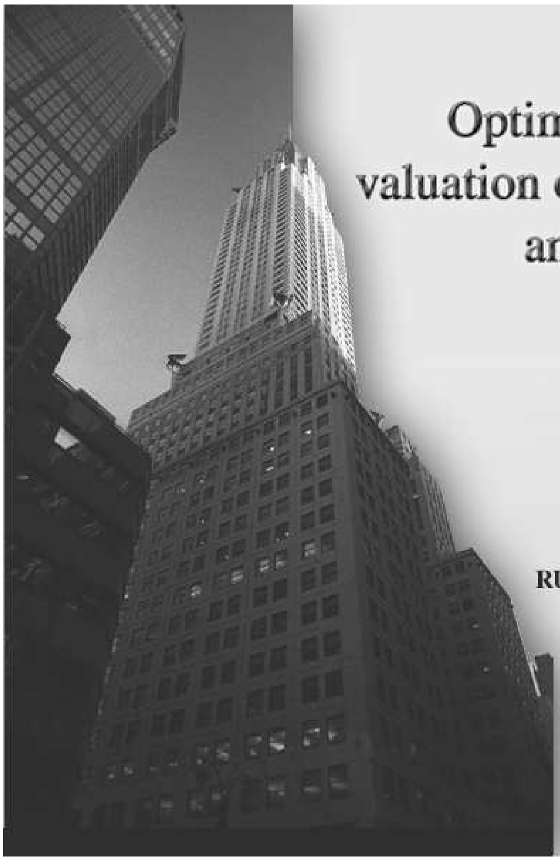

Optimization methods in valuation of financial derivatives and financial planning

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1 Dynamic Programming Model for the Valuation of the Bermudan and American Options

The paper that serves as starting point of our investigation is the one by Geske and Johnson (1984). These authors look at the Bermudan put option P_n on an asset that does not pay dividend, n is the number of equidistant times until expiration, where the option can be exercised. Then write up formulas for $n = 2, 3$, suggest a general formula, compute numerically P_1, P_2, P_3, P_4 and then apply the Richardson extrapolation to approximate the value of the American put.

The celebrated Black–Scholes–Merton (BSM) formulas for the prices of the European call and put options are:

$$c = Se^{-D(T-t)}N(d_1) - Xe^{-r(T-t)}N(d_2)$$
$$p = Xe^{-r(T-t)}N(-d_2) - Se^{-D(T-t)}N(-d_1),$$

where

$$d_1 = \frac{\ln\dfrac{S}{X} + \left(r - D + \dfrac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln\dfrac{S}{X} + \left(r - D - \dfrac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t},$$

where t = current time, T = expiration, S = current asset price, r = rate of interest, D = dividend, σ = volatility.

The geometric Brownian motion of the stock price process is of the form

$$S(t) = S(0)e^{\sigma B(t)+\mu t}, \qquad t \geq 0,$$

where $\sigma > 0$ is the already introduced constant, $\mu = r - \dfrac{\sigma^2}{2}$ and $B(t)$, $t \geq 0$ is a standard Brownian motion process.

The price dynamics is characterized by the stochastic differential equation:

$$\frac{\mathrm{d}S(t)}{S(t)} = \sigma \mathrm{d}B(t) + (r - D)\mathrm{d}t.$$

Black and Scholes (1973) and Merton (1973) derived the parabolic PDE, for the case of $D = 0$, valid for the value function $V = V(t)$ of any derivative:

$$\frac{\partial V}{\partial t} + (r - D)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV.$$

Formulas have been obtained by the solutions of the above equation, with the boundary conditions $V = [S - X]_+$, $V = [X - S]_+$, assumed to hold at time T, for the European call and put options, respectively.

Dynamic Programming Recursion for the Value of the Bermudan Put Option

We assume that the asset price process is a multiplicative Brownian motion process. We use risk neutral valuation which means we assume that $\mu = r - \sigma^2/2$. We also assume that the asset pays dividend continuously, at a constant rate D.

Let us subdivide the time interval $[t, T]$ into n parts and let the subdividing points be $t_1, \ldots, t_{n-1}$, where $t < t_1 < \cdots < t_{n-1} < T$. Introduce the notations: $t_0 = t$, $t_n = T$, $\Delta t_i = t_i - t_{i-1}$, $i = 1, \ldots, n$. If the only possible exercise times are $t_0, t_1, \ldots, t_n$, then the option is called Bermudan. Its value can be taken as an approximate value of the American option. Let p, P designate the European and American put prices, and c, C the European and American call prices, respectively. In case of n subdividing intervals the Bermudan option prices are designated by P_n and C_n, respectively.

At any time t the option (spot) payoff is defined as the value $[X - S]_+$, where S is the spot price of the asset. Assume that the Bermudan option will be exercised whenever the spot payoff becomes at least as large as the current value of the option. If, at time t_i, $i = 1, \dots, n$, the spot payoff becomes at least as large as the current value of the option, then S is called a critical price corresponding to t_i. At time t_n the value of the option is 0, hence the critical price is equal to X. Let $V_n(S), V_{n-1}(S), \dots, V_0(S)$ designate the option values corresponding to $t_n, t_{n-1}, \dots, t_0$, respectively, where in each $V_i(S)$, the value S represents the spot price of the asset. It is a variable and $V_i(S)$ is its function which provides us with the option values for all possible values of $S > 0$. For $S = 0$ the above functions are defined by continuity. In view of our assumptions, we have the following recursion formulas:

$$
\begin{aligned}
V_n(S) &= [X - S]_+ \\
V_i(S) &= \max\left([X - S]_+, e^{-r\Delta t_{i+1}} E(V_{i+1}(Se^{-D\Delta t_{i+1}} Y_{i+1}))\right) \\
&\qquad i = 0, \dots, n-1,
\end{aligned}
$$

where

$$
\begin{aligned}
\log Y_i &\sim N\left(\left(r - \frac{\sigma^2}{2}\right)\Delta t_i, \sigma^2 \Delta t_i\right) \\
&\qquad i = 1, \dots, n.
\end{aligned}
$$

The price of the Bermudan option is $P_n = V_0(S)$. Let $V(t) = V_0(t)$.

Some properties of the functions can immediately be derived.

Theorem *The following assertion holds true*

(a) $V_i(0) = X,\ i = 0, \ldots, n,$

(b) $V_i(S)$ *and* $E(V_i(Se^{-D\Delta t_{i+1}}Y_i))$ *are continuous and strictly decreasing and convex for* $S \geq 0,\ i = 0, \ldots, n-1,$

(c) $E(V_i(Se^{-D\Delta t_{i+1}}Y_i))$ *is strictly convex for* $S \geq 0,\ i = 0, \ldots, n-1.$

Theorem implies that for every $i = 1, \ldots, n-1$ there exists a unique S such that

$$X - S = e^{-r\Delta t_i} E(V_i(Se^{-D\Delta t_{i+1}}Y_i)).$$

Let q_{n-i} designate this value. Let $q_0 = X$ and call $q_0, \ldots, q_{n-1}$ critical prices.

THEOREM *Suppose that* $\Delta t_i = \Delta t = (T-t)/n$, $i = 1, \ldots, n$. *Then for any* $S \geq 0$ *we have the relation*

$$V_{i-1}(S) \geq V_i(S).$$

The inequality is strict if $S < q_{n-i+1}$. *In addition, we have the inequalities*

$$q_0 = X > q_1 > \cdots > q_{n-1}.$$

A subdivision $t_0 < t_1 < \cdots < t_n$ of the interval $[t_0, t_n]$ will be designated by τ. A subdivision is called equidistant if $t_i - t_{i-1} = (t_n - t_0)/n$, $i = 1, \ldots, n$. If τ_1 and τ_2 are two equidistant subdivisions such that each point in τ_1 appears also in τ_2, then we write $\tau_1 \prec \tau_2$.

THEOREM *If* τ_1, τ_2 *are two equidistant subdivisions and* $\tau_1 \prec \tau_2$, *then, the value of the Bermudan option corresponding to* τ_1 *is not greater than the value corresponding to* τ_2.

Formulas for the Price of the Bermudan Put Option

Let us introduce the notations

$$
\begin{aligned}
d_1(S,q,\tau) &= \frac{\log \dfrac{S}{q} + \left(r + \dfrac{\sigma^2}{2} \right) \tau}{\sigma \sqrt{\tau}} \\
d_2(S,q,\tau) &= \frac{\log \dfrac{S}{q} + \left(r - \dfrac{\sigma^2}{2} \right) \tau}{\sigma \sqrt{\tau}} = d_1(S,q,\tau) - \sigma\sqrt{\tau}.
\end{aligned}
$$

