Arithmetic Geometry and Hyperbolic Geometry

Pit-Mann Wong
University of Notre Dame
Notre Dame, Indiana, USA

Lecture at Fields Institute
Toronto, Canada
June 26, 2008

Numbers

In the beginning there were the natural numbers $\mathbb{N}: 1, 2, 3, \ldots$
Then came fractions : $1/2, 2/3, 4/3, \ldots$ etc..
It took awhile, but eventually, the integers

$$\mathbb{Z}: \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

and the rational numbers

$$\mathbb{Q}: -4/3, -1, -2/3, 0, 2/3, 1, 4/3, \ldots$$

were also understood. The Archimedean absolute value $| \ |_\infty$ converts a negative number to a positive number: $|a|_\infty = a$ if $a \geq 0$ and $|a|_\infty = -a$ if $a \leq 0$.

Numbers

In the beginning there were the natural numbers $\mathbb{N}: 1, 2, 3, \ldots$
Then came fractions : $1/2, 2/3, 4/3, \ldots$ etc..
It took awhile, but eventually, the integers

$$\mathbb{Z}: \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

and the rational numbers

$$\mathbb{Q}: -4/3, -1, -2/3, 0, 2/3, 1, 4/3, \ldots$$

were also understood. The Archimedean absolute value $| \ |_\infty$ converts a negative number to a positive number: $|a|_\infty = a$ if $a \geq 0$ and $|a|_\infty = -a$ if $a \leq 0$.

Numbers

In the beginning there were the natural numbers $\mathbb{N} : 1, 2, 3, \ldots$
Then came fractions : $1/2, 2/3, 4/3, \ldots$ etc..
It took awhile, but eventually, the integers

$$\mathbb{Z} : \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

and the rational numbers

$$\mathbb{Q} : -4/3, -1, -2/3, 0, 2/3, 1, 4/3, \ldots$$

were also understood. The Archimedean absolute value $| \ |_\infty$ converts a negative number to a positive number: $|a|_\infty = a$ if $a \geq 0$ and $|a|_\infty = -a$ if $a \leq 0$.

As our ancestors started measuring distance, relationship among numbers began to emerge. Among these is the Pythagorean Theorem, relating the length c of the hypothenuse of a right-angled triangle with the lengths of the two legs, a and b, of a right-angled triangle:

$$c^2 = a^2 + b^2.$$

Taking $a = b = 1$ results in the equation:

$$c^2 = 2.$$

Soon it was realized that the solution c does not behave like any rational numbers. That is, it cannot be represented as a fraction. Euclid, for example, gave a completely rigorous proof of this. This marks the discovery of irrational numbers.

As our ancestors started measuring distance, relationship among numbers began to emerge. Among these is the Pythagorean Theorem, relating the length c of the hypothenuse of a right-angled triangle with the lengths of the two legs, a and b, of a right-angled triangle:

$$c^2 = a^2 + b^2.$$

Taking $a = b = 1$ results in the equation:

$$c^2 = 2.$$

Soon it was realized that the solution c does not behave like any rational numbers. That is, it cannot be represented as a fraction. Euclid, for example, gave a completely rigorous proof of this. This marks the discovery of irrational numbers.

As our ancestors started measuring distance, relationship among numbers began to emerge. Among these is the Pythagorean Theorem, relating the length c of the hypothenuse of a right-angled triangle with the lengths of the two legs, a and b, of a right-angled triangle:

$$c^2 = a^2 + b^2.$$

Taking $a = b = 1$ results in the equation:

$$c^2 = 2.$$

Soon it was realized that the solution c does not behave like any rational numbers. That is, it cannot be represented as a fraction. Euclid, for example, gave a completely rigorous proof of this. This marks the discovery of irrational numbers.

Other irrational numbers were also discovered, among these, the number π is probably the most important. This, no doubt, came with the invention of the wheels. However, it was not until 1776 that Lambert (and Lagrange) gave a proof that π is indeed an irrational number. It took even longer (Lindemann 1882) for mathematicians to realize that π belongs to a different category of numbers known as the transcendental numbers.

Definition

A number is said to be an algebraic number if it is the solution of a monic polynomial with integer coefficients. A number is said to be transcendental if it is not algebraic.

For example the solution $x = \sqrt{2}$ of the equation $x^2 = 2$ is an algebraic number.

