)] \cdots [\mathcal{D}, \pi(b_d)] \end{aligned}$$

satisfies $\pi(\mathbf{c})^2 = 1$.

$\pi(\mathbf{c})$ defines the **volume form** on $\mathcal{A}$, i.e.

$$\begin{aligned} & \phi_{\pi(\mathbf{c})}(a_0, \dots, a_d) \\ & = \text{Tr}_\omega(\pi(\mathbf{c})\pi(a_0)[\mathcal{D}, \pi(a_1)] \cdots [\mathcal{D}, \pi(a_d)]\langle \mathcal{D} \rangle^{-d}) \end{aligned}$$

provides a **non-vanishing Hochschild d -cocycle** on $\mathcal{A}$.

The Dirac operator for the harmonic oscillator

Hamiltonian of **one-dimensional** harmonic oscillator:

$$H = -\frac{d^2}{dx^2} + \omega^2 x^2 \text{ on } L^2(\mathbb{R})$$

- H^{-1} has eigenvalues $\mu_n = \frac{1}{\omega(2n+1)}$, $n \in \mathbb{N}$
- H generalises Laplacian $-\Delta$, Dirac operator is a square root, so **eigenvalues of $H^{-1} = \mathcal{D}^{-2}$ are of order $\mathcal{O}(\frac{1}{n})$**

suggests: **$\mathcal{D} \sim H^{\frac{1}{2}}$ seems to be of spectral dimension two!**

Construct $\mathcal{D}$ in **two Clifford dimensions**

$$\begin{aligned} \mathcal{D}_1 &= i\sigma_1 \frac{d}{dx} + \sigma_2 \omega x \\ &= \begin{pmatrix} 0 & i(\frac{d}{dx} + \omega x) \\ i(\frac{d}{dx} - \omega x) & 0 \end{pmatrix} = \begin{pmatrix} 0 & i\sqrt{2\omega}a \\ -i\sqrt{2\omega}a^\dagger & 0 \end{pmatrix} \end{aligned}$$

$$[a, a^\dagger] = 1$$

Generalisation to d -dimensional harmonic oscillator

d **bosonic** and **fermionic** creation and annihilation operators:

$$\begin{aligned} \textcircled{1} \quad [a_\mu, a_\nu] &= [a_\mu^\dagger, a_\nu^\dagger] = 0 & [a_\mu, a_\nu^\dagger] &= \delta_{\mu\nu} & \mu, \nu &= 1, \dots, d \\ \{b_\mu, b_\nu\} &= \{b_\mu^\dagger, b_\nu^\dagger\} = 0 & \{b_\mu, b_\nu^\dagger\} &= \delta_{\mu\nu} \end{aligned}$$

$$\begin{aligned} \textcircled{2} \quad \mathcal{D}_d &= \sqrt{2}\omega \sum_{\mu=1}^d (ia_\mu \otimes b_\mu^\dagger - ia_\mu^\dagger \otimes b_\mu) & (\text{supersymmetry}) \\ &= i \frac{\partial}{\partial x_\mu} \otimes (b_\mu + b_\mu^\dagger) + \omega x^\mu \otimes (ib_\mu^\dagger - ib_\mu) \\ &= i\Gamma_\mu \frac{\partial}{\partial x_\mu} + \omega \Gamma_{d+\mu} x^\mu & \text{Cliff}(\mathbb{C}^{2d}) \end{aligned}$$

$$\textcircled{3} \quad \mathcal{H}_d = L^2(\mathbb{R}^d) \otimes \wedge(\mathbb{C}^d) \text{ (bosonic } \otimes \text{ fermionic Hilbert spaces)}$$

$$\chi_d = 1 \otimes \prod_{\mu=1}^d (b_\mu^\dagger b_\mu - b_\mu b_\mu^\dagger) = (-i)^d (-1)^{\frac{d(d+1)}{2}} \Gamma_1 \cdots \Gamma_{2d}$$

$$\begin{aligned} \textcircled{4} \quad \mathcal{D}_n^2 &= \omega \sum_{\mu=1}^d (\{a_\mu, a_\mu^\dagger\} \otimes 1 + 1 \otimes \{b_\mu^\dagger, b_\mu\}) = H_d \otimes 1 + \omega \otimes \Sigma_d \\ H_d &= -\frac{\partial^2}{\partial x_\mu \partial x^\mu} + \omega^2 x_\mu x^\mu & \Sigma_d &= \sum_{\mu=1}^d [b_\mu^\dagger, b_\mu] \end{aligned}$$