We have already defined $q_0 = X$, $q_1, \ldots, q_{n-1}$. Now we give them new definitions and later on show that the two definitions provide us with the same numbers. First, for a given, not necessarily equidistant subdivision of the interval $[t,T]$, and given r, D, σ, $q_0 > q_1 > \cdots > q_{n-1}$ we formally write up the formulas of the function sequences $U_i(S)$, $W_i(S)$:

$$
\begin{aligned}
&U_i(S)\\
&\quad = e^{-r(t_{n-i+1}-t_{n-i})}N_1(-d_2(S,q_{i-1},t_{n-i+1}-t_{n-i}))\\
&\qquad +e^{-r(t_{n-i+2}-t_{n-i})}N_2(d_2(S,q_{i-1},t_{n-i+1}-t_{n-i}),\\
&\qquad -d_2(S,q_{i-2},t_{n-i+2}-t_{n-i});R^{(2)})\\
&\qquad +e^{-r(t_{n-i+3}-t_{n-i})}N_3(d_2(S,q_{i-1},t_{n-i+1}-t_{n-i}),d_2(S,q_{i-2},t_{n-i+2}-t_{n-i}),\\
&\qquad -d_2(S,q_{i-3},t_{n-i+3}-t_{n-i});R^{(3)})+\cdots\\
&\qquad +e^{-r(t_{n-i+h}-t_{n-i})}N_h(d_2(S,q_{i-1},t_{n-i+1}-t_{n-i}),\ldots,d_2(S,q_{i-h+1},t_{n-i+h-1}-t_{n-i}),\\
&\qquad -d_2(S,q_{i-h},t_{n-i+h}-t_{n-h});R^{(h)})+\cdots\\
&\qquad +e^{-r(t_{n}-t_{n-i})}N_i(d_2(S,q_{i-1},t_{n-i+1}-t_{n-i}),\ldots,d_2(S,q_1,t_{n-1}-t_{n-i}),\\
&\qquad -d_2(S,q_0,t_n-t_{n-i});R^{(i)}),\\
&\qquad\quad i=1,\ldots,n,
\end{aligned}
$$

$$
\begin{aligned}
&W_i(S)\\
&\quad = e^{-D(t_{n-i+1}-t_{n-i})}N_1(-d_1(S,q_{i-1},t_{n-i+1}-t_{n-i}))\\
&\qquad +e^{-D(t_{n-i+2}-t_{n-i})}N_2(d_1(S,q_{i-1},t_{n-i+1}-t_{n-i}),-d_1(S,q_{i-2},t_{n-i+2}-t_{n-i});R^{(2)})\\
&\qquad +e^{-D(t_{n-i+3}-t_{n-i})}N_3(d_1(S,q_{i-1},t_{n-i+1}-t_{n-i}),d_1(S,q_{i-2},t_{n-i+2}-t_{n-i}),\\
&\qquad -d_1(S,q_{i-3},t_{n-i+3}-t_{n-i});R^{(3)})+\cdots\\
&\qquad +e^{-D(t_{n-i+h}-t_{n-i})}N_h(d_1(S,q_{i-1},t_{n-i+1}-t_{n-i}),\ldots,d_1(S,q_{i-h+1},t_{n-i+h+1}-t_{n-i}),\\
&\qquad -d_1(S,q_{i-h},t_{n-i+h}-t_{n-i});R^{(h)})+\cdots\\
&\qquad +e^{-D(t_n-t_{n-i})}N_i(d_1(S,q_{i-1},t_{n-i+1}-t_{n-i}),\ldots,d_1(S,q_1,t_{n-1}-t_{n-i}),\\
&\qquad -d_1(S,q_0,t_n-t_{n-i});R^{(i)}),\\
&\qquad\quad i=1,\ldots,n.
\end{aligned}
$$

The covariance matrix $R^{(h)}$ has entries

$$\rho_{jk}^{(h)} = \sqrt{\frac{t_{n-i+j} - t_{n-i}}{t_{n-i+k} - t_{n-i}}} \quad \text{if} \quad 1 \le j \le k \le h-1$$

$$\rho_{jh}^{(h)} = -\sqrt{\frac{t_{n-i+j} - t_{n-i}}{t_{n-i+h} - t_{n-i}}} \quad \text{if} \quad 1 \le j \le h.$$

Now, the definitions of $q_0, \ldots, q_{n-1}$ and $U_1(S), \ldots, U_n(S)$, $W_1(S), \ldots, W_n(S)$ are as follows. First we define $q_0 = X$ and

$$\begin{aligned} U_1(S) &= e^{-r(t_n - t_{n-1})} N_1(-d_2(S, q_0, t_n - t_{n-1})) \\ W_1(S) &= e^{-D(t_n - t_{n-1})} N_1(-d_1(S, q_0, t_n - t_{n-1})). \end{aligned}$$

Suppose that $U_1(S), \ldots, U_i(S), W_1(S), \ldots, W_i(S), q_1, \ldots, q_{i-1}$ have already been defined; write up the equation

$$X - S = XU_i(S) - SW_i(S)$$

and designate its unique solution (with respect to S) by q_i. Then we define $U_{i+1}(S)$ and $W_{i+1}(S)$, respectively. The existence and unity of q_i is a byproduct of the proof of the following

Theorem *We have the equations*

$$e^{-r\Delta t_{n-i+1}} E(V_{n-i+1}(Se^{-D\Delta t_{n-i+1}} Y_{n-i+1})) = XU_i(S) - SW_i(S), \qquad i = 1, \ldots, n$$

and the price of the Bermudan put option is given by

$$P_n = \max(X - S, XU_n(S) - SW_n(S)).$$

Probabilistic Interpretation of $U_n(S)$, $W_n(S)$

$$
\begin{aligned}
&U_n(S)\\
&\quad = e^{-r(t_1-t_0)} P\left(Se^{\sigma(B(t_1)-B(t_0))+\left(r-D-\frac{\sigma^2}{2}\right)(t_1-t_0)} > q_{n-1}\right)\\
&\qquad + \sum_{h=2}^{n} e^{-r(t_h-t_0)} P\left(Se^{\sigma(B(t_l)-B(t_0))+\left(r-D-\frac{\sigma^2}{2}\right)(t_l-t_0)} < q_{n-l},\right.\\
&\qquad\quad \left. l = 1,\ldots,h-1,\ Se^{\sigma(B(t_h)-B(t_0))+\left(r-D-\frac{\sigma^2}{2}\right)(t_h-t_0)} > q_{n-h}\right)
\end{aligned}
$$

$= P$ (present value of \$ 1.00 discounted at the constant rate r, paid upon the time the geometric Brownian motion

$$X(t) = Se^{\sigma(B(t)-B(t_0))+\left(r-D-\frac{\sigma^2}{2}\right)(t-t_0)}$$

first reaches or exceeds the discrete critical function q_{n-h}, $h = 1,\ldots,n$, at some point t_h, $h = 1,\ldots,n$),

where $B(t)$, $t \geq 0$ is a standard Brownian motion process.

Similarly, we have that

$$
\begin{aligned}
&W_n(S) \\
&= e^{-D(t_1-t_0)} P\left(S e^{\sigma(B(t_1)-B(t_0))+\left(r-D+\frac{\sigma^2}{2}\right)(t_1-t_0)} > q_{n-1} \right) \\
&\quad + \sum_{h=2}^{n} e^{-D(t_h-t_0)} P\Bigg(S e^{\sigma(B(t_l)-B(t_0))+\left(r-D+\frac{\sigma^2}{2}\right)(t_l-t_0)} < q_{n-l}, \\
&\qquad l = 1, \ldots, h-1,\ S e^{\sigma(B(t_h)-B(t_0))+\left(r-D+\frac{\sigma^2}{2}\right)(t_h-t_0)} > q_{n-h} \Bigg)
\end{aligned}
$$

$= P$ (present value of \$ 1.00, discounted at the constant rate D, paid upon the time the geometric Brownian motion

$$X(t) = S e^{\sigma(B(t)-B(t_0))+\left(r-D+\frac{\sigma^2}{2}\right)(t-t_0)}$$

first reaches or exceeds the discrete critical function q_{n-h}, $h = 1, \ldots, n$, at some point t_h, $h = 1, \ldots, n$),

where $B(t)$, $t \geq 0$ is the same as before.

Brief Description of the Applied Integration and Simulation Techniques. Numerical Results.

In order to obtain the numerical values of the Bermudan put and call options, we need numerical integration technique that provides us with the values of the normal c.d.f. in higher dimensions. With the dimension n we go up to the point where the calculations of the Bermudan option values P_n, C_n are still reliable.

Based on extensive experience with the integration of the multivariate normal p.d.f., we have chosen three methods to apply here. One is due to Genz (1992), Genz and Kwong (2000), the other one is due to Szántai (2000), Gassmann et al. (2002) and the third one is due to Ambartzumian et al. (1998).

The second one is a collection of methods and works in such a way that we take lower and upper bounds for the probability of a union of events and then use simulation for the difference. If A_i is the event $\{X_i > x_i\}$, $i = 1, \ldots, n$ and we want to estimate $P(X_i \leq x_i,\ i = 1, \ldots, n)$, then first we estimate the probability of the union $\bigcup_{i=1}^{n} A_i$ and then estimate

$$P(X_i \leq x_i,\ i = 1, \ldots, n) = 1 - P(A_1 \cup \ldots \cup A_n).$$

For the probability of the union prominent bounds, based on binomial moments (the $S_1, S_2, \ldots$ that appear in the inclusion-exclusion formula), and graph structures are available. The binomial moment bounds that are used in this context are those, presented in Boros and Prékopa (1989), for the cases, where only S_1, S_2, S_3, S_4 are used and in Prékopa (1988), where $S_1, \ldots, S_m$, $m > 4$ are used. The graph structure bounds utilized here are those of Bukszár and Prékopa (2001), Bukszár and Szántai (2002) and Bukszár (2002).

After realizing that the calculated multivariate normal probabilities are frequently very small, the sequential conditioned importance sampling (SCIS) procedure by Ambartzumian et al. (1998) was also tested.

All the three simulation and numerical integration procedures produced essentially the same results. As regards accuracy, the individual integrals have been computed with 5 digit precision on a confidence level of 99%. We claim 3 digit accuracy in the values of the Bermudan options.

In Tables 2, 3, 4, 5 the "True" values are those obtained by the authors of the mentioned papers, by the use of the binomial tree method using large numbers of steps.