Other irrational numbers were also discovered, among these, the number π is probably the most important. This, no doubt, came with the invention of the wheels. However, it was not until 1776 that Lambert (and Lagrange) gave a proof that π is indeed an irrational number. It took even longer (Lindemann 1882) for mathematicians to realize that π belongs to a different category of numbers known as the transcendental numbers.

Definition

A number is said to be an algebraic number if it is the solution of a monic polynomial with integer coefficients. A number is said to be transcendental if it is not algebraic.

For example the solution $x = \sqrt{2}$ of the equation $x^2 = 2$ is an algebraic number.

Other irrational numbers were also discovered, among these, the number π is probably the most important. This, no doubt, came with the invention of the wheels. However, it was not until 1776 that Lambert (and Lagrange) gave a proof that π is indeed an irrational number. It took even longer (Lindemann 1882) for mathematicians to realize that π belongs to a different category of numbers known as the transcendental numbers.

Definition

A number is said to be an algebraic number if it is the solution of a monic polynomial with integer coefficients. A number is said to be transcendental if it is not algebraic.

For example the solution $x = \sqrt{2}$ of the equation $x^2 = 2$ is an algebraic number.

For many years (more than two thousand years) mathematicians pondered the question whether one can square a circle? What it means is this. The area of a circle of radius r is πr^2; so the area of a circle of radius 1 is π. The area of a square (with each side equals x) is x^2. The equation we are dealing with is $x^2 = \pi$ and the question is:

- Is it possible to construct, using only straight edge and compass, the solution $x = \sqrt{\pi}$?

Note that this is possible (indeed very easy) for the algebraic number $\sqrt{2}$. Lindemann's work showed that this is not possible for $\sqrt{\pi}$, hence it cannot be algebraic.

With the advances in Calculus, mathematicians understood the concept of limit and the real numbers $\mathbb{R}$ are just the limits, with respect to the Archimedean absolute value, of the rational numbers. We say that $\mathbb{R}$ is the completion of $\mathbb{Q}$ with respect to $|\ |_\infty$. This coupled with the imaginary number $\sqrt{-1}$, namely, solution of the equation $x^2 = -1$, lead us to the complex numbers $\mathbb{C}$. The rational and irrational, algebraic or transcendental, numbers together constitute the complex numbers $\mathbb{C}$.

Over time more and more transcendental numbers were discovered. Here is a very brief history:

- Lambert (1766) (building upon earlier work by Euler) : The number π is transcendental. The natural number e is transcendental (the proof that we frequently see nowadays in 'elementary' textbooks is due to Fourier (1815)).
- Liouville (1840): The number $1/e$ is transcendental.
- Hermite (1873): The number e^π is transcendental.

- Lindemann (1882) If $\alpha \neq 0$ is an algebraic number then e^{α} (e.g., $e, e^2, \sqrt{e}, e^{\sqrt{2}}$) is transcendental. If a non-zero number is the logarithmic of an algebraic number then it is transcendental (e.g., $\pi = \frac{1}{\sqrt{-1}} \ln(-1)$).
- Hilbert (1900): Hilbert's 7-th problem raised the question whether α^{β} is transcendental for algebraic numbers α and β (e.g., $(-1)^{-\sqrt{-1}}$, $2^{\sqrt{2}}$). (Hilbert proposed 23 problems in the Second International Congress of Mathematicians in 1900.) It was reported that Hilbert thought that this problem is harder than Fermat's last theorem and the Riemann hypothesis.

- A. O. Gelfond (1929): α^β is transcendental for any algebraic number $\alpha(\neq 0, 1)$ and β any imaginary quadratic irrational; e. g., $(-1)^{-\sqrt{-1}}$ is transcendental.
- Kuzmin (1930): α^β is transcendental for any algebraic number $\alpha(\neq 0, 1)$ and β any quadratic irrational; e.g., $2^{\sqrt{2}}$ is transcendental.
- Hilbert's 7-th problem was solved, in the affirmative, in 1934 by A. O. Gelfond and Th. Schneider independently.

Shortly after that Siegel began to look at the the problem through function theory. He looked at a transcendental function $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ and an algebraic number field $\mathbb{K}$and asked how big is the set $f^{-1}(\mathbb{K})$? He showed that, for many explicit transcendental functions, the set of algebraic values $f^{-1}(\mathbb{K})$ is finite.