Spectral data for harmonic oscillator Moyal space

$$\textcircled{1} \quad \mathcal{D}_4 = (i\Gamma^\mu \partial_\mu + \Omega \Gamma^{\mu+4} \tilde{X}_\mu) \quad \mu = 1, \dots, 4; \quad \tilde{X}_\mu = 2(\Theta^{-1})_{\mu\nu} X^\nu$$

$$\mathcal{D}_4^2 = (-\Delta + \Omega^2 \|\tilde{X}\|^2) \mathbf{1} + \tilde{\Omega} \Sigma_4 \quad (\tilde{\Omega} = \frac{2\Omega}{\theta}; \quad \Theta = \theta S; \quad S^2 = -1)$$

$$\textcircled{2} \quad \text{Hilbert space } \mathcal{H}_4 = L^2(\mathbb{R}^4) \otimes \mathbb{C}^{16}$$

$$\textcircled{3} \quad \text{algebra } \mathcal{A}_4 = \mathcal{S}(\mathbb{R}^4) \text{ with}$$

$$(f \star g)(x) = \int d^4 y \frac{d^4 k}{(2\pi)^4} f(x + \frac{1}{2}\Theta k) g(x + y) e^{iky}$$

$\mathcal{A}_4$ acts on $\mathcal{H}_4$ componentwise by $\pi(\mathbf{f})\psi = L_\star(\mathbf{f})\psi = \mathbf{f} \star \psi$

$\mathcal{B}_4 = \{ \text{smooth bounded functions on } \mathbb{R}^4 \\ \text{with all derivatives bounded} \} \quad \star \text{ extends to } \mathcal{B}_4$

$$\textcircled{4} \quad \chi_4 = \Gamma_9 \quad J_4 \psi = \chi_4 \bar{\psi}$$

orientation form is $\pi(\mathbf{c}_4) = \frac{1}{(1+\Omega^2)^2} \prod_{\mu=1}^4 (\Gamma^\mu + \Omega \Gamma^{\mu+4}), \quad \text{not } \Gamma_9$

Verification of the axioms

$L_\star(b)$ and $[\mathcal{D}, L_\star(b)] = i(\Gamma^\mu + \Omega \Gamma^{\mu+4}) L_\star(\partial_\mu b)$ are **bounded**

0 Compactness

$(\mathcal{D} - \lambda)^{-1}$ **itself is compact** for $\lambda \notin sp(\mathcal{D})$

(for spectral action one should probably require $(\mathcal{D} - \lambda)^{-1}$ compact)

1 Regularity

- define $\langle \mathcal{D} \rangle = (\mathcal{D}^2 + 1)^{\frac{1}{2}}$, e.g. $\langle \mathcal{D} \rangle = \frac{1}{\pi} \int_0^\infty \frac{d\lambda}{\sqrt{\lambda}} \frac{\mathcal{D}^2 + 1 + (\lambda - \mathcal{D})}{\mathcal{D}^2 + 1 + \lambda}$
- $L_\star(b)$ and $[\mathcal{D}, L_\star(b)]$ belong for all n to the **domain of δ^n** :

$$\begin{aligned} \text{key: } [\langle \mathcal{D} \rangle^2, L_\star(b)] &= -L_\star(\Delta b) - 2L_\star(\partial_\mu b) \nabla_\mu^{(\Omega)} \\ [\nabla_\mu^{(\Omega)}, L_\star(b)] &= L_\star(\partial_\mu b) \quad [\langle \mathcal{D} \rangle^2, \nabla_\mu^{(\Omega)}] = c_{\mu\nu} \nabla^{(\Omega')\nu} \end{aligned}$$

$$\begin{aligned} \delta^n L_\star(b) &= \frac{1}{\pi^n} \sum_{k=0}^n \binom{n}{k} (-1)^k \\ &\quad \times \int_0^\infty \left(\prod_{i=1}^k \frac{1}{\langle \mathcal{D} \rangle^2 + \lambda_i} \right) P_b^{n+k}(\nabla) \langle \mathcal{D} \rangle^{-n} \prod_{j=1}^n \frac{d\lambda_j \sqrt{\lambda_j} \langle \mathcal{D} \rangle}{(\langle \mathcal{D} \rangle^2 + \lambda_j)^2} \end{aligned}$$

