Boundaries of discrete quantum groups

Fields Institute Workshop

Operator Spaces and Quantum Groups

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Plan of the talk

- Introduction to discrete and compact quantum groups.
- Boundaries at infinity and applications (joint work with Vergnioux)

  $\rightsquigarrow$ exactness of certain C^*-algebras,

  $\rightsquigarrow$ Ozawa's solidity of certain von Neumann algebras.
- Identification of Poisson boundaries for random walks (joint work with Vander Vennet).

Compact quantum groups (Woronowicz)

A compact quantum group $\mathbb{G}$ is a pair $(C(\mathbb{G}), \Delta)$ where

- $C(\mathbb{G})$ is a unital C*-algebra,
- $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes_{\min} C(\mathbb{G})$ is a unital *-homomorphism,

satisfying

- co-associativity : $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$,
- the density conditions :
  $$\Delta(C(\mathbb{G}))(1 \otimes C(\mathbb{G})) \quad \text{and} \quad \Delta(C(\mathbb{G}))(C(\mathbb{G}) \otimes 1)$$
  are total in $C(\mathbb{G}) \otimes_{\min} C(\mathbb{G})$.

If $C(\mathbb{G})$ is commutative, this corresponds to

- $\mathbb{G}$ being a compact group,
- $C(\mathbb{G})$ being the C*-algebra of continuous functions,
- Δ given by $(\Delta(f))(x, y) = f(xy)$.

Operator algebra point of view

Theorem (Woronowicz)

If $\mathbb{G}$ is a compact quantum group, $C(\mathbb{G})$ admits a unique Haar state h.

⤳ Reduced C*-algebra $\pi_h(C(\mathbb{G}))$ denoted as $C_{\text{red}}(\mathbb{G})$.

⤳ Von Neumann algebra $\pi_h(C(\mathbb{G}))''$ denoted as $L^\infty(\mathbb{G})$.

Potential interest, because we can take $\mathbb{G} = \widehat{\Gamma}$, namely

- $C_{\text{red}}(\mathbb{G}) = C^*_{\text{red}}(\Gamma)$ and $L^\infty(\mathbb{G}) = \mathcal{L}(\Gamma)$,
- $\Delta(u_g) = u_g \otimes u_g$.

⤳ Concrete compact quantum groups can lead to interesting operator algebras.

Representation theory

Let $\mathbb{G}$ be a compact quantum group.

Definition

An *n-dimensional unitary representation* of $\mathbb{G}$ is

- an $n \times n$ unitary matrix (U_{ij}) with matrix coefficients in $C(\mathbb{G})$,
- satisfying $\Delta(U_{ij}) = \sum_k U_{ik} \otimes U_{kj}$.

All that you expect, holds :

⇝ Direct sums, irreducibles, Peter-Weyl, ...

⇝ Tensor product $U \textcircled{\scriptsize T} V := (U_{ij} V_{kl})$.

⇝ The representation $\overline{U} := (U_{ij}^*)$ is not necessarily unitary, but can be unitarized :

contragredient $U^c := F\overline{U}F^{-1}$ for some $F \in \mathrm{GL}_n(\mathbb{C})$.

⇝ Enveloping C*-algebra $C_u(\mathbb{G})$ and notion of co-amenability.

Universal compact quantum groups

Definition (Van Daele-Wang, Banica)

We define two families of compact quantum groups $\mathbb{G}$.

$\mathbb{G} = A_o(F)$ for $F \in \mathrm{GL}_n(\mathbb{C})$ and $F\overline{F} = \pm 1$.

$\mathbb{G} = A_u(F)$ for $F \in \mathrm{GL}_n(\mathbb{C})$.

Let $C(\mathbb{G})$ be the universal unital C*-algebra with generators (U_{ij}) subject to the relations

- U is unitary,
- $U = F\overline{U}F^{-1}$. (for $A_o(F)$)
- $F\overline{U}F^{-1}$ is also unitary. (for $A_u(F)$)

with comultiplication $\Delta(U_{ij}) = \sum_k U_{ik} \otimes U_{kj}$.