Siegel's work were reformulated and extended by Lang. The most general form at this time is due to Bombieri. It takes the following form:

Theorem

Let $\mathbb{K}$ be an algebraic number field and $f_1, \ldots, f_n$ be meromorphic functions of finite order defined on $\mathbb{C}^m$ considered as a map $F = (f_1, ..., f_n) : \mathbb{C}^m \to \mathbb{C}^n$. Assume that the transcendence degree of $\mathbb{K}(f_1, ..., f_n)$ is at least $n + 1$ and

$$\frac{\partial}{\partial z_i} : \mathbb{K}(f_1, ..., f_n) \to \mathbb{K}(f_1, ..., f_n)$$

for $1 \le i \le n$. Then there exists a polynomial P such that $F^{-1}(\mathbb{K}^n) \subset \{P = 0\}$.

For $m = 1$ a polynomial of one variable can only have a finite number of roots so $F^{-1}(\mathbb{K}^n)$ is finite, recovering the results of Siegel.

Bombieri's proof of this theorem is a master piece! He looked at the problem first from the number theoretic point of view and then he brought in complex analysis (the deep and powerful theory of L^2-estimate of Hörmander for complex analytic functions) and magically, the two theory begin to merge in front of your eyes and the theorem falls onto your lap naturally.

Notice that Bombieri's result did not claim finiteness in higher dimension ($m \geq 2$). One of the problem that we propose for our program is to find natural conditions so that we do get finiteness.

Fix a prime number p. Every rational number x can be factorize as $x = p^s a/b$ where s, a and b are integers with a and n relatively prime to p. The p-adic absolute value $|x|_p$ is defined to be

$$|x|_p = p^{-r}.$$

These are non-Archimedean absolute values. They behave like the usual Archimedean absolute value but satisfied a stronger form of the triangle inequality:

$$|a + b|_p \leq \max\{|a|_p, |b|_p\}.$$

Analogous to the construction of the field of real numbers $\mathbb{R}$ from the rational numbers via the Archimedean absolute value, we get from the p-adic absolute value the field $\mathbb{R}_p$. The complex number field $\mathbb{C}$ is the algebraic closure of $\mathbb{R}$. The Archimedean absolute extends to $\mathbb{C}$ and $(\mathbb{C}, |\ \ |_\infty)$ is a complete metric space. Analogously, we take the algebraic closure of $\mathbb{R}_p$ with the extended p-adic absolute value. It turns out that this is not complete as a metric space. Thus we must take the completion, the resulting field is algebraically closed and complete as a metric space. This field, $\mathbb{C}_p$, is the field of p-adic numbers.

Diophantine Equations and Diophantine Approximation

The basic problem in Diophantine Equations is the investigation of whether an equation, say defined over integers, admits any integer solutions? This seemingly innocent problem turns out to be very difficult in general. As a first step, we ask an easier question. Does the equation admit only a finite number of integer solutions?

Example

Consider the equation

$$x^3 - 2y^3 = 1.$$

If $y = 0$ then the equation is reduced to $x^3 - 1 = 0$. The left hand side factorize into $(x-1)(x-\theta)(x-\theta^2)$ where $\theta = e^{2\pi\sqrt{-1}/3}$ and we get 3 roots, $x = 1, x = \theta$ and $x = \theta^2$. Only one of them is an integer.

For the general case we divide the original equation by y^3

$$\frac{x^3}{y^3} - 2 = \frac{1}{y^3}$$

and factorize the left hand side:

$$(\frac{x}{y} - 2^{1/3})(\frac{x}{y} - 2^{1/3}\theta)(\frac{x}{y} - 2^{1/3}\theta^2) = \frac{1}{y^3}.$$

From this one deduces that

$$|\frac{x}{y} - 2^{1/3}| \leq \frac{C}{|y|^3}.$$

Thus the problem is reduced to a problem in approximation of the number $2^{1/3}$ by rational numbers x/y, namely how many rational numbers x/y satisfy the inequality? The answer is provided by the famous Roth's Lemma.

Theorem (Roth)

Let r be an irrational algebraic number. Then for any $\epsilon > 0$, there exists a constant $C > 0$ such that there only finitely many rational numbers $x/y, x, y \in \mathbb{Z}$ satisfying the estimate

$$|\frac{x}{y} - r| \leq \frac{C}{|y|^{2+\epsilon}}.$$

Roth's theorem applied to the preceding example shows that there are only finitely many integer solutions to the equation $x^3 - 2y^3 = 1$.