for $P_b^m(\nabla)$ – operator polynomial of order in m in $\nabla^{(\Omega)}$

1 dimension spectrum

- $\Psi_0(\mathcal{A}) =$ algebra generated by $\delta^n L_*(a)$ and $\delta^n [\mathcal{D}, L_*(a)]$
 $\Psi_k(\mathcal{A}) := \Psi_0(\mathcal{A}) \langle \mathcal{D} \rangle^k$ pseudo-differential op's of order $\leq k$
- study zeta-function $\zeta_\alpha(z) := \text{Tr}(\alpha \langle \mathcal{D} \rangle^{-z})$ for $\alpha \in \Psi_0(\mathcal{A})$
 important: $\langle \mathcal{D} \rangle^{-z}$ is trace class for $\text{Re}(z) > 2d$

integral kernel for resolvent

$$\left(\frac{1}{\langle \mathcal{D} \rangle^2 + \lambda_i} \right) (x, y) = \int_0^\infty dt e^{-t-t\lambda_i-t\tilde{\Omega}\Sigma_d} \cdot e^{-tH_d}(x, y)$$

involves Mehler kernel

$$e^{-tH_d}(x, y) = \left(\frac{\tilde{\Omega}}{2\pi \sinh(2\tilde{\Omega}t)} \right)^{\frac{d}{2}} e^{-\frac{\tilde{\Omega}}{4} \coth(\tilde{\Omega}t) \|x-y\|^2 - \frac{\tilde{\Omega}}{4} \tanh(\tilde{\Omega}t) \|x+y\|^2}$$

(solves $(\frac{d}{dt} + H_{d,x})e^{-tH_d}(x, y) = 0$, $\lim_{t \rightarrow 0} e^{-tH_d}(x, y) = \delta(x-y)$)

integral kernel for Moyal multiplication ($d = 4$)

$$(L_*(a))(x, y) = \frac{1}{\pi^4 \theta^4} \int d^4 w a(w) e^{i\langle x-y, \Theta^{-1}(x+y) \rangle + 2i\langle w, \Theta^{-1}(x-y) \rangle}$$

1 dimension spectrum (continued)

$$\begin{aligned}
 & (\delta^n L_\star(a))(x, y) \\
 &= \frac{1}{(2\sqrt{\pi})^n} \int_0^\infty \prod_{i=1}^n \frac{ds_i dt_i}{(s_i + t_i)^{\frac{3}{2}}} e^{-\sum_{j=1}^n (s_j + t_j)(1 + \check{\Omega}\Sigma_4)} \\
 &\times \int d^4x' d^4y' e^{-\sum_{j=1}^n s_j H_4(x, x')} ((\text{ad}(H_4))^n(L_\star(a)))(x', y') e^{-\sum_{j=1}^n t_j H_4(y', y)}
 \end{aligned}$$

with

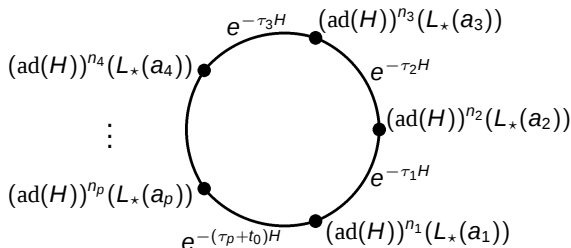
$$\begin{aligned}
 & ((\text{ad}(H_4))^n(L_\star(a)))(x, y) \\
 &= \sum \int d^4w a^{\nu_1 \dots \nu_r}(w) (\tilde{x} + \tilde{y})_{\nu_1} \dots (\tilde{x} + \tilde{y})_{\nu_r} e^{i\langle x-y, \Theta^{-1}(x+y) \rangle + 2i\langle w, \Theta^{-1}(x-y) \rangle}
 \end{aligned}$$

$$a^{\nu_1 \dots \nu_r} \in \mathcal{A} \text{ and } r \leq n$$

1 dimension spectrum (continued)

$$\begin{aligned}
 & \text{Tr}((\delta^{n_1} L_*(a_1)) \cdots (\delta^{n_p} L_*(a_p)) \langle \mathcal{D} \rangle^{-z}) \\
 &= \frac{1}{\Gamma(\frac{z}{2})} \int_0^\infty dt_0 \, t_0^{\frac{z}{2}-1} \int d^4 x_0 d^4 x_1 \dots d^4 x_p \\
 & \quad \times (\delta^{n_1} L_*(a_1))(x_1, x_2) \cdots (\delta^{n_p} L_*(a_p))(x_p, x_0) (e^{-t_0 \mathcal{D}_4^2})(x_0, x_1) .
 \end{aligned}$$

is concatenation of Mehler kernels and Moyal kernel polynomials
 resembles one-loop Feynman graph, all integrals are Gaussian