⇝ $\mathrm{SU}_q(2) = A_o\begin{pmatrix} 0 & \sqrt{|q|} \\ -\frac{\mathrm{sign}(q)}{\sqrt{|q|}} & 0 \end{pmatrix}$ for $q \in [-1,1] \setminus \{0\}$.

⇝ Apart from the $\mathrm{SU}_q(2)$, all $A_{o,u}(F)$ are non co-amenable.

Representation th. of $A_o(F)$ and $A_u(F)$ (Banica)

Let $\mathbb{G} = A_o(F)$.

Irred $\mathbb{G} = \frac{1}{2}\mathbb{N}$ and

$$U_n ⓣ U_m \;\cong\; U_{|n-m|} \oplus U_{|n-m|+1} \oplus \cdots \oplus U_{n+m}\,.$$

⟿ Same fusion rules as SU(2).

⟿ Fusion rules are abelian.

Let $\mathbb{G} = A_u(F)$.

Irred $\mathbb{G} = \mathbb{N} * \mathbb{N}$, the free monoid generated by α and β,

$$U_x ⓣ U_y = \bigoplus_{z,\; x=az,\; y=\overline{z}b} U_{ab}\,,$$

where $x \mapsto \overline{x}$ is the involution on $\mathbb{N} * \mathbb{N}$ satisfying $\overline{\alpha} = \beta$. Also,

contragredient of U_x is $U_{\overline{x}}$.

Operator algebraic properties

We discuss $\mathbb{G} = A_o(F)$ (for F at least 3×3) and $\mathbb{G} = A_u(F)$.

What is known.

Work of Banica :

- $L^\infty(A_u(F))$ is a factor.
- $C_{\text{red}}(A_u(F))$ is simple.
- $L^\infty(A_u(I_2)) \cong \mathcal{L}(\mathbb{F}_2)$.

Work of V & Vergnioux :

- $L^\infty(A_{o,u}(F))$ is solid.
- $C_{\text{red}}(A_{o,u}(F))$ is exact.
- At least for certain F, $L^\infty(A_o(F))$ is a full factor, $C_{\text{red}}(A_o(F))$ is simple.

What is open :

- Are all $L^\infty(A_{o,u}(I_n))$ free group factors?
- Are all $L^\infty(A_{o,u}(F))$ and $L^\infty(A_{o,u}(F))$ free Araki-Woods factors?
- Are the $C_{\text{red}}(A_{o,u}(I_n))$ projectionless?
- Do the $L^\infty(A_{o,u}(I_n))$ share more with the $\mathcal{L}(\mathbb{F}_k)$? (Haagerup property, complete metric approximation, absence of Cartan, ...)

Exact C*-algebras

Definition

A unital C*-algebra A is called exact if the minimal tensor product $A \otimes_{\min} \cdot$ with A, preserves short exact sequences.

Theorem (Ozawa, Anantharaman-Delaroche)

Let Γ be a discrete group. Then, $C^*_{\mathrm{red}}(\Gamma)$ is exact if and only if Γ admits an amenable action on a compact space.

Amenability of Γ :
$\exists \xi_n \in \ell^2(\Gamma)$ with $\|\xi_n\|_2 = 1$ and $\|\lambda_g \xi_n - \xi_n\|_2 \to 0$ for all g.

Amenability of $\Gamma \curvearrowright X$:
$\exists \xi_n : X \to \ell^2(\Gamma)$ continuous, with $\|\xi_n(x)\|_2 = 1$ and $\|\lambda_g \xi_n(x) - \xi_n(g \cdot x)\|_2 \to 0$ uniformly in $x \in X$, for all $g \in \Gamma$.