In the calculation of $P_1, \ldots, P_n$ and $C_1, \ldots, C_n$ it is unavoidable that some numerical errors occur. To overcome this difficulty we use of the following discrete exponential function

$$f(n) = k - \sum_{i=1}^{m} \alpha_i e^{-\beta_i n}$$

to smooth the above sequences. The constants $k, \alpha_i, \beta_i,\ i = 1, \ldots, m$ are determined by the least squares principle

$$\min_{\substack{k,\alpha_1,\ldots,\alpha_m,\\ \beta_1,\ldots,\beta_m}} \sum_{i=1}^{n} (f(i) - P_i)^2.$$

In our examples it was enough to choose $m = 1$ or $m = 2$, to obtain very good fit.

The use of the type of function $f(n)$ is supported by the empirical fact that all jth order differences of the sequences $\{P_i\}$, $\{C_i\}$ are of the same sign and the signs are alternating as j varies (see Table 8). The same property is enjoyed by the function $f(n)$.

In the tables that follow we compare our numerical results to those in Geske and Johnson (1984), Broadie and Detemple (1996), Ju (1998), Arciniega and Allen (2004) and create new numerical examples. Our figures, obtained by the exponential smoothing procedure, and labeled as Exp. smoothed in the tables, are smaller than those, obtained by others, for the same input data and significantly smaller than the "True" values. The latters are obtained by the binomial tree calculation, using $10,000 - 15,000$ steps.

The fact that the binomial tree method overestimates the option price, at least when n is not sufficiently large, can be explained as follows. The dynamic programming recursions describe the binomial tree recursions as well. In that case the random variable Y_{i+1} takes the values d and u, with probabilities p and $q = 1 - p$, respectively, at each node. If d and u are chosen in such a way that $P(Sd \leq Se^{-D\Delta t_i} Y_i \leq Su)$ is very close to 1, then by the Edmundson–Madansky inequality (e.g. Prékopa 1995) we have that

$$E(V_i(Se^{-D\Delta t_i} Y_i)) < pV_i(Sde^{-D\Delta t_i}) + qV_i(Sue^{-D\Delta t_i})$$

and strict inequality holds if V_i is nonlinear in the interval $[Sd, Su]$. The Berry–Esséen theorem tells us, that the difference between the standardized binomial c.d.f. and the standard normal c.d.f. is of order of magnitude $1/\sqrt{n}$, hence $n = 10,000$ gives only two digit accuracy.

Table 1: Results for some problems in Geske and Johnson (1984).

	Problem No.									
	15	16	17	18	20	21	23	24	25	26
S	40	40	40	40	40	40	40	40	40	40
r	0.0490	0.0490	0.0490	0.0490	0.0490	0.0490	0.0490	0.0490	0.0490	0.0490
X	35	40	40	40	45	45	35	35	40	40
σ	0.2	0.2	0.2	0.2	0.2	0.2	0.3	0.3	0.3	0.3
T	0.5833	0.0833	0.3333	0.5833	0.3333	0.5833	0.3333	0.5833	0.0833	0.3333
G–J	0.4321	0.8528	1.5807	1.9905	5.0951	5.2719	0.6969	1.2194	1.3100	2.4817
Exp. smoothed	0.4306	0.8511	1.5750	1.9841	5.0822	5.2496	0.6945	1.2157	1.3090	2.4780
$P(1\ldots4)$	0.4318	0.8523	1.5807	1.9904	5.0953	5.2713	0.6967	1.2187	1.3102	2.4834
$P(1\ldots5)$	0.4328	0.8519	1.5817	1.9920	5.0882	5.2667	0.6987	1.2202	1.3095	2.4833
$P(1\ldots6)$	0.4334	0.8513	1.5791	1.9893	5.0954	5.2611	0.6932	1.2190	1.3146	2.4780
$P(1\ldots7)$	0.4253	0.8659	1.5640	1.9944	5.0759	5.2800	0.7103	1.2101	1.2814	2.4998
$P(1\ldots8)$	0.4903	0.7879	1.6048	1.9728	5.0928	5.2348	0.6300	1.2822	1.3544	2.4462
$P(124)$	0.4324	0.8521	1.5770	1.9846	5.1027	5.2848	0.6970	1.2196	1.3100	2.4809
$P(1248)$	0.4325	0.8522	1.5801	1.9909	5.0915	5.2679	0.6971	1.2194	1.3096	2.4830
	Problem No.									
	27	28	29	30	33	34	36	37	38	39
S	40	40	40	40	40	40	40	40	40	40
r	0.0490	0.0490	0.0490	0.0490	0.0490	0.0490	0.0490	0.0490	0.0490	0.0490
X	40	45	45	45	35	40	40	45	45	45
σ	0.3	0.3	0.3	0.3	0.4	0.4	0.4	0.4	0.4	0.4
T	0.5833	0.0833	0.3333	0.5833	0.5833	0.0833	0.5833	0.0833	0.3333	0.5833
G–J	3.1733	5.0599	5.7012	6.2365	2.1568	1.7679	4.3556	5.2855	6.5093	7.3831
Exp. smoothed	3.1622	5.0576	5.7001	6.2314	2.1508	1.7670	4.3427	5.2836	6.5003	7.3705
$P(1\ldots4)$	3.1715	5.0599	5.7011	6.2357	2.1531	1.7680	4.3555	5.2860	6.5095	7.3812
$P(1\ldots5)$	3.1696	5.0589	5.7045	6.2455	2.1548	1.7683	4.3518	5.2856	6.5108	7.3853
$P(1\ldots6)$	3.1700	5.0594	5.7039	6.2372	2.1553	1.7665	4.3549	5.2934	6.5143	7.3846
$P(1\ldots7)$	3.1655	5.0516	5.7282	6.2673	2.1514	1.7760	4.3273	5.2700	6.4985	7.3931
$P(1\ldots8)$	3.1974	5.1220	5.6217	6.2066	2.1598	1.7548	4.4861	5.3345	6.5257	7.3464
$P(124)$	3.1649	5.0622	5.7016	6.2365	2.1549	1.7683	4.3494	5.2849	6.5034	7.3724
$P(1248)$	3.1704	5.0595	5.7037	6.2412	2.1541	1.7682	4.3535	5.2867	6.5104	7.3838

Table 2: Results for the problems of Table 2 in Broadie and Detemple (1996).

	Problem No. 1	2	3	4	5	6	7	8	9	10
S	80	90	100	110	120	80	90	100	110	120
r	0.0300	0.0300	0.0300	0.0300	0.0300	0.0300	0.0300	0.0300	0.0300	0.0300
X	100	100	100	100	100	100	100	100	100	100
σ	0.2	0.2	0.2	0.2	0.2	0.4	0.4	0.4	0.4	0.4
d	0.0700	0.0700	0.0700	0.0700	0.0700	0.0700	0.0700	0.0700	0.0700	0.0700
T	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000
"True" values	2.5800	5.1670	9.0660	14.4430	21.4140	11.3260	15.7220	20.7930	26.4950	32.7810
Exp. smoothed	2.5531	5.1214	8.9908	14.3246	21.2415	11.2649	15.6423	20.6896	26.3683	32.4997
$C(1\ldots4)$	2.5875	5.1551	9.0134	14.4454	21.5784	11.3437	15.7135	20.7406	26.4021	32.6765
$C(1\ldots5)$	2.5821	5.1788	9.0483	14.4072	21.4820	11.3458	15.7534	20.8132	26.4779	32.7203
$C(1\ldots6)$	2.5790	5.1736	9.0771	14.4055	21.4349	11.3143	15.7116	20.8199	26.5234	32.7799
$C(1\ldots7)$	2.5533	5.1559	9.0371	14.4130	21.4649	11.2800	15.7957	20.8274	26.4947	32.8260
$C(1\ldots8)$	2.6649	5.2134	9.2526	14.4626	21.2436	11.5417	15.4307	20.6523	26.6471	32.7397
$C(124)$	2.5618	5.1220	9.0462	14.5688	21.7059	11.2739	15.6401	20.7010	26.4295	32.7913
$C(1248)$	2.5827	5.1680	9.0412	14.4219	21.5073	11.3375	15.7309	20.7867	26.4596	32.7199
	Problem No. 11	12	13	14	15	16	17	18	19	20
S	80	90	100	110	120	80	90	100	110	120
r	0.0000	0.0000	0.0000	0.0000	0.0000	0.0700	0.0700	0.0700	0.0700	0.0700
X	100	100	100	100	100	100	100	100	100	100
σ	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3
d	0.0700	0.0700	0.0700	0.0700	0.0700	0.0300	0.0300	0.0300	0.0300	0.0300
T	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000
"True" values	5.5180	8.8420	13.1420	18.4530	24.7910	12.1450	17.3690	23.3480	29.9640	37.1040
Exp. smoothed	5.4623	8.7594	13.0235	18.2863	24.5751	12.1443	17.3661	23.3450	29.9583	37.0955
$C(1\ldots4)$	5.5138	8.7942	13.0652	18.4180	24.8934	12.1444	17.3695	23.3442	29.9618	37.1070
$C(1\ldots5)$	5.5303	8.8444	13.1009	18.3905	24.7790	12.1460	17.3575	23.3554	29.9636	37.1022
$C(1\ldots6)$	5.5320	8.8872	13.1524	18.4041	24.7461	12.1425	17.3913	23.3236	29.9675	37.0986
$C(1\ldots7)$	5.4790	8.7499	13.1774	18.4209	24.7338	12.1602	17.3406	23.4384	29.9152	37.0970
$C(1\ldots8)$	5.3916	9.1701	13.0275	18.5811	24.7014	12.0994	17.3378	23.0717	30.1828	37.2436
$C(124)$	5.4629	8.7740	13.1250	18.5719	25.0974	12.1484	17.3752	23.3575	29.9747	37.1141
$C(1248)$	5.5208	8.8306	13.0999	18.4062	24.8132	12.1452	17.3669	23.3478	29.9627	37.1041

Table 3: Results for the problems of Table 1 in Ju (1998), the same as those of Table 1 in Broadie and Detemple (1996).