1 dimension spectrum (continued)

- result for $p = 1$: finite linear combination of integrals

$$I_q(z) = \frac{1}{\Gamma(\frac{z+n_1}{2})} \int_0^\infty dt t^{\frac{z+n_1+2q}{2}-1} e^{-t} \\ \times \int d^4w g_q(w) \left(\frac{\tanh(\tilde{\Omega}t)}{t} \right)^q (1 + \tanh^2(\tilde{\Omega}t))^h e^{-\frac{2\Omega \tanh(\tilde{\Omega}t)}{(1+\Omega^2)\theta} |w|^2}$$

for $q \in \{-[\frac{n_1}{2}] - 2, -[\frac{n_1}{2}] - 1, \dots, n_1 - 2\}$, $h \in \mathbb{Z}$ and $g_q \in \mathcal{A}$

Taylor expansion: $I_q(z)$ holomorphic for $z + n_1 \notin \{2, 4, \dots, -2q\}$
 $-n_1 - 2q \leq 4 \Rightarrow \text{Tr}(\delta^{n_1} L_\star(a) \langle D \rangle^{-z})$ holomorphic on $\mathbb{C} \setminus (4 - \mathbb{N})$

- residues are local, i.e. $\int d^4w g(w)$
- general case $p \geq 2$: holomorphic extension to $\mathbb{C} \setminus (4 - \mathbb{N})$

residues are again local, i.e. $\int d^4w (g_1 \star \dots \star g_p)(w)$

- relates to locality of counterterms in QFT
 $n_i \in \{0, 1\} \Rightarrow$ only finitely many residues

2 Metric dimension.

- $d := \sup\{k \in \mathbb{R} : k \in Sd\} = 4$ (= oscillator dimension!)
- Connes' trace theorem (generalised)

$$\begin{aligned} \mathrm{Tr}_\omega(L_\star(a)\langle\mathcal{D}\rangle^{-d}) &= \lim_{s \rightarrow 1} (s-1) \mathrm{Tr}(L_\star(a)\langle\mathcal{D}\rangle^{-ds}) \\ &= \frac{1}{2\pi^2(1+\Omega^2)^2} \int d^4x \, a(x) \end{aligned}$$

3 Finiteness

- smooth spinors $\psi \in \mathcal{H}^\infty := \bigcap_{k \in \mathbb{N}} \mathrm{dom}(\mathcal{D}^k)$ are Schwartzian

consequence: $\mathcal{H}^\infty = (\mathcal{A})^{16}$ is a free module

- hermitian structure $\langle \cdot, \cdot \rangle_{\mathcal{A}} : \mathcal{H}^\infty \times \mathcal{H}^\infty \rightarrow \mathcal{A}$
- scalar product from hermitian structure and Dixmier trace:
 $(\xi, \eta)_{\mathcal{H}^\infty} = \mathrm{Tr}_\omega(\langle \xi, \eta \rangle_{\mathcal{A}} \langle \mathcal{D} \rangle^{-4})$

④ **Reality:** $J\psi = \Gamma_9 \bar{\psi}$ defines real structure of **KO-dimension 0, independent of d**

⑤ **First order:** by construction

⑥ **Orientability**

- **unitisation $\mathcal{B}$** of $\mathcal{A}$ contains plane waves $u_\mu = e^{ix_\mu}$

- **Hochschild 4-cycle on $\mathcal{B}$**

$$\mathbf{c}_4 = \frac{1}{4!(1+\Omega^2)^2} \sum_{\sigma \in S_4} (-1)^\sigma \times (u_{\sigma(1)} \star u_{\sigma(2)} \star u_{\sigma(3)} \star u_{\sigma(4)})^{-1\star} \otimes u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes u_{\sigma(3)} \otimes u_{\sigma(4)}$$