Example : boundary action $\mathbb{F}_k \curvearrowright \{$infinite reduced words$\}$,

$$\xi_n(x_1 x_2 x_3 \cdots) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \delta_{x_1 \cdots x_j}.$$

An application : solid factors

Definition (Ozawa)

A von Neumann algebra M with tracial state τ is called solid if the relative commutant of any diffuse subalgebra is injective.

⇝ In particular, a solid non-hyperfinite II_1 factor M is prime : if $M = M_1 \,\overline{\otimes}\, M_2$, one of both is a matrix algebra.

⇝ If τ is non-tracial : consider subalgebras with state preserving conditional expectation.

Theorem (Ozawa)

Let Γ be a discrete group. The group von Neumann algebra $\mathcal{L}(\Gamma)$ is solid if Γ admits a compactification X such that

- left action $\Gamma \curvearrowright \Gamma$ extends to amenable action $\Gamma \curvearrowright X$,
- right action $\Gamma \curvearrowright \Gamma$ extends to action on X trivial on $\partial\Gamma = X \setminus \Gamma$.

⇝ We shall produce such actions for the duals of $A_o(F)$.

Discrete quantum groups

Let $\mathbb{G}$ be a compact quantum group

$$\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes_{\min} C(\mathbb{G}).$$

Unitary representation is $n \times n$ matrix (U_{ij}) with $\Delta(U_{ij}) = \sum_k U_{ik} \otimes U_{kj}$.

$\rightsquigarrow$ Set $H_U = \mathbb{C}^n$ and view $U \in \mathrm{B}(H_U) \otimes C(\mathbb{G})$.

The dual discrete quantum group

Let $\mathrm{Irred}\,\mathbb{G}$ be the set of all irreducible rep. of $\mathbb{G}$.

Write $U_x \in \mathrm{B}(H_x) \otimes C(\mathbb{G})$ for all $x \in \mathrm{Irred}\,\mathbb{G}$. Set

$$\ell^\infty(\hat{\mathbb{G}}) = \prod_{x \in \mathrm{Irred}\,\mathbb{G}} \mathrm{B}(H_x).$$

$\rightsquigarrow$ Comultiplication $\hat{\Delta} : \ell^\infty(\hat{\mathbb{G}}) \to \ell^\infty(\hat{\mathbb{G}}) \overline{\otimes} \ell^\infty(\hat{\mathbb{G}})$ such that $\hat{\Delta}(a)_{x,y}\, S = S\, a_z$ whenever S intertwines U_z and $U_x \circledT U_y$.

Remark. If $\mathbb{G} = \hat{\Gamma}$, of course $\ell^\infty(\hat{\mathbb{G}}) = \ell^\infty(\Gamma)$ and $\Delta(a)(x, y) = a(xy)$.

Exactness and solidity for dual of $A_o(F)$

Let $\mathbb{G} = A_o(F)$. We have Irred $\mathbb{G} = \frac{1}{2}\mathbb{N}$ with U_n on H_n and

$$U_n \textcircled{\scriptsize T} U_m \;\cong\; U_{|n-m|} \oplus U_{|n-m|+1} \oplus \cdots \oplus U_{n+m} \,.$$

Compactification of $\hat{\mathbb{G}}$: C*-algebra B with $c_0(\hat{\mathbb{G}}) \subset B \subset \ell^\infty(\hat{\mathbb{G}})$.

- Let $V_n : H_{n+1} \to H_n \otimes H_1$ be an isometric intertwiner.
- Define $\psi_n : \mathrm{B}(H_n) \to \mathrm{B}(H_{n+1}) : \psi_n(a) = V_n^*(a \otimes 1)V_n$, UCP.
- In fact, $\psi_{n,m} : \mathrm{B}(H_n) \to \mathrm{B}(H_m)$ for $m \geq n$.
- C*-algebra B as 'direct limit'. (Non-trivial: ψ_n are not homom.)

