Embedding of noncommutative L_p

Marius Junge

Joint work with J. Parcet

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Fields Institute, December 2007

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- (Maurey-Pisier-Krivine)
  For every infinite dimensional Banach space there exist p_0, q_0 such that ℓ_p^n is $(1+\varepsilon)$–isomorphic to a subspace of X for all

$$p \in [p_0, 2] \cup \{q_0\} \ .$$

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- Corollary: $L_p(0,\infty) \cap L_2(0,\infty) \cong L_p([0,1])$ $p \geq 2$,
  $L_p(0,\infty) + L_2(0,\infty) \cong L_p([0,1])$ $p \geq 2$.

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- $x_k = s_k$, $x_k = s_k + is_{-k}$ for semi-circular s_k.

Starting point $\sim$ 98/99

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Corollary (Junge 06) OH is a subspace of $R(\lambda) + C(\mu)$ for suitable λ and μ.

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Xu's work made this very precise.

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Here $L_4^c = [N, L_2^c]_{\frac{1}{2}}$ satisfies

$$C_m \otimes_h L_4^c \otimes_h C_m \;=\; L_4(M_m(N))S_4^m\ .$$

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Theorem Let $x_1, \dots, x_n$ be mean 0 variables in $N = *_{i,B} A_i$. Then

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Remark: The L_p version is harder.

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Technique: Decompose a in elements which start with symbols in A_k, end with symbols in A_k, remainder terms.

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b) Let $1 \leq q \leq p \leq 2$ and N a von Neumann algebra. Then for some M

$$L_p(N) \subset_{cb} L_q(M) \ .$$

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