It is convenient to express Roth's theorem in logarithmic form:

Theorem (Roth)

Let r be an irrational algebraic number. Then for any $\epsilon > 0$, the estimate

$$\log \frac{1}{|\frac{x}{y} - r|} \leq (2+\epsilon) h(\frac{x}{y}) + O(1)$$

holds for all but finitely many rational numbers x/y where $h(x/y) = \log|y|$ is the height of x/y.

In this form Roth's theorem looks very much like the Second Main Theorem in Nevanlinna theory to be introduced in the next section.

The problem can be formulated geometrically. The equation in the example can be written in terms of homogeneous coordinates:

$$x^3 - 2y^3 = z^3.$$

This is an equation in the projective 2-space $\mathbb{P}^2$. Thus $C = \{[x, y, z] \mid x^3 - 2y^3 = z^3\}$ is a curve of degree 3 in $\mathbb{P}^2$. There is a very important invariant, known as the genus (denoted by g), attached to a curve. A curve of genus zero (resp. one) is called a rational curve (resp. an elliptic curve). Curves of genus greater than one are called general type. The genus formula on $\mathbb{P}^2$ asserts that, for a smooth curve of degree d

$$g = \frac{(d-1)(d-2)}{2}.$$

For the curve C the genus is one, i.e., it is an elliptic curve. The curve $L = \{[x, y, z] \mid z = 0\}$ is a line (degree one) so the genus is zero, i.e., a rational curve (the curve at infinity). The example we examined is in $C \setminus L$ (as we have $z = 1$). The implication of Roth's Theorem asserts that the number of integer points on $C \setminus L$ is finite. It is known that

Theorem (Siegel)

(a) *Let L be a rational curve. Then*

$$L \setminus \{\textit{three distinct points on } L\}$$

has finitely many integer points.
(b) *Let E be an elliptic curve. Then*

$$E \setminus \{\textit{one point on } E\}$$

has finitely many integer points.

Theorem (Faltings)

Let C be a projective curve of genus $g \geq 2$. Then C has only finitely many rational points.

The preceding theorem was first conjectured by Mordell.
The proof of these theorems are not effective in the sense that no bound of the number of points (solutions) are given. A breakthrough on the effective aspect of Roth's Theorem is due to Baker, for example, he showed that

$$|\frac{x}{y} - 2^{1/3}| > 10^{-}6\frac{1}{|y|^{2.955}}.$$

With this all solutions of the equation of the example were found.

Roth's Theorem was extended to higher dimension (multiple variables, simultaneous Diophantine approximation) by Schmidt and is known as Schmidt's Subspace Theorem:

Theorem (Schmidt)

Let $L_0, L_1, \ldots, L_n$ be homogeneous linear forms in $x_1, \ldots, x_n$ with algebraic coefficients, linearly independent over $\mathbb{Q}$. Then, for any $\epsilon > 0$, there exists a finite number of hyperplanes $H_1, \ldots, H_N$ in $\mathbb{P}(\mathbb{Q})^n$ and a constant $C > 0$ such that all integer solutions of the inequality

$$|L_0(\frac{\mathbf{x}}{|\mathbf{x}|}) \cdots L_n(\frac{\mathbf{x}}{|\mathbf{x}|})| \leq \frac{C}{|\mathbf{x}|^{n+1+\epsilon}}$$

are contained in $\cup_{i=1}^{N} H_i$.

We prefer the logarithmic version of Schmidt's theorem:

Theorem (Schmidt)

Let $L_0, L_1, \ldots, L_n$ be homogeneous linear forms in $x_1, \ldots, x_n$ with algebraic coefficients, linearly independent over $\mathbb{Q}$. Then, for any $\epsilon > 0$, there exists a finite number of hyperplanes $H_1, \ldots, H_N$ in $\mathbb{P}(\mathbb{Q})^n$ such that all integer solutions of the inequality

$$\sum_{i=0}^{n} \log \frac{1}{|L_i(\mathbf{x})|} \leq (n+1+\epsilon)h(\mathbf{x}) + C$$

holds provided that $\mathbf{x} \notin \cup_{i=1}^{N} H_i$.