Theorem (V – Vergnioux)

The left action $\hat{\mathbb{G}} \curvearrowright c_0(\hat{\mathbb{G}})$ extends to an amenable action on B, while the right action extends to an action on B that is trivial on $B/c_0(\hat{\mathbb{G}})$.

$\rightsquigarrow$ $C_{\mathrm{red}}(\mathbb{G})$ is exact and $L^\infty(\mathbb{G})$ is (generalized) solid.

Remarks :

- It is non-obvious to define amenable actions.
- Extends to $A_u(F)$: joint work with Vander Vennet.

Probabilistic boundaries

Let μ be a probability measure on a discrete group Γ.

$\rightsquigarrow$ Random walk with transition probab. $p(x,y) = \mu(x^{-1}y)$.

$\rightsquigarrow$ Markov operator on $\ell^\infty(\Gamma)$: $(Pa)(x) = \sum_y p(x,y)a(y)$.

Bounded harmonic functions $H^\infty(\Gamma,\mu) = \{a \in \ell^\infty(\Gamma) \mid Pa = a\}$.

Assume : transience and $\operatorname{supp}\mu$ generates Γ as semi-group.

Poisson boundary

$\Gamma \curvearrowright (Y,\eta)$.

- Characterization.
- Properties.
- Definition.

Action of Γ on probability space (Y,η).

Poisson integral

$$\Theta(a)(x) = \int_Y a(x\cdot y)\, d\eta(y)$$

is a bijection $L^\infty(Y,\eta) \to H^\infty(\Gamma,\mu)$.

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Poisson boundary

$\Gamma \curvearrowright (Y,\eta)$.

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- Properties.
- Definition.

We can take a compactification $\Gamma \subset X$ such that

- Almost every path converges to a point in $Y := X \setminus \Gamma$.
- Probability to end up in $\mathcal{U}$ is $\eta(\mathcal{U})$.

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- Characterization.
- Properties.
- Definition.

Equipped with the product

$$a \cdot b = \text{strong}^*\text{-}\lim_n P^n(ab)$$

$H^\infty(\Gamma,\mu)$ is a von Neumann algebra.

Set $H^\infty(\Gamma,\mu) = L^\infty(Y,\eta)$ with

$$\int a \, d\eta = a(e).$$

Probabilistic boundaries

Let μ be a probability measure on a discrete group Γ.

⤳ Random walk with transition probab. $p(x,y) = \mu(x^{-1}y)$.

⤳ Markov operator on $\ell^\infty(\Gamma)$: $(Pa)(x) = \sum_y p(x,y)a(y)$.

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$H^\infty(\Gamma,\mu)$ is a von Neumann algebra.

Set $H^\infty(\Gamma,\mu) = L^\infty(Y,\eta)$ with

$$\int a \, d\eta = a(e).$$

⤳ **Basic problem :** identify $\Gamma \curvearrowright (Y,\eta)$.

Poisson boundaries of discrete quantum groups

Remember : $\ell^\infty(\hat{\mathbb{G}}) = \prod_{x \in \mathrm{Irred}\,\mathbb{G}} \mathrm{B}(H_x)$.

- $\mathrm{B}(H_x)$ has favorite state ψ_x coming from unitarizing $\overline{U_x}$.
- Fix a probability measure μ on $\mathrm{Irred}\,\mathbb{G}$
  - ⇝ State ψ_μ on $\ell^\infty(\hat{\mathbb{G}})$ given by $\psi_\mu(a) = \sum_x \mu(x)\psi_x(a_x)$.
  - ⇝ Markov operator on $\ell^\infty(\hat{\mathbb{G}})$: $P(a) = (\mathrm{id} \otimes \psi_\mu)\hat{\Delta}(a)$.
- Assume transience and generating property of μ and consider

$$H^\infty(\hat{\mathbb{G}}, \mu) = \{a \in \ell^\infty(\hat{\mathbb{G}}) \mid Pa = a\}.$$

Von Neumann algebra with product $a \cdot b = \text{strong}^*\text{-}\lim_n P^n(ab)$.