	Problem No.									
	1	2	3	4	5	6	7	8	9	10
S	80	90	100	110	120	80	90	100	110	120
r	0.0300	0.0300	0.0300	0.0300	0.0300	0.0300	0.0300	0.0300	0.0300	0.0300
X	100	100	100	100	100	100	100	100	100	100
σ	0.2	0.2	0.2	0.2	0.2	0.4	0.4	0.4	0.4	0.4
d	0.0700	0.0700	0.0700	0.0700	0.0700	0.0700	0.0700	0.0700	0.0700	0.0700
T	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
Ju's results	0.2196	1.3872	4.7837	11.0993	20.0005	2.6899	5.7237	10.2404	16.1831	23.3622
"True" values	0.2194	1.3864	4.7825	11.0978	20.0004	2.6889	5.7223	10.2385	16.1812	23.3598
Exp. smoothed	0.2185	1.3809	4.7702	11.0740	20.0000	2.6835	5.7104	10.2211	16.1601	23.3328
$C(1\dots4)$	0.2212	1.3852	4.7853	11.0891	20.0078	2.6863	5.7217	10.2446	16.1835	23.3411
$C(1\dots5)$	0.2158	1.3838	4.7799	11.0848	20.0041	2.6888	5.7158	10.2399	16.1876	23.3616
$C(1\dots6)$	0.2191	1.3955	4.8036	11.1016	20.0032	2.6929	5.7357	10.2290	16.1838	23.3585
$C(1\dots7)$	0.2279	1.3704	4.7349	11.1078	20.0029	2.6491	5.6901	10.2430	16.1579	23.3749
$C(1\dots8)$	0.1810	1.3460	4.8351	10.9405	20.0771	2.8248	5.7940	10.2627	16.3131	23.3733
$C(124)$	0.2186	1.3860	4.7725	11.1086	20.0125	2.6879	5.7233	10.2336	16.1608	23.3391
$C(1248)$	0.2188	1.3854	4.7842	11.0895	20.0058	2.6875	5.7208	10.2401	16.1848	23.3542
	Problem No.									
	11	12	13	14	15	16	17	18	19	20
S	80	90	100	110	120	80	90	100	110	120
r	0.0000	0.0000	0.0000	0.0000	0.0000	0.0700	0.0700	0.0700	0.0700	0.0700
X	100	100	100	100	100	100	100	100	100	100
σ	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3
d	0.0700	0.0700	0.0700	0.0700	0.0700	0.0300	0.0300	0.0300	0.0300	0.0300
T	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
Ju's results	1.0381	3.1247	7.0371	12.9574	20.7194	1.6644	4.4947	9.2506	15.7975	23.7062
"True" values	1.0373	3.1233	7.0354	12.9552	20.7173	1.6644	4.4947	9.2504	15.7977	23.7061
Exp. smoothed	1.0329	3.1104	7.0106	12.9219	20.6657	1.6644	4.4948	9.2507	15.7973	23.7063
$C(1\dots4)$	1.0337	3.1239	7.0374	12.9346	20.7425	1.6644	4.4947	9.2536	15.8228	23.7136
$C(1\dots5)$	1.0415	3.1208	7.0385	12.9457	20.7193	1.6644	4.4955	9.2532	15.6970	23.6965
$C(1\dots6)$	1.0247	3.1231	7.0426	12.9660	20.7078	1.6642	4.4945	9.2521	15.7387	23.7199
$C(1\dots7)$	1.0535	3.1078	7.0169	12.9105	20.7385	1.6658	4.4942	9.0140	17.0787	23.6793
$C(1\dots8)$	1.1109	3.2148	7.1118	13.1302	20.5338	1.6584	4.4989	10.4784	11.3676	23.7361
$C(124)$	1.0355	3.1210	7.0174	12.9431	20.7878	1.6644	4.4947	9.2514	15.8038	23.7059
$C(1248)$	1.0367	3.1224	7.0382	12.9447	20.7256	1.6644	4.4950	9.2499	15.7906	23.7064

Table 4: Results for the problems of Table 2 in Ju (1998), the same as those of Table 5 in Barone-Adesi and Whaley (1987).

	Problem No.									
	1	2	3	4	5	6	7	8	9	10
S	80	90	100	110	120	80	90	100	110	120
r	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800
X	100	100	100	100	100	100	100	100	100	100
σ	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
d	0.1200	0.1200	0.1200	0.1200	0.1200	0.0800	0.0800	0.0800	0.0800	0.0800
T	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000
Ju's results	25.6570	20.0817	15.4970	11.8022	8.8850	22.2084	16.2106	11.7066	8.3695	5.9323
"True" values	25.6577	20.0832	15.4981	11.8032	8.8856	22.2050	16.2071	11.7037	8.3671	5.9299
Exp. smoothed	25.6346	20.0675	15.4857	11.7948	8.8805	22.1287	16.1546	11.6663	8.3411	5.9118
$P(1\ldots4)$	25.6537	20.1084	15.5116	11.8032	8.8810	22.2083	16.1654	11.7047	8.3892	5.9437
$P(1\ldots5)$	25.6767	20.0888	15.4934	11.7967	8.8866	22.1757	16.1969	11.7257	8.3708	5.9232
$P(1\ldots6)$	25.6685	20.0706	15.4870	11.8118	8.8723	22.1861	16.2360	11.7101	8.3618	5.9293
$P(1\ldots7)$	25.6564	20.1139	15.5541	11.8160	8.9634	22.1821	16.1966	11.6621	8.3920	5.9394
$P(1\ldots8)$	25.6340	20.0484	15.5113	11.7865	8.6922	22.2679	16.2153	11.9086	8.2090	5.9764
$P(124)$	25.6030	20.0653	15.5094	11.8218	8.9010	22.3167	16.1800	11.6616	8.3519	5.9338
$P(1248)$	25.6653	20.0940	15.5023	11.8027	8.8845	22.1907	16.1921	11.7132	8.3760	5.9332
	Problem No.									
	11	12	13	14	15	16	17	18	19	20
S	80	90	100	110	120	80	90	100	110	120
r	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800
X	100	100	100	100	100	100	100	100	100	100
σ	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
d	0.0400	0.0400	0.0400	0.0400	0.0400	0.0000	0.0000	0.0000	0.0000	0.0000
T	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000
Ju's results	20.3511	13.5000	8.9474	5.9146	3.8997	20.0000	11.6991	6.9346	4.1571	2.5119
"True" values	20.3500	13.4968	8.9438	5.9119	3.8975	20.0000	11.6974	6.9320	4.1550	2.5102
Exp. smoothed	20.0338	13.3803	8.8655	5.8582	3.8591	20.0000	11.5934	6.8157	4.0821	2.4644
$P(1\ldots4)$	20.5123	13.5246	8.8909	5.8901	3.9042	19.7315	11.8840	6.9263	4.1039	2.4916
$P(1\ldots5)$	20.4568	13.4702	8.9181	5.9274	3.9086	19.8255	11.7968	6.8916	4.1333	2.5184
$P(1\ldots6)$	20.4286	13.4586	8.9556	5.9188	3.8828	19.8851	11.7335	6.8717	4.1831	2.5145
$P(1\ldots7)$	20.3666	13.4804	8.9220	5.8936	3.9320	19.8651	11.6892	6.9800	4.0767	2.5156
$P(1\ldots8)$	20.5382	13.4704	9.0945	6.0700	3.7601	20.0703	11.7581	6.8590	4.5005	2.5993
$P(124)$	20.5755	13.6494	8.9315	5.8599	3.8624	19.5135	11.9430	7.0237	4.1248	2.4624
$P(1248)$	20.4686	13.4891	8.9160	5.9112	3.9025	19.8045	11.8126	6.9054	4.1308	2.5081

Table 5: Results for the problems of Table 2 in Arciniega and Allen (2004).

	Problem No.		
	1	2	3
S	90	100	110
r	0.0300	0.0300	0.0300
X	100	100	100
σ	0.4	0.4	0.4
d	0.0700	0.0700	0.0700
T	0.5000	0.5000	0.5000
C_1	5.6221	10.0211	15.7676
C_2	5.6491	10.1095	15.9815
C_3	5.6686	10.1461	16.0434
C_4	5.6802	10.1671	16.0755
C_5	5.6875	10.1805	16.0954
C_6	5.6929	10.1896	16.1095
C_7	5.6967	10.1964	16.1194
C_8	5.6996	10.2013	16.1271
C_9	5.7020	10.2055	16.1329
C_{10}	5.7038	10.2087	16.1377
C_{11}	5.7055	10.2112	16.1415
C_{12}	5.7065	10.2133	16.1448
C_{13}	5.7080	10.2153	16.1477
C_{14}	5.7088	10.2170	16.1500
C_{15}	5.7095	10.2183	16.1522
C_{16}	5.7104	10.2197	16.1539
C_{17}	5.7111	10.2208	16.1552
C_{18}	5.7114	10.2216	16.1568
C_{19}	5.7120	10.2226	16.1581
C_{20}	5.7126	10.2234	16.1596
"True" value	5.7220	10.2390	16.1810
Exp. smoothed	5.7110	10.2205	16.1597
$C(1\ldots4)$	5.7223	10.2447	16.1831
$C(1\ldots5)$	5.7157	10.2385	16.1843
$C(1\ldots6)$	5.7425	10.2275	16.2085
$C(1\ldots7)$	5.6674	10.2893	16.0858
$C(1\ldots8)$	5.8389	10.0333	16.4124
$C(124)$	5.7231	10.2337	16.1608
$C(1248)$	5.7213	10.2400	16.1847