- $\pi(\mathbf{c}_4) = \frac{1}{(1+\Omega^2)^2} (\Gamma_1 + \Omega\Gamma_5)(\Gamma_2 + \Omega\Gamma_6)(\Gamma_3 + \Omega\Gamma_7)(\Gamma_4 + \Omega\Gamma_8)$

- **Hochschild 4-cocycle on $\mathcal{A}$** is **volume form** ($a \equiv L_\star(a)$)

$$\begin{aligned} \phi_{\pi(\mathbf{c}_4)}(a_0, \dots, a_4) &= \text{Tr}_\omega(\pi(\mathbf{c}_4) a_0 [\mathcal{D}, a_1] \cdots [\mathcal{D}, a_4] \langle \mathcal{D} \rangle^{-4}) \\ &= \frac{1}{2\pi^2} \int a_0 \star da_1 \wedge_\star da_2 \wedge_\star da_3 \wedge_\star da_4 \end{aligned}$$

U(1)-Higgs model

tensor $(\mathcal{A}_4, \mathcal{H}_4, \mathcal{D}_4, \chi_4)$ with $(\mathbb{C} \oplus \mathbb{C}, \mathbb{C}^2, M\sigma_1)$ [Connes+Lott]

- $\mathcal{D} = \mathcal{D}_4 \otimes \mathbf{1}_2 + \Gamma_9 \otimes \sigma_1 M = \begin{pmatrix} \mathcal{D}_4 & M\Gamma_9 \\ M\Gamma_9 & \mathcal{D}_4 \end{pmatrix}$

- selfadjoint **fluctuated Dirac operators** $\mathcal{D}_A := \mathcal{D} + \sum_i a_i [\mathcal{D}, b_i]$, $a_i, b_i \in \mathcal{A} = \mathcal{A}_4 \oplus \mathcal{A}_4$ are of the form

$$\mathcal{D}_A = \begin{pmatrix} \mathcal{D}_4 + (\Gamma^\mu + \Omega \Gamma^{\mu+4}) L_\star(A_\mu) & \Gamma_9 L_\star(\phi) \\ \Gamma_9 L_\star(\bar{\phi}) & \mathcal{D}_4 + (\Gamma^\mu + \Omega \Gamma^{\mu+4}) L_\star(B_\mu) \end{pmatrix}$$

for $A_\mu = \overline{A}_\mu, B_\mu = \overline{B}_\mu, \phi \in \mathcal{A}_4$

- $\mathcal{D}_A^2 = \begin{pmatrix} (H_4 + L_\star(\phi \star \bar{\phi})) \mathbf{1} + \Sigma_4 + F_A & i(\Gamma^\mu + \Omega \Gamma^{\mu+4}) \Gamma_9 L_\star(D_\mu \phi) \\ i(\Gamma^\mu + \Omega \Gamma^{\mu+4}) \Gamma_9 L_\star(\overline{D_\mu \phi}) & (H_4 + L_\star(\bar{\phi} \star \phi)) \mathbf{1} + \Sigma_4 + F_B \end{pmatrix}$

- $D_\mu \phi = \partial_\mu \phi - iA \star \phi + i\phi \star B$
 $F_A = \{L_\star(A^\mu), i\partial_\mu + \Omega^2 M_\bullet(\tilde{x}_\mu)\} + (1 + \Omega^2) L_\star(A_\mu \star A^\mu)$
 $+ i(\frac{1}{4}[\Gamma^\mu, \Gamma^\nu] + \frac{1}{4}\Omega^2[\Gamma^{\mu+4}, \Gamma^{\nu+4}] + \Omega \Gamma^\mu \Gamma^{\nu+4}) L_\star(F_{\mu\nu}^A)$

Spectral action principle

most general form of bosonic action is $S(\mathcal{D}_A) = \text{Tr}(\chi(\mathcal{D}_A^2))$

- Laplace transf. + asympt. expansion $e^{-t\mathcal{D}_A^2} = \sum_{n=-d/2}^{\infty} a_n(\mathcal{D}_A^2) t^n$
 $(d = \text{dimension})$
- lead to $S(\mathcal{D}_A) = \sum_{n=-d/2}^{\infty} \chi_n \text{Tr}(a_n(\mathcal{D}_A^2))$

with $\chi_z = \frac{1}{\Gamma(-z)} \int_0^\infty ds s^{-z-1} \chi(s)$ for $z \notin \mathbb{N}$