- Evaluation in the trivial rep. ε : harmonic state η on $H^\infty(\hat{\mathbb{G}}, \mu)$.

⇝ We get the Poisson boundary $H^\infty(\hat{\mathbb{G}}, \mu)$, equipped with actions of $\hat{\mathbb{G}}$ and $\mathbb{G}$.

⇝ Definition due to Izumi (study of infinite product actions).

Central problem : identify $\hat{\mathbb{G}} \curvearrowright H^\infty(\hat{\mathbb{G}}, \mu)$ in examples.

Identification of Poisson boundaries

- Restriction of Markov operator : random walk on Irred $\mathbb{G}$.
- If fusion rules are commutative, the Poisson boundary of Irred $\mathbb{G}$ is trivial and $H^\infty(\widehat{\mathbb{G}}, \mu)$ independent of μ.

Theorem (Izumi)

For any generating measure μ on Irred$(SU_q(2))$, we get
$H^\infty(\widehat{\mathbb{G}}, \mu) \cong L^\infty(\mathbb{T} \backslash SU_q(2))$ (Podles' sphere).

⟿ (Izumi-Neshveyev-Tuset) Generalization to $SU_q(n)$.

Theorem (Tomatsu, see next talk)

If $\mathbb{G}$ is co-amenable compact quantum group with commutative fusion rules, $H^\infty(\widehat{\mathbb{G}}, \mu) \cong L^\infty(\mathbb{K} \backslash SU_q(2))$ where $\mathbb{K}$ is the maximal compact subgroup of Kac type.

Tool : Izumi's Poisson integral $\Theta : L^\infty(\mathbb{G}) \to H^\infty(\widehat{\mathbb{G}}, \mu)$

$$\Theta(a) = (\mathrm{id} \otimes h)(\mathbb{V}^*(1 \otimes a)\mathbb{V}).$$

Poisson boundary of $\widehat{A_o(F)}$

$F\overline{F} = \pm 1$ and $C(A_o(F))$ generated by U_{ij} with $U = F\overline{U}F^{-1}$ unitary.

When $F_q = \begin{pmatrix} 0 & \sqrt{|q|} \\ -\frac{\text{sign}(q)}{\sqrt{|q|}} & 0 \end{pmatrix}$, we get $\text{SU}_q(2)$.

⟿ Let $\text{Tr}(F^*F) = |q + \frac{1}{q}|$ and $F\overline{F} = -\text{sign}(q)$.
Take the unital C*-algebra $\mathcal{L}$ generated by the entries of a $2 \times n$ matrix L with $L^*L = 1$, $LL^* = 1$ and $L = F_q\overline{L}F^{-1}$.

⟿ Commuting ergodic actions $\text{SU}_q(2) \curvearrowright \mathcal{L}$ on the left and $A_o(F) \curvearrowright \mathcal{L}$ on the right.

Theorem (V – Vander Vennet)

The Poisson boundary of $\widehat{A_o(F)}$ can be identified with $(\mathbb{T}\backslash\mathcal{L})''$.

Remarks.

- $\mathbb{T}\backslash\mathcal{L} \cong$ boundary C*-algebra $B/c_0(\widehat{\mathbb{G}})$ of $\widehat{A_o(F)}$.
- $\mathcal{L}$ is nuclear. Is it simple?
- (De Rijdt - Vander Vennet) Behavior of Poisson boundaries under monoidal equivalence.