Table 6: Results for new problems (Call options, $r > d$)

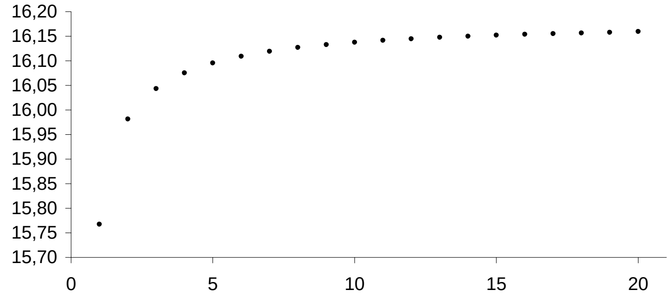
	Problem No.								
	1	2	3	4	5	6	7	8	9
S	70	70	70	70	70	70	70	70	70
r	0.0600	0.0600	0.0600	0.0600	0.0600	0.0600	0.0600	0.0600	0.0600
X	60	60	60	70	70	70	80	80	80
σ	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3
d	0.0400	0.0400	0.0400	0.0400	0.0400	0.0400	0.0400	0.0400	0.0400
T	0.2500	1.0000	3.0000	0.2500	1.0000	3.0000	0.2500	1.0000	3.0000
C_1	10.9067	13.9497	18.4211	4.3081	8.6232	14.2442	1.2021	5.0546	10.9725
C_2	10.9068	13.9672	18.6291	4.3081	8.6265	14.3429	1.2021	5.0552	11.0190
C_3	10.9069	13.9734	18.6825	4.3081	8.6293	14.3796	1.2021	5.0562	11.0450
C_4	10.9070	13.9765	18.7097	4.3081	8.6305	14.3992	1.2018	5.0567	11.0586
C_5	10.9072	13.9783	18.7261	4.3081	8.6312	14.4107	1.2021	5.0568	11.0665
C_6	10.9070	13.9795	18.7369	4.3082	8.6317	14.4181	1.2020	5.0571	11.0718
C_7	10.9070	13.9804	18.7444	4.3081	8.6321	14.4235	1.2020	5.0571	11.0752
C_8	10.9072	13.9809	18.7501	4.3081	8.6323	14.4273	1.2021	5.0573	11.0779
C_9	10.9059	13.9822	18.7530	4.3076	8.6323	14.4299	1.2019	5.0579	11.0799
C_{10}	10.9068	13.9810	18.7568	4.3079	8.6331	14.4325	1.2019	5.0572	11.0818
C_{11}	10.9073	13.9833	18.7605	4.3086	8.6335	14.4347	1.2023	5.0576	11.0831
C_{12}	10.9077	13.9822	18.7630	4.3072	8.6328	14.4352	1.2018	5.0581	11.0840
C_{13}	10.9069	13.9822	18.7646	4.3083	8.6326	14.4364	1.2018	5.0577	11.0846
C_{14}	10.9068	13.9830	18.7663	4.3082	8.6329	14.4376	1.2018	5.0573	11.0862
C_{15}	10.9073	13.9821	18.7681	4.3083	8.6333	14.4388	1.2019	5.0584	11.0862
C_{16}	10.9070	13.9827	18.7699	4.3082	8.6323	14.4396	1.2019	5.0584	11.0872
C_{17}	10.9069	13.9820	18.7694	4.3081	8.6334	14.4408	1.2018	5.0581	11.0874
C_{18}	10.9073	13.9843	18.7707	4.3083	8.6319	14.4419	1.2019	5.0573	11.0872
C_{19}	10.9080	13.9835	18.7719	4.3081	8.6334	14.4409	1.2025	5.0572	11.0870
C_{20}	10.9078	13.9836	18.7718	4.3082	8.6340	14.4432	1.2028	5.0576	11.0876
Exp. smoothed	10.9072	13.9829	18.7720	4.3081	8.6330	14.4426	1.2020	5.0579	11.0888
$C(1\ldots4)$	10.9067	13.9856	18.8017	4.3080	8.6327	14.4645	1.1993	5.0576	11.0977
$C(1\ldots5)$	10.9116	13.9840	18.7904	4.3084	8.6324	14.4451	1.2145	5.0549	11.0920
$C(1\ldots6)$	10.8832	13.9849	18.7809	4.3105	8.6436	14.4521	1.1693	5.0720	11.1043
$C(1\ldots7)$	10.9759	14.0121	18.7877	4.2851	8.6101	14.4836	1.2552	5.0060	11.0342
$C(1\ldots8)$	10.8495	13.8221	18.8055	4.3978	8.6706	14.3329	1.1991	5.2209	11.3472
$C(124)$	10.9072	13.9862	18.7745	4.3082	8.6362	14.4601	1.2014	5.0590	11.1092
$C(1248)$	10.9076	13.9850	18.7929	4.3081	8.6336	14.4547	1.2028	5.0577	11.0952

Table 7: Results for new problems (Put options, $r < d$)

	Problem No. 1	2	3	4	5	6	7	8	9
S	70	70	70	70	70	70	70	70	70
r	0.0200	0.0200	0.0200	0.0200	0.0200	0.0200	0.0200	0.0200	0.0200
X	60	60	60	70	70	70	80	80	80
σ	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3
d	0.0400	0.0400	0.0400	0.0400	0.0400	0.0400	0.0400	0.0400	0.0400
T	0.2500	1.0000	3.0000	0.2500	1.0000	3.0000	0.2500	1.0000	3.0000
P_1	0.8111	4.0264	9.5599	4.3297	8.7974	15.1250	11.3671	15.3670	21.6543
P_2	0.8111	4.0264	9.5636	4.3297	8.7974	15.1379	11.3671	15.3673	21.6878
P_3	0.8111	4.0265	9.5678	4.3297	8.7976	15.1466	11.3671	15.3677	21.7023
P_4	0.8111	4.0265	9.5698	4.3297	8.7976	15.1507	11.3672	15.3680	21.7096
P_5	0.8111	4.0265	9.5708	4.3297	8.7977	15.1530	11.3681	15.3681	21.7137
P_6	0.8111	4.0265	9.5717	4.3297	8.7976	15.1545	11.3677	15.3682	21.7164
P_7	0.8111	4.0265	9.5722	4.3297	8.7977	15.1556	11.3658	15.3683	21.7183
P_8	0.8111	4.0265	9.5725	4.3297	8.7977	15.1564	11.3652	15.3683	21.7197
P_9	0.8111	4.0265	9.5722	4.3297	8.7976	15.1568	11.3679	15.3683	21.7217
P_{10}	0.8111	4.0260	9.5727	4.3297	8.7975	15.1574	11.3667	15.3681	21.7221
P_{11}	0.8111	4.0260	9.5729	4.3297	8.7979	15.1583	11.3702	15.3681	21.7226
P_{12}	0.8111	4.0266	9.5733	4.3298	8.7979	15.1573	11.3645	15.3682	21.7228
P_{13}	0.8111	4.0264	9.5733	4.3298	8.7976	15.1583	11.3641	15.3684	21.7230
P_{14}	0.8111	4.0261	9.5741	4.3298	8.7970	15.1589	11.3709	15.3678	21.7239
P_{15}	0.8111	4.0261	9.5736	4.3298	8.7972	15.1591	11.3672	15.3686	21.7247
P_{16}	0.8111	4.0262	9.5735	4.3298	8.7979	15.1586	11.3677	15.3686	21.7247
P_{17}	0.8111	4.0268	9.5738	4.3298	8.7965	15.1601	11.3627	15.3686	21.7238
P_{18}	0.8111	4.0267	9.5749	4.3298	8.7975	15.1597	11.3671	15.3675	21.7259
P_{19}	0.8111	4.0265	9.5750	4.3298	8.7979	15.1600	11.3668	15.3685	21.7240
P_{20}	0.8111	4.0261	9.5744	4.3298	8.7978	15.1594	11.3668	15.3677	21.7241
Exp. smoothed	0.8111	4.0264	9.5739	4.3298	8.7976	15.1587	11.3669	15.3682	21.7233
$P(1\ldots4)$	0.8111	4.0266	9.5731	4.3297	8.7967	15.1598	11.3684	15.3690	21.7298
$P(1\ldots5)$	0.8111	4.0255	9.5733	4.3297	8.8001	15.1586	11.3872	15.3682	21.7265
$P(1\ldots6)$	0.8111	4.0286	9.5946	4.3307	8.7900	15.1698	11.2913	15.3668	21.7301
$P(1\ldots7)$	0.8111	4.0249	9.5034	4.3275	8.8230	15.1609	11.2145	15.3825	21.7218
$P(1\ldots8)$	0.8111	4.0284	9.7474	4.3340	8.7402	15.1238	12.2916	15.2985	21.7822
$P(124)$	0.8111	4.0266	9.5787	4.3297	8.7979	15.1678	11.3674	15.3690	21.7347
$P(1248)$	0.8111	4.0264	9.5747	4.3298	8.7978	15.1609	11.3611	15.3685	21.7285

