Multipliers and the Second Dual of a Banach Algebra

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Let A be a Banach algebra with a faithful multiplication.

Recall: $\mu \in B(A)$ is called a **left** (**right**) **multiplier** on A if $\mu(ab) = \mu(a)b$ ($\mu(ab) = a\mu(b)$) for all $a, b \in A$.

- For each fixed $a \in A$,

$$l_a : x \longmapsto ax \text{ is a left multiplier on } A,$$
$$r_a : x \longmapsto xa \text{ is a right multiplier on } A.$$

- $LM(A)$:= the left multiplier algebra of A ($\subseteq B(A)$)
  $RM(A)$:= the right multiplier algebra of A ($\subseteq B(A)^{op}$)
  Then $LM(A)$ and $RM(A)$ are Banach algebras.

- $a \longmapsto l_a$ and $a \longmapsto r_a$ are injective and contractive.

- If A has a bounded approximate identity (**BAI**), then

$$\|\cdot\|_{LM(A)} \sim \|\cdot\|_A \sim \|\cdot\|_{RM(A)} \text{ on } A.$$

In this case, A can be identified with a left closed ideal in $LM(A)$, and a right closed ideal in $RM(A)$.

For $\mu \in LM(A)$ (resp. $\mu \in RM(A)$), we write $\mu \in A$ if $\mu = l_a$ (resp. $\mu = r_a$) for some $a \in A$.

Question: How can A be characterized inside $LM(A)$ (resp. $RM(A)$)?

Some representation theorems of LCQGs.

Let G be a locally compact group.

Let $L_1(G)$ be the group algebra of G and $A(G)$ the Fourier algebra of G. As preduals of Hopf-von Neumann algebras $L_\infty(G)$ and $L(G)$, resp., $L_1(G)$ and $A(G)$ are completely contractive Banach algebras, where $L(G)$ is the von Neumann algebra generated by the left regular representation of G.

Let $M(G)$ be the measure algebra of G and $M_{cb}A(G)$ the algebra of completely bounded multipliers on $A(G)$. Then $M(G)$ ($\cong LM(L_1(G)) \cong LM_{cb}(L_1(G))$) and $M_{cb}A(G)$ are completely contractive Banach algebras.

Theorem (**Ghahramani** 78; **Størmer** 80). There exists a weak*-weak* continuous completely isometric homomorphism $\Theta : M(G) \longrightarrow CB^{\sigma}_{L(G)}(B(L_2(G)))$.

Theorem (**Haagerup** 80; **Spronk** 02). There exists a weak*-weak* continuous completely isometric homomorphism $\widehat{\Theta} : M_{cb}\,A(G) \longrightarrow CB^{\sigma}_{L_\infty(G)}(B(L_2(G)))$.

Theorem (**Neufang** 00). We have

$$\Theta(M(G)) = CB^{\sigma, L_\infty(G)}_{L(G)}(B(L_2(G))).$$

Theorem (**Neufang-Ruan-Spronk** 04). We have

$$\widehat{\Theta}(M_{cb}\,A(G)) = CB^{\sigma, L(G)}_{L_\infty(G)}(B(L_2(G))).$$

Note that in the framework of locally compact quantum groups (**LCQGs**),

$$\widehat{L_\infty(G)} = L(G) \text{ and } \widehat{L(G)} = L_\infty(G).$$

Let $\mathcal{G} = \{\mathcal{M}, \Gamma, \varphi, \psi\}$ be von Neumann algebraic LCQG (**Kustermans**-**Vaes**).

Following the abelian case $\mathcal{G}_a = (L_\infty(G), \Gamma, \varphi, \psi)$, we write $L_\infty(\mathcal{G}) = \mathcal{M}$, $L_1(\mathcal{G}) = \mathcal{M}_*$, and $L_2(\mathcal{G})$ for the Hilbert space associated with φ.

Then $L_1(\mathcal{G})$ is a completely contractive Banach algebra. $\mathcal{G}$ is said to be **co-amenable** if $L_1(\mathcal{G})$ has a BAI.

It is known that for all locally compact groups G,

- $\mathcal{G}_a = (L_\infty(G), \Gamma, \varphi, \psi)$ is co-amenable;
- $\widehat{\mathcal{G}_a} = (L(G), \widehat{\Gamma}, \widehat{\varphi}, \widehat{\psi})$ is co-amenable $\Longleftrightarrow$ G is amenable.

In both cases, $L_1(\mathcal{G})$ has a BAI consisting of **states** on $L_\infty(\mathcal{G})$.

We show that it is true for all LCQGs.

Theorem 1 (**H.**-**Neufang**-**Ruan**). Let $\mathcal{G}$ be a LCQG. Then $\mathcal{G}$ is co-amenable $\Longleftrightarrow$ $L_1(\mathcal{G})$ has a BAI consisting of states on $L_\infty(\mathcal{G})$.