Nevanlinna Theory

We now turn our attention to function theory, more precisely, the theory of holomorphic (complex analytic) functions. We are also interested in solutions of equations, just that we are now seeking solutions in functions spaces. In the 1920's, Nevanlinna introduced a theory, now known as Nevanlinna theory to deal with such problems. For example $x = \sin\theta, y = \cos\theta$ is a solution of the equation:

$$x^2 + y^2 = 1.$$

It can also be shown that the only holomorphic solutions to the equation

$$x^3 - 2y^3 = 1$$

are constants.

Definition

A complex space X is said to be hyperbolic if every holomorphic map $f : \mathbb{C} \to X$ is a constant.

The following results in hyperbolic geometry are classical:

Theorem

(a) Let L be a rational curve. Then

$$L \setminus \{\textit{three distinct points on } L\}$$

is hyperbolic.
(b) Let E be an elliptic curve. Then

$$E \setminus \{\textit{one point on } E\}$$

is hyperbolic.
(c) Every projective (or compact) curve of genus $g \geq 2$ is hyperbolic.

There are various different proofs of the preceding classical theorem. One of the proof is based on the Second Main Theorem (abbrev. SMT) of Nevanlinna Theory:

Theorem (SMT, $n = 1$)

Let $f : \mathbb{C} \to \mathbb{C}$ be a non-algebraic meromorphic function. Then, for any q distinct points $a_1, \ldots, a_q \in \mathbb{P}^1$,

$$\sum_{i=1}^{q} \int_0^{2\pi} \log^+ \frac{1}{|f(re^{\sqrt{-1}\theta}) - a_i|} \frac{d\theta}{2\pi} \leq (2+\epsilon) T_f(r) + O(\log r)$$

where the estimate holds for all r outside an exceptional set of finite Lebesgue measure.

This is the analogue of Roth's theorem. The LHS is known as the proximity function and $T_t(r)$ is called the characteristic function. It is the analogue of the height function in number theory.

The SMT can be generalized to higher dimension:

Theorem (SMT, $n \geq 2$)

Let $L_1, \ldots, L_q$ be q hyperplanes in general position in $\mathbb{P}^n$. Then, for any linearly non-degenerate holomorphic map $f : \mathbb{C} \to \mathbb{P}^n$,

$$\sum_{i=1}^{q} \int_0^{2\pi} \log^+ \frac{1}{|L_i(f(re^{\sqrt{-1}\theta}))|} \frac{d\theta}{2\pi} \leq (n+1+\epsilon)T_f(r) + O(\log r)$$

where the estimate holds for all r outside an exceptional set of finite Lebesgue measure.

This is the analogue of Schmidt's subspace theorem.

Diophantine Geometry vs Hyperbolic Geometry

The resemblance of the Roth-Schmidt's theorem with the SMT is quite striking. It certainly suggest a deeper relationship between Diophantine geometry and hyperbolic geometry. Vojta is the first to introduce a dictionary between the two theories. Using the dictionary, an assertion in Diophantine approximation can be translated to an assertion in Nevanlinna theory and vice versa. However, at this time, the correspondence is at the 'formal' level; no one has been able to rigorously unify the two theories. What we do have are many results form each theory which corresponds to each other in a very precise manner.

Even more striking is that many of the proofs can be translated from one theory to the other via the following Principle:

- If a statement in hyperbolic geometry can be proved using only the SMT then the corresponding statement in Diophantine geometry can be proved in the same manner using only Roth-Schmidt's theorem. The same is true the other way around.

For example:

- It was discovered that the SMT of Nevanlinna implies that the space

$$\mathbb{P}^n \setminus \{2n+1 \text{ hyperplanes in general position}\}$$

  is hyperbolic. It was observed that the proof carries over to Diophantine geometry and yields the higher dimensional analogue of Siegel's theorem in dimension one. Namely, the complement of $2n+1$ hyperplanes, defined over $\mathbb{Q}$, in $\mathbb{P}^n$ admits only finitely many integral points.
- Corvaja and Zannier extends Schmidt's subspace theorem for hyperplanes to the case of hypersurfaces of degree d. The proof is translated by Ru and yields the SMT for hypersurfaces.