Recall the quantum group $A_u(F)$

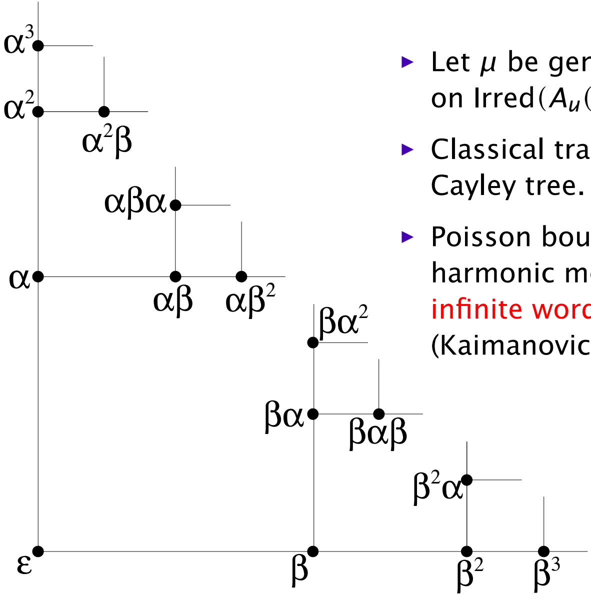
$\mathbb{G} = A_u(F)$ and $C(\mathbb{G})$ generated by U_{ij} with U and $F\overline{U}F^{-1}$ unitary.

Recall. Irred $\mathbb{G} = \mathbb{N} * \mathbb{N}$, the free monoid generated by α and β,

$$U_x \circledT U_y = \bigoplus_{z,\ x=az,\ y=\overline{z}b} U_{ab}\ ,$$

where $x \mapsto \overline{x}$ is the involution on $\mathbb{N} * \mathbb{N}$ satisfying $\overline{\alpha} = \beta$.

Cayley tree of $\mathrm{Irred}(A_u(F))$



- Let μ be generating prob. measure on $\mathrm{Irred}(A_u(F))$ with finite support.
- Classical transient random walk on Cayley tree.
- Poisson boundary is given by harmonic measure η on set Y of infinite words in α, β. (Kaimanovich et al.)

Poisson boundary of $\widehat{A_u(F)}$

Let μ be a generating probability measure on $\mathrm{Irred}(A_u(F))$ with finite support

Let η be the harmonic measure on the set Y of infinite words in α, β.

⤳ An infinite word x in α, β can be of the form $x = x_1 \otimes x_2 \otimes x_3 \otimes \cdots$ or has tail $\alpha\beta\alpha\beta\alpha\beta\cdots$.

⤳ η has no atoms in words with tail $\alpha\beta\alpha\beta\alpha\beta\cdots$.

⤳ It suffices to consider infinite words of the form $x = x_1 \otimes x_2 \otimes x_3 \otimes \cdots$.

Theorem (V - Vander Vennet)

The Poisson boundary $H^\infty(\hat{\mathbb{G}}, \mu)$ can be identified with the measurable field of $\mathrm{ITPFI}_{\text{finite}}$ factors over (Y, η) with fiber in $x = x_1 \otimes x_2 \otimes x_3 \otimes \cdots$ given by $\overline{\bigotimes_k}(\mathrm{B}(H_{x_k}), \psi_{x_k})$.

Back to exactness and solidity for $\mathbb{G} = A_u(F)$

(Work in progress)

Consider the set Y of infinite words in α, β as a compact space.

$\rightsquigarrow$ Continuous field B of unital C*-algebras over Y with

- fiber in $x = x_1 \otimes x_2 \otimes x_3 \otimes \cdots$ given by $\bigotimes_k \mathrm{B}(H_{x_k})$,
- fiber in $x = y \otimes \alpha\beta\alpha\beta\cdots$ given by $\mathrm{B}(H_y) \otimes$ boundary $\widehat{A_o}$.

$\rightsquigarrow$ We can view B as a boundary of $\widehat{\mathbb{G}}$.

$\rightsquigarrow$ Left action of $\widehat{\mathbb{G}}$ becomes amenable on B, while the right action becomes trivial on B.

$\rightsquigarrow$ Exactness for $C_{\mathrm{red}}(A_u(F))$ and solidity for $L^\infty(A_u(F))$.