$\chi_k = (-1)^k \chi^{(k)}(0)$ for $k \in \mathbb{N}$

- a_n – Seeley coefficients, must be computed from scratch

Duhamel expansion: $\mathcal{D}_A^2 = H_0 - V$

$$\begin{aligned} e^{-t(H_0 - V)} &= e^{-tH_0} - \int_0^t dt_1 \frac{d}{dt_1} (e^{-(t-t_1)(H_0 - V)} e^{-t_1 H_0}) \\ &= e^{-tH_0} + \int_0^t dt_1 (e^{-(t-t_1)(H_0 - V)} V e^{-t_1 H_0}) \quad \dots \text{iteration} \end{aligned}$$

Position space kernels

Mehler kernel

$$e^{-H_0 t}(x, y) = \frac{\tilde{\Omega}^2(1 - \tanh^2(\tilde{\Omega}t))^2}{16\pi^2 \tanh^2(\tilde{\Omega}t)} e^{-t\tilde{\Omega}\Sigma_4 1_2 - \frac{\tilde{\Omega}}{4} \frac{\|x-y\|^2}{\tanh(\tilde{\Omega}t)} - \frac{\tilde{\Omega}}{4} \tanh(\tilde{\Omega}t) \|x+y\|^2}$$

$$(\text{with } \tilde{\Omega} = \frac{2\Omega}{\theta} \quad \Theta = \theta S \quad S^2 = -1)$$

vacuum trace

$$\begin{aligned} \text{Tr}(e^{-tH_0}) &= \text{tr}_{cl} \int d^4x (e^{-tH_0})(x, x) = \frac{1}{8 \sinh^4(\tilde{\Omega}t)} \text{tr}_{cl}(e^{-t\tilde{\Omega}\Sigma_4}) \\ &= \frac{2}{\tanh^4(\tilde{\Omega}t)} = 2(\tilde{\Omega}t)^{-4} + \frac{8}{3}(\tilde{\Omega}t)^{-2} + \frac{52}{45} + \mathcal{O}(t^2) \end{aligned}$$

- expansion starts with $t^{-4} \Rightarrow$ corresponds to **8-dim. space**
- **spectral action is finite**, in contrast to pure Moyal at $\Omega = 0$
- **view spacial regularisation as part of geometry**

Vertex kernels (with $x \wedge y = x^\mu (\Theta^{-1})_{\mu\nu} y^\nu$)

$$(L_\star(f))(x, y) = \int \frac{d^4 z}{\pi^4 \theta^4} f(z) e^{2i(x \wedge y + y \wedge z + z \wedge x)}$$

$$\begin{aligned} & \{L_\star(A^\mu), i\partial_\mu + \Omega^2 M_\bullet(\tilde{X}_\mu)\}(x, y) \\ &= \int \frac{d^4 z}{\pi^4 \theta^4} (2\tilde{Z}^\mu - (1 - \Omega^2)(\tilde{X}^\mu + \tilde{Y}^\mu)) A_\mu(z) e^{2i(x \wedge y + y \wedge z + z \wedge x)} \end{aligned}$$

- one-vertex trace

$$\begin{aligned} & \int_0^t dt_1 \operatorname{Tr}(e^{-(t-t_1)H_0} L_\star(f) e^{-t_1 H_0}) \\ &= \frac{\tilde{\Omega}^2 t}{16\pi^2 (1 + \Omega^2)^2 \sinh^2(\tilde{\Omega} t)} \int d^4 z f(z) e^{-\frac{\tilde{\Omega} \tanh(\tilde{\Omega} t)}{1 + \Omega^2} |z|^2} \end{aligned}$$

