

Proper infiniteness and K_1 -injectivity

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Infiniteness

Definition. [Cuntz]

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- p infinite $\Leftrightarrow \exists \mathcal{T}_1 = \langle s_1 \rangle \hookrightarrow pAp$ unital $*$ -homom.
- p finite $\Leftrightarrow p$ is not infinite
- p properly infinite $\Leftrightarrow \exists \mathcal{T}_2 = \langle s_1, s_2 \rangle \hookrightarrow pAp$ unital $*$ -homom.

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Proposition. [Cuntz]

If there exists a full prop. inf. proj. $p \in \mathcal{P}_{\text{full prop. inf.}}(A)$,

- $\forall g \in K_0(A), \exists q \in \mathcal{P}_{\text{full prop. inf.}}(A)$ with $[q] = g$
- If $p, q \in \mathcal{P}_{\text{full prop. inf.}}(A)$, $p \sim q \Leftrightarrow [p] = [q]$ in $K_0(A)$

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$$\mathbf{A}_x := \mathbf{A} / [C_x(X).\mathbf{A}] \quad \text{and} \quad a \in A \longmapsto \mathbf{a}_x \in \mathbf{A}_x$$

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$$\begin{aligned} x \mapsto \|a_x\| &= \|a + C_x(X)A\| \\ &= \inf \{ \| [1 - f + f(x)]a \|, f \in C(X) \} \end{aligned}$$

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Definition. The $C(X)$ -algebra A is **continuous**
(or is a **continuous C^* -bundle over X with fibres A_x**)
iff

$\forall a \in A$, the function $x \mapsto \|a_x\|$ is continuous.

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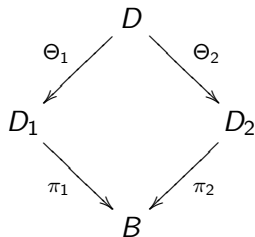
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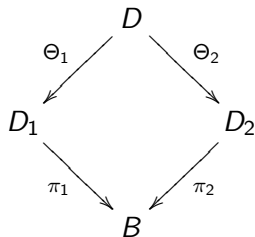
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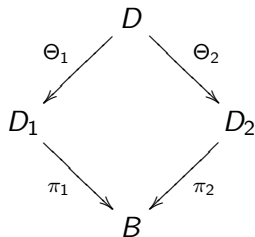
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Note that **Q2** $\Rightarrow$ **Q1**

Stability of proper infiniteness under deformation (2)

Let $\sigma_1 : \mathcal{T}_3 \rightarrow D_1$ and $\sigma_2 : \mathcal{T}_3 \rightarrow D_2$ be unital $*$ -homom.

Then $\boxed{v = \sum_{j=1}^2 (\pi_1 \sigma_1)(s_j) (\pi_2 \sigma_2)(s_j)^*}$ partial isom. in B
s.t. $(\pi_1 \sigma_1)(s_j) = v (\pi_2 \sigma_2)(s_j)$

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Lemma.

Let $v \in B$ partial isom. s.t. $1 - vv^*$ and $1 - v^*v$ are full + prop. inf.

Then $\boxed{\exists u \in \mathcal{U}(B)}$ s.t. a) $v = u v^* v$ and
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$$\Rightarrow \exists \tilde{\sigma} = (\sigma_1, \sigma'_2) : \mathcal{T}_2 \rightarrow D = D_1 \oplus_B D_2$$

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Lemma. [B., Rohde, Rørdam]

Let B be a unital C^* -algebra.

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$$\tilde{u} = \begin{pmatrix} u & \\ & u \end{pmatrix} \in \mathcal{U}(M_2(B)) \text{ and } \tilde{p} = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \in \mathcal{P}(M_2(B)).$$

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$\rightsquigarrow$ If A unital continuous $C(X)$ -algebra with prop. inf. fibres,

$\exists n \geq 1$ s.t. $M_{2^n}(A)$ prop. inf.

K_1 -injectivity

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► In diagramme ($\clubsuit$), B K_1 -injective $\Rightarrow D$ prop.inf.
Hence **Q3** $\Rightarrow$ **Q2**

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Proposition. Let B be a unital **prop. inf.** C^* -algebra

and $v \in \mathcal{U}(B)$ a unitary s.t. $\begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in $\mathcal{U}_2(B)$

i.e. $\exists u \in C([0, 1], \mathcal{U}_2(B))$ with $u(0) = \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}$, $u(1) = 1_2$.

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Proof. Put $w_t = u_t \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (0 \leq t \leq 1)$

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But are these projections equivalent in $C(\mathbb{T}; M_2(B))$?

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(ii) p is **prop. inf.** in $C(\mathbb{T}; M_2(B))$

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Hence **Q1** $\Rightarrow$ **Q3**

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- d) All unital prop. inf. C^* -alg. are K_1 -injective.
- e) $\mathcal{O}_\infty * \mathcal{O}_\infty$ is K_1 -injective

Concluding remarks

Proposition. [Cuntz]

Every purely infinite simple unital C^* -algebra is K_1 -injective.

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- X be a contractible compact Hausdorff space,
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Then p is prop. inf. $\Leftrightarrow p_{x_0}$ is prop. inf.