Table 8: Option values and their differences for Problem 3 of Table 5

C_1	15.7676									
C_2	15.9815	0.2139								
C_3	16.0434	0.0620	-0.1519							
C_4	16.0755	0.0320	-0.0299	0.1220						
C_5	16.0956	0.0202	-0.0119	0.0180	-0.1040					
C_6	16.1094	0.0138	-0.0064	0.0055	-0.0125	0.0915				
C_7	16.1195	0.0101	-0.0037	0.0027	-0.0028	0.0097	-0.0818			
C_8	16.1271	0.0076	-0.0025	0.0013	-0.0014	0.0014	-0.0083	0.0735		
C_9	16.1329	0.0058	-0.0018	0.0007	-0.0006	0.0008	-0.0006	0.0076	-0.0659	
C_{10}	16.1377	0.0048	-0.0011	0.0007	0.0000	0.0006	-0.0002	0.0004	-0.0072	0.0587
C_{11}	16.1415	0.0038	-0.0010	0.0001	-0.0006	-0.0006	-0.0012	-0.0010	-0.0014	0.0058
C_{12}	16.1448	0.0034	-0.0004	0.0005	0.0004	0.0010	0.0016	0.0028	0.0038	0.0052
C_{13}	16.1477	0.0029	-0.0004	0.0000	-0.0005	-0.0009	-0.0019	-0.0035	-0.0063	-0.0100
C_{14}	16.1500	0.0022	-0.0007	-0.0002	-0.0002	0.0003	0.0012	0.0031	0.0066	0.0128
C_{15}	16.1522	0.0022	0.0000	0.0007	0.0009	0.0011	0.0009	-0.0003	-0.0034	-0.0100
C_{16}	16.1539	0.0017	-0.0005	-0.0005	-0.0012	-0.0021	-0.0032	-0.0041	-0.0037	-0.0003
C_{17}	16.1552	0.0013	-0.0004	0.0001	0.0006	0.0017	0.0038	0.0070	0.0111	0.0148
C_{18}	16.1568	0.0016	0.0003	0.0007	0.0007	0.0001	-0.0016	-0.0054	-0.0124	-0.0235
C_{19}	16.1581	0.0013	-0.0003	-0.0006	-0.0013	-0.0020	-0.0021	-0.0005	0.0049	0.0173
C_{20}	16.1596	0.0015	0.0002	0.0005	0.0011	0.0025	0.0045	0.0066	0.0071	0.0022



Option values for Problem 3 of Table 5

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2 Bounds on the Values of Financial Derivatives under Partial Knowledge of the Asset Price Distribution

- Lo and MecKinlay, 1999,

  A non-random walk down wall street, Princeton Univ. Press

- These authors state that stock market prices do not follow random walks: evidence from a simple specification test.

  Stock market prices are not memoryless.

- Ritchken (1985), lower and upper bounds on the price of the European options, μ, σ known

- Lo (1987) upper bounds, European options, μ, σ known

- Grundy (1991) upper bound, European options, n^{th} moment alone is known

- Zheng (1994) moment problem formulation of the option price bounding problem, μ, σ known

- Bertsimas–Popescu (2002) prices of options with different strikes as well as μ, σ are known.

General Method of Moments Semi Infinite LP

Primal problem

$$\min\,(\max) \int f(z)\,\mathrm{d}F(z)$$

subject to

$$\int z^k\,\mathrm{d}F(z) = \mu_k, \qquad k = 0, \dots, m$$

$F(\cdot)$ is a c.d.f.

Dual of problem

$$\max\,(\min) \sum_{k=0}^{m} y_k \mu_k$$

subject to

$$\sum_{k=0}^{m} y_k z^k \le f(z) \qquad \text{all } Z.$$

Kemperman, 1968, attributes the method, using the dual problem, to Markov. Moment problem of the above form was originated by Chebyshev, Stieltjes, Markov.

Solutions of moment problems can be carried out by

(a) presenting formulas, available only if μ_1, μ_2 are known or in special cases a few more moments are known.

(b) Solving semi-infinite LP usually goes by discretization, thus, the first job is to look at discrete moment problems.

- Apart from above necessity discrete moment problems came to prominence by the discovery that a class of sharp probability bounds can be obtained as opt. sol. of discrete moment problems (Prékopa 1988, 1990).
- Discrete moment problems usually provide us with good approximation (not only bounds) for the value (in our case the value of a derivative) in question.

Discrete Moment Problem

DMP

Univariate

$$
\begin{aligned}
&\min\,(\max) \sum_{i=0}^{n} f(z_i)\, x_i \\
&\text{subject to} \\
&\quad \sum_{i=0}^{n} z_i^k x_i = \mu_k, \qquad k = 0, \dots, m \\
&\qquad\qquad x \geq 0, \qquad \mu_0 = 1
\end{aligned}
$$

$F(\cdot)$, $z_0, \dots, z_n$ known

$\mu_1, \dots, \mu_m$ also known

x decision variable.

If X is a random variable with possible values $\{z_i\}$ and first m moments $\mu_1, \dots, \mu_m$ are known, then

$$\min \; \leq E(f(X)) \leq \max.$$

Important special cases for $f(z)$

(1) $f(z)$ has nonnegative divided differences of order $m+1$. Recall: first m moments known.

(2)
$$f(z) = \begin{cases} 1 & \text{if} \quad z \geq z_r \\ 0 & \text{if} \quad z < z_r. \end{cases}$$

Write up moment problem as

$$\min\,(\max) \sum_{i=0}^{n} f_i x_i, \quad f_i = f(z_i)$$

subject to

$$\sum_{i=0}^{n} a_i x_i = b$$
$$x \geq 0.$$

Theorem (Prékopa 1988, 1990). *Assume $m+1$st div. diff. are strictly positive. A basis in the DMP is dual feasible iff it has structure*

	$m+1$ *even*	$m+1$ *odd*
min	$i, i+1, \ldots, j, j+1$	$0, i, i+1, \ldots, j, j+1$
max	$0, i, i+1, \ldots, j, j+1, n$	$i, i+1, \ldots, j, j+1, n$

Remark If $f(z)$ has nonnegative div. diff. only, then the above basis structure is only a sufficient condition for dual feasibility.

If $m = 1$ or $m = 2$, then it is not difficult to find a basis that is both dual and primal feasible and come up with formulas for the bounds.

Otherwise simple dual algorithms solves the problem where the pricing in is replaced by a simple search to restore dual feasibility.

THEOREM (Prékopa 1988, 1990). *Under the same conditions the dual feasible bases for function* (2) *are*

<u>*min problem $m+1$ even*</u>

$$\begin{array}{ll}
I \subset \{0,\ldots,r-1\} & \textit{if} \;\; r \geq m+1 \\
\{0,i,i+1,\ldots,j,j+1,r-1,k,k+1,\ldots,t,t+1\} & \textit{if} \;\; 2 \leq r \leq n \\
\{i,i+1,\ldots,j,j+1,r-1,k,k+1,\ldots,t,t+1,n\} & \textit{if} \;\; 1 \leq r \leq n
\end{array}$$

<u>*min problem $m+1$ odd*</u>

$$\begin{array}{ll}
I \subset \{0,\ldots,r-1\} & \textit{if} \;\; r \geq m+1 \\
\{0,i,i+1,\ldots,j,j+1,r-1,k,k+1,\ldots,t,t+1,n\} & \textit{if} \;\; 2 \leq r \leq n \\
\{i,i+1,\ldots,j,j+1,r-1,k,k+1,\ldots,t,t+1\} & \textit{if} \;\; 1 \leq r \leq n-1
\end{array}$$

<u>*max problem $m+1$ even*</u>

$$\begin{array}{ll}
I \subset \{r,\ldots,m\} & \textit{if} \;\; n-r \geq m \\
\{i,i+1,\ldots,j,j+1,r,k,k+1,\ldots,t,t+1,n\} & \textit{if} \;\; 1 \leq r \leq n-1 \\
\{0,i,i+1,\ldots,j,j+1,r,k,k+1,\ldots,t,t+1\} & \textit{if} \;\; 1 \leq r \leq n
\end{array}$$

<u>*max problem $m+1$ odd*</u>

$$\begin{array}{ll}
I \subset \{r,\ldots,n\} & \textit{if} \;\; n-r \geq m \\
\{i,i+1,\ldots,j,j+1,r,k,k+1,\ldots,t,t+1\} & \textit{if} \;\; 1 \leq r \leq n \\
\{0,i,i+1,\ldots,j,j+1,r,k,k+1,\ldots,t,t+1,n\} & \textit{if} \;\; 1 \leq r \leq n
\end{array}$$

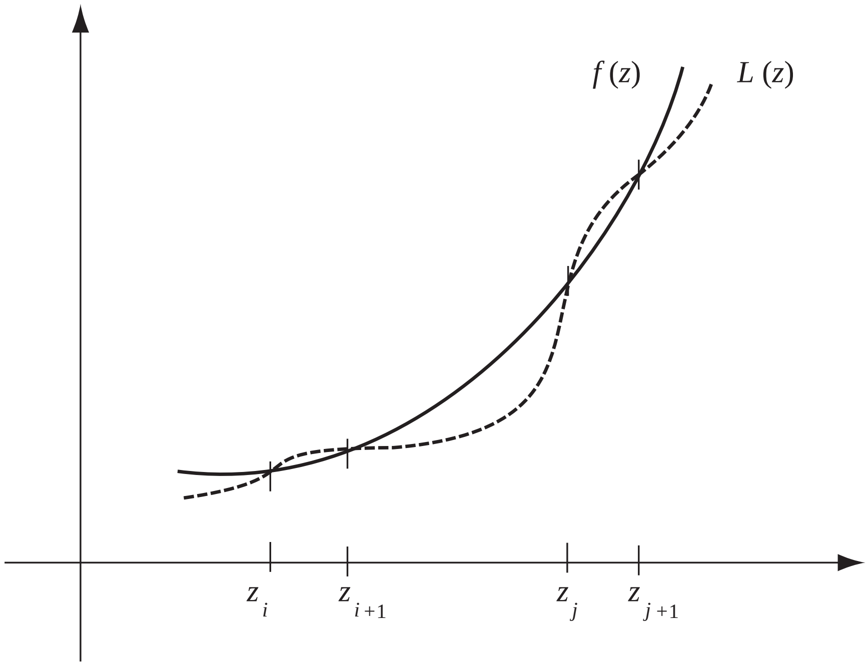
$I =$ *basis subscript set.*

- If $B_1(B_2)$ is any dual feasible basis in the min problem (max problem), then we have:

$$\begin{aligned} &\text{obj. function value in min pr.} \\ &\leq E(f(X)) \\ &\leq \text{obj. function value in max pr.} \end{aligned}$$

- So we may have a variety of bounding formulas.
- The sharp bounds can be obtained by formulas (known only for $m \leq 4$) or by dual algorithms.