The algebra $M_{cb}^r(L_1(\mathcal{G}))$ of completely bounded right multipliers of $L_1(\mathcal{G})$ was recently introduced by **Junge-Neufang-Ruan**, which is defined to be the set of all $q \in L_\infty(\widehat{\mathcal{G}})'$ such that $\rho(f)q \in \rho(L_1(\mathcal{G}))$ for all $f \in L_1(\mathcal{G})$ and the map

$$m_q^r : L_1(\mathcal{G}) \longrightarrow L_1(\mathcal{G}), f \longmapsto \rho^{-1}(\rho(f)q) \text{ is c.b.,}$$

where $L_\infty(\widehat{\mathcal{G}})'$ is the commutant of $L_\infty(\widehat{\mathcal{G}})$ in $B(L_2(\mathcal{G}))$ and ρ is the right regular representation of $\mathcal{G}$.

Under the identification $q \longleftrightarrow m_q^r \in RM_{cb}(L_1(\mathcal{G}))$, $M_{cb}^r(L_1(\mathcal{G}))$ becomes a completely contractive Banach algebra.

Unifying and generalizing the representation theorems mentioned earlier, Junge-Neufang-Ruan showed

Theorem (**Junge-Neufang-Ruan** 06). Let $\mathcal{G}$ be a LCQG. There is a completely isometric homomorphism

$$\pi : M_{cb}^r(L_1(\mathcal{G})) \longrightarrow CB^{\sigma}_{L_\infty(\widehat{\mathcal{G}})}(B(L_2(\mathcal{G})))$$

such that $\pi(M_{cb}^r(L_1(\mathcal{G}))) = CB^{\sigma, L_\infty(\mathcal{G})}_{L_\infty(\widehat{\mathcal{G}})}(B(L_2(\mathcal{G})))$.

Using a measure theoretic proof, **Neufang-Ruan-Spronk** (04) showed that for all locally compact groups G,

$$\pi(L_1(G)) = CB_{L(G)}^{\sigma,(L_\infty(G),C_b(G))}(B(L_2(G)))$$

$$= CB_{L(G)}^{\sigma,(L_\infty(G),RUC(G))}(B(L_2(G))),$$

where $C_b(G)$ (resp. $RUC(G)$) is the space of bounded cont. (resp. right uniformly cont.) functions on G.

Question: Let $\mathcal{G}$ be a co-amenable LCQG. What is $\pi(L_1(\mathcal{G}))$? Is there a subspace Y of $L_\infty(\mathcal{G})$ such that

$$\pi(L_1(\mathcal{G})) = CB_{L_\infty(\widehat{\mathcal{G}})}^{\sigma,(L_\infty(\mathcal{G}),Y)}(B(L_2(\mathcal{G})))?$$

This was open even for the co-commutative LCQG $\mathcal{G} = (\widehat{L_\infty(G)}, \widehat{\Gamma}, \widehat{\varphi}, \widehat{\psi})$ (**Neufang-Ruan-Spronk** 04).

We will consider a *Banach algebraic* approach to this image problem.

To characterize a Banach algebra A inside $RM(A)$ and $LM(A)$, respectively, we introduce the following classes of Banach algebras.

A class of Banach algebras.

Definition 2 (**H.-Neufang-Ruan**). Let A be a Banach algebra with a BAI. Assume that for every $\mu \in RM(A)$, there is a closed subalgebra B of A with a BAI satisfying the following conditions.

(I) $\mu|_B \in RM(B)$.

(II) $f|_B \in BB^*$ for all $f \in AA^*$.

(III) There is a family $\{B_i\}$ of closed right ideals in B such that

(i) each B_i is weakly sequentially complete (**WSC**) with a *sequential* BAI;

(ii) for all i, there exists a left B_i-module projection from B onto B_i;

(iii) $\mu \in A$ if $\mu|_{B_i} \in B_i$ for all i.

Then A is said to be of type (RM).

Similarly, Banach algebras of type (LM) can be defined.

A is said to be of type (M) if A is both of type (LM) and of type (RM).

Examples of Banach algebras of type (M)

(1) All WSC Banach algebras with a *sequential* BAI, in particular, separable $L_1(\mathcal{G})$ of co-amenable LCQGs $\mathcal{G}$.

(2) All group algebras $L_1(G)$ of locally compact groups.

(3) All weighted convolution (Beurling) algebras $L_1(G, \omega)$.

(4) Fourier algebras $A(G)$ of all amenable groups G.

(5) The algebras $L_1(\mathcal{G})$ of some co-amenable LCQGs $\mathcal{G}$.

Theorem 3 (**H.-Neufang-Ruan**). Let A be a Banach algebra of type (RM) and $\mu \in RM(A)$. T.F.A.E.

(i) $\mu \in A$.

(ii) $\mu^*(A^*) \subseteq \langle AA^* \rangle$.

(iii) There exists an $m \in A^{**}$ such that for all $n \in A^{**}$,
$\mu^{**}(n) = n \triangle m$ (the *right Arens product* on A^{**}).

The left version of the theorem holds if A is a Banach algebra of type (LM).

The completely isometric representation $\pi|_{L_1(\mathcal{G})}$.