Essentially all of the corresponding results whose proofs, though not directly translatable at this time, are quite similar in spirit. Among these are the following results, considered to be major milestones:

- (Hyperbolic Geometry) Bloch's Theorem (Bloch 1926) Let X be a subvariety of an abelian variety. Assume that X is not a translate of an abelian subvariety. Then every holomorphic map $f : \mathbb{C} \to X$ is algebraically degenerate.
- (Hyperbolic Geometry) Lang's Conjecture (established in the affirmative by Siu and Yeung) Let A be an abelian variety and D an ample divisor in A. Then $A \setminus D$ is hyperbolic.
- (Arithmetic Geometry) The arithmetic analogues of both Bloch's and Siu-Yeung's theorems are due to Faltings. (These are the generalization of the classical results about elliptic curves mentioned earlier.)

Open Problems

- The higher dimensional extension of Faltings Theorem. To formulate this precisely we need the following terminology. A projective variety X is said to be Mordellic if the number of $\mathbb{K}$ rational points is finite for an algebraic number field $\mathbb{K}$. A variety X is said to be algebraically hyperbolic if every rational map f from a group variety into X is constant. The following conjecture is due to Lang:

  Let X be a projective variety. Then X is Mordellic if and only if X is algebraically hyperbolic if and only if every subvariety of X is of general type.

- The SMT in Nevanlinna Theory mentioned earlier is actually a weak version. A stronger version, due to Ahlfors, known as the Ramified Second main Theorem asserts that

Theorem (Ahlfors)

Let $L_1, \ldots, L_q$ be q hyperplanes in general position in $\mathbb{P}^n$. Then, for any linearly non-degenerate holomorphic map $f : \mathbb{C} \to \mathbb{P}^n$,

$$N(\mathrm{Ram}\, f, r) + \sum_{i=1}^{q} \int_0^{2\pi} \frac{1}{|L(f(re^{\sqrt{-1}\theta}))|} \frac{d\theta}{2\pi}$$
$$\leq (n+1+\epsilon) T_f(r) + O(\log r)$$

where the estimate holds for all r outside an exceptional set of finite Lebesgue measure.

Roughly speaking the term $N(\mathrm{Ram}\, f, r)$ counts the number of tangents to the image of f with order of contact exceeding the dimension n. For example when $n = 1$ this counts the number of points of inflection.

At this time it is not known how exactly to extend Roth-Schmidt's theorem in an analogous fashion. Some formulation of the analogue of the ramified term were proposed by Lang and Vojta. However, there has been no substantial progress so far.

Some of the results in hyperbolic geometry can only be established with the stronger form of SMT. The arithmetic analogues of all of these remain open.

- For p-adic numbers the concept of holomorphic functions also makes sense; namely, convergent power series with coefficients in $\mathbb{C}_p$:

$$\sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}_p, |a_n|_p \to 0.$$

There is a well-developed p-adic Nevanlinna Theory and p-adic hyperbolic geometry. In general, the results in the p-adic case are at least as strong, and often stronger, than the complex analogue. The relationship between p-adic Diophantine geometry and hyperbolic geometry remains, to my knowledge, relatively unexplored.

- Complex hyperbolic geometry plays an important role in the theory of complex dynamics, currently a very active field in complex analysis. Recently, p-adic dynamics has also received much attention as it has a p-adic hyperbolicity as well as an arithmetic component. The exploration of the arithmetic aspect of complex dynamics is, at this time, still at the beginning stage. Hopefully this will also receives the attention it deserves.

Special Program in AG/HG at Fields

The purpose of the special program "Arithmetic Geometry and Hyperbolic Geometry" in the coming semester, sponsored by the Fields Institute, is to bring researchers in these two area together providing them opportunities for interactions. The following organized activities are scheduled:

- Two major workshops, one in Arithmetic Geometry and one in Hyperbolic Geometry. Lectures will be delivered by some of the leading researchers in the respective fields.
- Distinguished Lecture Series, by Y.-T. Siu (Harvard) in Complex geometry and Coxeter Lecture Series by S. Zhang (Columbia) in Arithmetic geometry.
- Two mini-workshops, one in complex dynamics and one in p-adic dynamics.

We'll also have many distinguished short term visitors from around the world, who will be giving many seminar talks during their visit here in the Fall.

We believe strongly that a research program should include an educational component. This is a very substantial part of our program. We offer

- Three mini-courses for graduate students:
  Arithmetic Geometry (Gillet),
  Diophantine Approximation and Nevanlinna Theory (Ru),
  Hyperbolic Geometry (Wong).

THANK YOU!