  The algorithms are simplified dual algorithms but are numerically sensitive.



Optimality Condition:

$$f(z) - L(z) \geq 0, \qquad z \in [a, b].$$

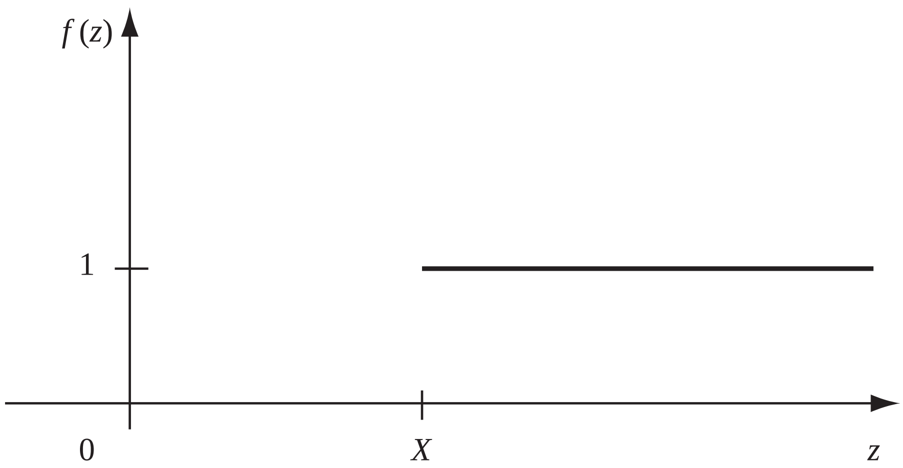
Violated only between

$$(z_i, z_{i+1}), \ldots, (z_j, z_{j+1}).$$

Application to cash-or-nothing option simplest case

$$f(z) = [z - X]_+$$

$$\begin{aligned} \text{option price} &= e^{-rT} E([f(S(T))]) \\ &= e^{-rT} P(S(T) \geq X) \end{aligned}$$



Risk neutral valuation

Application to Bounding European Call/Put Options. Consider the Call

$$
\begin{aligned}
t &= 0 \text{ time now, } T = \text{future time} \\
S(t) &= \text{price of underlying asset} \\
X &= \text{striking price} \\
r &= \text{rate of interest} \\
c &= \text{price of option.}
\end{aligned}
$$

Let $S(t) = e^{\alpha Z(t)+\beta}$, $\alpha > 0$, β real and assume that the first m moment of $Z(T)$ are known, given the value of $Z(t)$. Using the unknown conditional distribution of $S(T)$,

$$
\begin{aligned}
c &= e^{-rT} E([S(T) - X]_+) \\
&= e^{-rT} E([S(T) - X]_+ \mid S(T) > X) P(S(T) > X) \\
&= e^{-rT} E\left(e^{\alpha Z(T)+\beta} - X \mid e^{\alpha Z(T)+\beta} > X\right) P\left(e^{\alpha Z(T)+\beta} > X\right)
\end{aligned}
$$

we can obtain bounds for each factor on the right hand side.

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3 Conditional Mean-Conditional Variance Portfolio Selection Models

Introduction

Consider n assets with random returns on unit investments $R_1, \ldots, R_n$, investment amounts $x_1, \ldots, x_n$ and introduce the notations:

$$R = (R_1, \ldots, R_n)^T, \quad x = (x_1, \ldots, x_n)^T$$
$$E(R_i) = m_i, \quad i = 1, \ldots, n, \quad m = (m_1, \ldots, m_n)^T$$
$$X = R^T x, \quad C = E[(R - m)(R - m)^T]$$

with these notations we have the equations:

$$E(X) = m^T x, \quad \mathrm{Var}\,(X) = x^T C x.$$

Markowitz's mean-variance model (1952, 1959) is usually formulated in three different ways:

Model I.

$$\begin{aligned} &\text{maximize } m^T x \\ &\text{subject to} \\ &x^T C x \le V \\ &\sum_{i=1}^{n} x_i = 1 \\ &\quad x \ge 0, \end{aligned}$$

Model II.

$$\begin{aligned} &\text{minimize } x^T C x \\ &\text{subject to} \\ &m^T x \le M \\ &\sum_{i=1}^{n} x_i = 1 \\ &\quad x \ge 0, \end{aligned}$$

Model III.

$$\begin{aligned} &\text{maximize } \{m^T x - \beta x^T C x\} \\ &\text{subject to} \\ &\sum_{i=1}^{n} x_i = 1 \\ &\quad x \ge 0, \end{aligned}$$

where M, V are some constant upper bounds, for the expectation and variance, respectively, of the total return and $\beta > 0$ is a constant. Sometimes lower and upper bounds are imposed individually on the variables x_i, $i = 1, \ldots, n$.

Kataoka's (1963) model is similar to Model III. We can state it as follows:

$$\begin{aligned}
&\text{maximize } v \\
&\text{subject to} \\
&P(R^T x \geq v) \geq p \\
&\qquad \sum_{i=1}^{n} x_i = 1 \\
&\qquad\quad x \geq 0,
\end{aligned}$$

where $p \in (0,1)$ is a fixed probability chosen by ourselves.

If $R_1, \ldots, R_n$ have a normal joint distribution, then, no matter if it is degenerate or nondegenerate, the problem can be rewritten in the following form:

$$\begin{aligned}
&\text{maximize } \left\{ m^T x + \Phi^{-1}(1-p)\sqrt{x^T C x} \right\} \\
&\text{subject to} \\
&\sum_{i=1}^{n} x_i = 1 \\
&\quad x \geq 0.
\end{aligned}$$

In what follows φ, Φ designate the standard normal density and distribution functions, respectively.

The term Value at Risk was proposed by Till Guldimann at J.P. Morgan in the late 1980's. The Group of Thirty which had a representative from J.P. Morgan had a discussion on best risk management practices. The term VaR found its way through the G–30 report published in 1993.

Definition Value at risk of confidence level 100 p% is defined as the optimum value of the optimization problem:

$$\begin{aligned}&\text{maximize } v\\ &\text{subject to}\\ &P(X \geq v) \geq p,\end{aligned}$$

where X is a random variable and $p \in (0,1)$ is a constant. Let $\mathrm{VaR}_p(X)$ designate this value. Sometimes we simply write VaR.

Another definition (see Pflug, 2000) takes it as the optimum value of the problem $(0 < \alpha < 1)$:

$$\begin{aligned}&\inf v\\ &\text{subject to}\\ &P(X \leq v) > \alpha.\end{aligned}$$

The two definitions provide us with the same optimum value if $\alpha = 1 - p$. However, the practical use of the two definitions are different. We use the first definition if X means revenue or return and use the second definition if X means loss. Both p and α are chosen large, in practice, because we want to impose lower bound on revenue, or return, and upper bound on loss, by large probabilities.

Note that the c.d.f. of a random variable X, i.e., the function $F(v) = P(X \leq v)$, $-\infty < v < \infty$, is right continuous. If we define its inverse by the equation:

$$F^{-1}(v) = \inf\{z \mid F(z) \geq v\}, \quad -\infty < v < \infty,$$

then the optimum value of problem the problem in the second definition is $F^{-1}(\alpha)$ and we have the equality $\mathrm{VaR}_p(X) = F^{-1}(1-p)$.

If $X = R^T X$, where R has normal distribution with $E(R) = m$ and covariance matrix C, then we easily derive the equation:

$$\mathrm{VaR}_p(X) = m^T x + \Phi^{-1}(1-p)\sqrt{x^T C x}.$$

The definition of the Conditional Value at Risk depends on the definition of the Value at Risk. Since VaR is defined by the first definition, we use the following

Definition (Rockafellar, Uryasev, 2002). Conditional Value at Risk is the value

$$E(X \mid X \leq v), \qquad v = \mathrm{VaR}_p(X).$$

We designate it by $\mathrm{CVaR}_p(X)$, or simply by CVaR, if it is clear what X and p are on the table.

If VaR were defined by the second definition, then we would define CVaR as

$$E(X \mid X \geq v),$$

where v is the optimum value of the problem in the second definition.

Under the condition of a normally distributed R we have the formula:

$$\mathrm{CVaR}_p(X) = m^T x - \frac{\varphi(\Phi^{-1}(1-p))}{1-p}\sqrt{x^T C x}.$$

Note that $\Phi^{-1}(1-p) = -\Phi^{-1}(p)$ which implies $\varphi(\Phi^{-1}(1-p)) = \varphi(\Phi^{-1}(p))$.

CVaR is closely connected with risk measures introduced earlier in stochastic optimization. If in the underlying problem we have a constraint

$$Tx \geq \xi,$$

where T is an $r \times n$ matrix and ξ is an r-component random vector, then in the probabilistic constrained formulation we prescribe that the constraint

$$P(Tx \geq \xi) \geq p$$

should be satisfied for every x that we consider feasible. As a relaxation of the constraint Prékopa (1973) introduced the conditional expectation constraints:

$$E(\xi_i - T_i x \mid \xi_i - T_i x > 0) \leq d_i, \qquad i = 1, \ldots, r,$$

where T_i is the ith row of T and ξ_i is the ith component of ξ. Prékopa (1973b) has also shown that if each ξ_i has continuous probability distribution with logarithmically concave p.d.f., then the constraints are linear.