Theorem 4 (**H.-Neufang-Ruan**). Let $\mathcal{G}$ and let π be the same as in the representation theorem of **Junge-Neufang-Ruan**. If $L_1(\mathcal{G})$ is of type (M) (e.g., $\mathcal{G}$ is co-amenable with $L_1(\mathcal{G})$ separable), then

$$\pi(L_1(\mathcal{G})) = CB^{\sigma,(L_\infty(\mathcal{G}),\,RUC(\mathcal{G}))}_{L_\infty(\widehat{\mathcal{G}})}(B(L_2(\mathcal{G}))),$$

where $RUC(\mathcal{G}) := \langle L_1(\mathcal{G}) \cdot L_\infty(\mathcal{G})\rangle$.

Corollary 5 (**H.-Neufang-Ruan**). Let G be an amenable locally compact group and let $\widehat{\Theta}$ be the representation of $M_{cb}A(G)$ onto $CB^{\sigma,\,L(G)}_{L_\infty(G)}(B(L_2(G)))$. Then

$$\widehat{\Theta}(A(G)) = CB^{\sigma,(L(G),\,UC(\widehat{G}\,))}_{L_\infty(G)}(B(L_2(G))),$$

where $UC(\widehat{G}\,) = \langle A(G) \cdot L(G)\rangle$, the C^*-algebra generated by operators in $L(G)$ with compact support.

Remark. This corollary answers an open question of **Neufang-Ruan-Spronk** (04).

An application to second dual Banach algebras.

Let A be a Banach algebra.

Then A^{**} is a Banach algebra under the **left**/the **right** **Arens products** defined as follows:

for m, $n \in A^{**}$, $f \in A^*$ and a, $b \in A$, we have

$$\langle m \cdot n, f \rangle = \langle m, n \cdot f \rangle, \qquad \langle f, m \triangle n \rangle = \langle f \cdot m, n \rangle,$$

$$\langle n \cdot f, a \rangle = \langle n, f \cdot a \rangle, \qquad \langle a, f \cdot m \rangle = \langle a \cdot f, m \rangle,$$

$$\langle f \cdot a, b \rangle = \langle f, ab \rangle. \qquad \langle b, a \cdot f \rangle = \langle ba, f \rangle.$$

Both $\cdot$ and $\triangle$ extend the multiplication on A.

A is called **Arens regular** if $\cdot$ and $\triangle$ coincide on A^{**}.

E.g., every C^*-algebra is Arens regular.

For any fixed $m \in A^{**}$, the maps

$$n \longmapsto n \cdot m \text{ and } n \longmapsto m \triangle n$$

are weak*-weak* continuous on A^{**}.

The **left** and the **right topological centres** of A^{**} are defined as

$$Z_t^{(l)}(A^{**}) = \{m \in A^{**} : n \longmapsto m \cdot n \text{ is w*-w* continuous}\},$$

$$Z_t^{(r)}(A^{**}) = \{m \in A^{**} : n \longmapsto n \triangle m \text{ is w*-w* continuous}\}.$$

Then $A \subseteq Z_t^{(l)}(A^{**}),\ Z_t^{(r)}(A^{**}) \subseteq A^{**}$.

- A is Arens regular $\Longleftrightarrow$ $[\, Z_t^{(l)}(A^{**}) = Z_t^{(r)}(A^{**}) = A^{**}\,]$.

- A is called **left (right) strongly Arens irregular** if

$$Z_t^{(l)}(A^{**}) = A \ (Z_t^{(r)}(A^{**}) = A) \text{ (**Dales-Lau** 05).}$$

E.g., for all locally compact groups G, $L_1(G)$ is strongly Arens irregular (**Lau-Losert** 88).

Multipliers and strong Arens irregularity.

Theorem 6 (**H.-Neufang-Ruan**). Let A be a Banach algebra.

(i) If A is of type (LM), then
A is left strongly Arens irreg. $\iff [Z_t^{(l)}(A^{**}) \cdot A \subseteq A]$.

(ii) If A is of type (RM), then
A is right strongly Arens irreg. $\iff [A \cdot Z_t^{(r)}(A^{**}) \subseteq A]$.

In particular, A is strongly Arens irregular if A is of type (M) and an ideal in A^{**}.

Corollary 7 (**H.-Neufang-Ruan**).
Let $\mathcal{G}$ be a co-amenable compact quantum group.
If either $L_1(\mathcal{G})$ is separable or $L_1(\mathcal{G})$ has a central BAI, then $L_1(\mathcal{G})$ is strongly Arens irregular.

Corollary 7 was proved when

(i) $L_1(\mathcal{G}) = L_1(G)$ of compact groups G by **Işik-Pym-Ülger** (87);

(ii) $L_1(\mathcal{G}) = A(G)$ of amenable discrete groups G by **Lau-Losert** (93).

Remarks.

(a) Theorem 6(i) was proved by **Lau-Ülger** (96) for WSC Banach algebras A with a sequential BAI.

(b) Theorem 6(i) was also proved by **Baker-Lau-Pym** (98) under the following condition:

A is WSC with a BAI and A is a right ideal in A^{**}.