- order is t^{-1} as in 4D-standard model

opposite sign of t^{-1} , $t^0 \Rightarrow$ spontaneous symmetry breaking

The spectral action

$$\begin{aligned}
 S(\mathcal{D}_A) = & \frac{\theta^4 \chi_{-4}}{8\Omega^4} + \frac{2\theta^2 \chi_{-2}}{3\Omega^2} + \frac{52\chi_0}{45} + \frac{\chi_0}{2\pi^2(1+\Omega^2)^2} \int d^4z \left\{ 2D_\mu \phi \star \overline{D_\mu \phi} \right. \\
 & + \left(\frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2} \right) (F_{\mu\nu}^A \star F_A^{\mu\nu} + F_{\mu\nu}^B \star F_B^{\mu\nu}) \\
 & + \left(\phi \star \bar{\phi} + \frac{4\Omega^2}{1+\Omega^2} \tilde{X}_A^\mu \star \tilde{X}_{A\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 \\
 & + \left(\bar{\phi} \star \phi + \frac{4\Omega^2}{1+\Omega^2} \tilde{X}_B^\mu \star \tilde{X}_{B\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 \\
 & \left. - 2 \left(\frac{4\Omega^2}{1+\Omega^2} \tilde{X}_0^\mu \star \tilde{X}_{0\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 \right\} (z) + \mathcal{O}(\chi_1)
 \end{aligned}$$

deeper entanglement of gauge and Higgs fields

covariant coordinates $\tilde{X}_{A\mu}(z) = (\Theta^{-1})_{\mu\nu} z^\nu + A_\mu(z)$ appear with Higgs field ϕ in **unified potential**; vacuum is non-trivial!

potential cannot be restricted to Higgs part if distinction into discrete and continuous geometries no longer possible

The spectral action

$$\begin{aligned}
 S(\mathcal{D}_A) = & \frac{\theta^4 \chi_{-4}}{8\Omega^4} + \frac{2\theta^2 \chi_{-2}}{3\Omega^2} + \frac{52\chi_0}{45} + \frac{\chi_0}{2\pi^2(1+\Omega^2)^2} \int d^4z \left\{ 2D_\mu \phi \star \overline{D_\mu \phi} \right. \\
 & + \left(\frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2} \right) (F_{\mu\nu}^A \star F_A^{\mu\nu} + F_{\mu\nu}^B \star F_B^{\mu\nu}) \\
 & + \left(\phi \star \bar{\phi} + \frac{4\Omega^2}{1+\Omega^2} \tilde{X}_A^\mu \star \tilde{X}_{A\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 \\
 & + \left(\bar{\phi} \star \phi + \frac{4\Omega^2}{1+\Omega^2} \tilde{X}_B^\mu \star \tilde{X}_{B\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 \\
 & \left. - 2 \left(\frac{4\Omega^2}{1+\Omega^2} \tilde{X}_0^\mu \star \tilde{X}_{0\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 \right\} (z) + \mathcal{O}(\chi_1)
 \end{aligned}$$

further remarks

- coefficient in front of Yang-Mills action is **positive**
- spectral action is **invariant under gauge transformations**
- **spectral action is translation-invariant!**
 $(\tilde{X}_{A\mu}(x))$ considered **as a whole**

The vacuum

vacuum field equations

$$(\phi^{vac} = \overline{\phi^{vac}}, \quad A_\mu^{vac} = B_\mu^{vac})$$

$$\begin{aligned} \frac{1}{g^2} [\tilde{X}_{A\nu}, [\tilde{X}_A^\mu, \tilde{X}_A^\nu]_\star]_\star + 2[\phi, [\tilde{X}_A^\mu, \phi]_\star]_\star \\ = \frac{4\Omega^2}{1+\Omega^2} \left\{ \phi \star \phi + \frac{4\Omega^2}{1+\Omega^2} \tilde{X}_A^\nu \star \tilde{X}_{A\nu} - \eta^2, \tilde{X}_A^\mu \right\}_\star \end{aligned}$$

$$\begin{aligned} 2[\tilde{X}_{A\nu}, [\phi, \tilde{X}_A^\nu]_\star]_\star = \left\{ \phi \star \phi + \frac{4\Omega^2}{1+\Omega^2} \tilde{X}_A^\nu \star \tilde{X}_{A\nu} - \eta^2, \phi \right\}_\star \\ \left(\text{with } \frac{1}{4g^2} = \frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2} \quad \eta^2 = \frac{\chi-1}{\chi_0} \right) \end{aligned}$$

spirit of **emerging geometry**

- $\Omega = 0 \Rightarrow$ solution: $[X_\mu, X_\nu] = \Theta_{\mu\nu} = \text{const}$, $\phi = \eta = \text{const}$
 $\Omega \neq 0$ gives some **dynamical geometry**
- analytical solution seems impossible
 $\Rightarrow$ **need numerical simulations**