A related constraint type, called integrated chance constraint, was introduced by Klein Haneveld (1986). As applied to our case, and as another relaxation of the single probabilistic constraint, we write

$$E([\xi_i - T_i x]_+) \le d_i, \qquad i = 1, \dots, r.$$

The constraining functions are linear in x, without any limitation regarding the probability distributions of the ξ_i, $i = 1, \dots, r$.

We introduce further risk measures, where those in the next definition paraphrase the CVaR.

Definition The functions of the variable v $(-\infty < v < \infty)$:

$$E(X \mid X \geq v)$$
$$E(X \mid X \leq v)$$

will be called conditional expectation functions.

In the next definition new risk measures are introduced.

Definition The functions of the variable v $(-\infty < v < \infty)$:

$$E(X^2 \mid X \geq v) - E^2(X \mid X \geq v)$$
$$E(X^2 \mid X \leq v) - E^2(X \mid X \leq v)$$

will be called conditional variance functions.

If we choose $v = \mathrm{VaR}_p(X)$ in the second conditional expectation function, then its value is $\mathrm{CVaR}_p(X)$.

Mathematical Properties of the Conditional Variance Functions

The most important theorem in connection with CVAR is the following

THEOREM *If the random variable X has continuous distribution and logconcave p.d.f., then*

$$E(X^2 \mid X > v) - (E(X \mid X > v))^2, \quad -\infty < v < \infty$$

is a decreasing function of the variable v.

THEOREM *If the random variable X has continuous distribution and logconcave p.d.f., then*

$$E(X^2 \mid X \leq v) - (E(X \mid X \leq v))^2, \qquad -\infty < v < \infty$$

is an increasing function of the variable v.

THEOREM *If the p.d.f. of X is strictly logconcave in the entire real line, then the function in the first Theorem is strictly decreasing and the function in the second Theorem is strictly increasing in the entire real line.*

Conditional Mean-Conditional Variance Portfolio Selection Models

Model I. maximize $E(X \mid X \geq v_0)$

subject to

$$E(X^2 \mid X \geq v_0) - E^2(X \mid X \geq v_0) \leq V$$
$$\mathrm{VaR}_p(X) \geq v_0$$
$$\sum_{i=1}^{n} x_i = 1$$
$$x \geq 0,$$

Model II. minimize $\{E(X^2 \mid X \geq v_0) - E^2(X \mid X \geq v_0)\}$

subject to

$$E(X \mid X \geq v_0) \geq M$$
$$\mathrm{VaR}_p(X) \geq v_0$$
$$\sum_{i=1}^{n} x_i = 1$$
$$x \geq 0,$$

Model III. maximize $\{E(X \mid X \geq v_0) - \beta[E(X^2 \mid X \geq v_0) - E^2(X \mid X \geq v_0)]\}$

subject to

$$\mathrm{VaR}_p(X) \geq v_0$$
$$\sum_{i=1}^{n} x_i = 1$$
$$x \geq 0,$$

where $\beta > 0$ is some constant. In Model III. we may replace the conditional variance by its square root. In the general case the above models are nonconvex optimization problems but using suitable relaxation we can obtain problems the solutions of which are easier.

Model I.

If $v_0 = \mathrm{VaR}_p(X)$, then Model I. specializes to:

Model I. (a) maximize $\left\{ m^T x + \frac{1}{p}\varphi(\Phi^{-1}(1-p))\sqrt{x^T C x} \right\}$

subject to

$$x^T C x \left(1 + \frac{1}{p}\varphi(\phi^{-1}(1-p))\Phi^{-1}(1-p) - \frac{1}{p^2}\varphi^2(\Phi^{-1}(1-p)) \right) \le V$$

$$m^T x + \Phi^{-1}(1-p)\sqrt{x^T C x} \ge v_0$$

$$\sum_{i=1}^{n} x_i = 1$$

$$x \ge 0.$$

The constant that multiplies $x^T C x$ in the first constraint is positive, for any p, hence the set of feasible solutions is convex and the objective function, to be maximized, is concave.

We can also create a tractable problem from the general Model I. if we choose some fixed $v = v_0$ and prescribe:

$$E(X^2 \mid X \geq v_0) - E^2(X \mid X \geq v_0) \leq d x^T C x,$$

where d is some constant, chosen by ourselves, $0 < d < 1$. Let $h_1(v) = 1 - h'(v)$, then the above requirement means

$$h_1\left(\frac{v_0 - m^T x}{\sqrt{x^T C x}}\right) \leq d$$

and our problem is:

Model I. (b)

$$\begin{aligned}
&\text{maximize } \left\{ m^T x + \frac{\varphi\left(\dfrac{v_0 - m^T x}{\sqrt{x^T C x}}\right)}{1 - \Phi\left(\dfrac{v_0 - m^T x}{\sqrt{x^T C x}}\right)} \sqrt{x^T C x} \right\} \\
&\text{subject to} \\
&\quad m^T x + h_1^{-1}(d) \sqrt{x^T C x} \geq v_0 \\
&\quad m^T x + \Phi^{-1}(1-p) \sqrt{x^T C x} \geq v_0 \\
&\quad \sum_{i=1}^{n} x_i = 1 \\
&\quad x \geq 0.
\end{aligned}$$

The problem is still nonconvex because the objective function is not concave in x, in general.

Model II.

If we interchange the objective function and the constraining function in the first constraint in Models I. (a), (b), then we obtain the new versions of Model II. We may disregard the constant factor of $x^T Cx$. Thus, our new problems are the following:

Model II. (a)

$$\begin{aligned}
&\text{minimize } x^T Cx \\
&\text{subject to} \\
&m^T x + \frac{1}{p}\varphi(\Phi^{-1}(1-p))\sqrt{x^T Cx} \geq M \\
&\qquad m^T x + \Phi^{-1}(1-p)\sqrt{x^T Cx} \geq v_0 \\
&\qquad\qquad \sum_{i=1}^{n} x_i = 1 \\
&\qquad\qquad x \geq 0
\end{aligned}$$

and

Model II. (b)

$$
\begin{aligned}
&\text{maximize } \{m^T x + h_1^{-1}(d)\sqrt{x^T C x}\} \\
&\text{subject to} \\
&m^T x + \frac{\varphi\left(\dfrac{v_0 - m^T x}{\sqrt{x^T C x}}\right)}{1 - \Phi\left(\dfrac{v_0 - m^T x}{\sqrt{x^T C x}}\right)}\sqrt{x^T C x} \geq M \\
&\qquad m^T x + \Phi^{-1}(1-p)\sqrt{x^T C x} \geq v_0 \\
&\qquad \sum_{i=1}^{n} x_i = 1 \\
&\qquad x \geq 0.
\end{aligned}
$$

Model II. (a) is a convex problem but Model II. (b) is not.

Model III.

If we use the formulas for the expectation and the variance of the return, then we can write Model III. as follows:

$$\text{Model III.} \qquad \text{maximize} \left\{ m^T x + \frac{\varphi\left(\dfrac{v_0 - m^T x}{\sqrt{x^T C x}}\right)}{1 - \Phi\left(\dfrac{v_0 - m^T x}{\sqrt{x^T C x}}\right)} \sqrt{x^T C x} \right.$$

$$- \beta x^T C x \left(1 + \frac{\varphi\left(\dfrac{v_0 - m^T x}{\sqrt{x^T C x}}\right)}{1 - \Phi\left(\dfrac{v_0 - m^T x}{\sqrt{x^T C x}}\right)} \frac{v_0 - m^T x}{\sqrt{x^T C x}} \right.$$

$$\left. \left. - \left(\frac{\varphi\left(\dfrac{v_0 - m^T x}{\sqrt{x^T C x}}\right)}{1 - \Phi\left(\dfrac{v_0 - m^T x}{\sqrt{x^T C x}}\right)} \right)^2 \right) \right\}$$

$$\begin{aligned} &\text{subject to} \\ &m^T x + \Phi^{-1}(1-p)\sqrt{x^T C x} \geq v_0 \\ &\qquad\qquad\qquad \sum_{i=1}^{n} x_i = 1 \\ &\qquad\qquad\qquad\quad x \geq 0, \end{aligned}$$

where β is some positive constant. Another model is obtained if we take the square root of the factor that multiplies β in the objective function.

If we take $v_0 = \mathrm{VaR}_p(X)$, then the model specializes as follows:

Model III.

$$\begin{aligned}
&\text{maximize } \left\{ m^T x + \frac{1}{p}\varphi(\Phi^{-1}(1-p))\sqrt{x^T C x} \right. \\
&\qquad \left. -\beta x^T C x \left(1 + \frac{1}{p}\varphi(\Phi^{-1}(1-p))\Phi^{-1}(1-p) - \frac{1}{p^2}\varphi^2(\Phi^{-1}(1-p)) \right) \right\} \\
&\text{subject to} \\
&\sum_{i=1}^{n} x_i = 1 \\
&\qquad x \geq 0,
\end{aligned}$$

where β is some positive constant. Model III. is a convex optimization problem because both $\sqrt{x^T C x}$ and $x^T C x$ are convex functions of the variable x.

The value of p should be chosen large, e.g., 0.9, 0.95, 0.99. Efficient frontiers, based on Models I. and II. can be constructed in a similar way as suggested by Markowitz (1952, 1959).

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