Proper infiniteness and K_1-injectivity

E. Blanchard	(CNRS)
R. Rohde	(IMADA)
M. Rørdam	(IMADA)

Infiniteness

Definition. [Cuntz]

$$\mathcal{T}_n = \langle s_1, \ldots, s_n \,;\, \forall\, k \leq n,\ s_k^* s_k = 1 \geq \sum_{j=1}^{n} s_j s_j^* \ \rangle$$

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- p infinite $\Leftrightarrow \exists\, \mathcal{T}_1 = \langle s_1 \rangle \hookrightarrow pAp$ unital $*$-homom.
- p finite $\Leftrightarrow p$ is not infinite
- p properly infinite $\Leftrightarrow \exists\, \mathcal{T}_2 = \langle s_1, s_2 \rangle \hookrightarrow pAp$ unital $*$-homom.

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Proposition. [Cuntz]
If there exists a full prop. inf. proj. $p \in \mathcal{P}_{\text{full prop.inf.}}(A)$,

– $\forall\, g \in K_0(A)$, $\exists\, q \in \mathcal{P}_{\text{full prop.inf.}}(A)$ with $[q] = g$
– If $p, q \in \mathcal{P}_{\text{full prop.inf.}}(A)$, $\boxed{p \sim q \Leftrightarrow [p] = [q] \text{ in } K_0(A)}$

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$$\mathbf{A_x := A/[C_x(X).A]} \quad \text{and} \quad a \in A \longmapsto \mathbf{a_x} \in \mathbf{A_x}$$

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$$\begin{aligned} x \mapsto \|a_x\| \quad &= \|a + C_x(X)A\| \\ &= \inf\{\|\,[1 - f + f(x)]a\|, f \in C(X)\} \end{aligned}$$

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Definition. The $C(X)$-algebra A is **continuous**
(or is a **continuous C*-bundle over X with fibres A_x**)
iff
$\forall\, a \in A$, the function $x \mapsto \|a_x\|$ is continuous.

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with D_1, D_2 unital prop. inf. C*-alg.

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Note that **Q2** $\Rightarrow$ **Q1**

Stability of proper infiniteness under deformation (2)

Let $\sigma_1 : \mathcal{T}_3 \to D_1$ and $\sigma_2 : \mathcal{T}_3 \to D_2$ be unital $*$-homom.

Then $\boxed{v = \sum_{j=1}^{2} (\pi_1\sigma_1)(s_j)\,(\pi_2\sigma_2)(s_j)^*}$ partial isom. in B

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Lemma.

Let $v \in B$ partial isom. s.t. $1 - vv^*$ and $1 - v^*v$ are full + prop. inf.

Then $\boxed{\exists\, u \in \mathcal{U}(B)}$ s.t. a) $v = u\,v^*v$ and

b) $[u] = 0$ in $K_1(B)$.

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$$\Rightarrow \exists\, \widetilde{\sigma} = (\sigma_1, \sigma_2') : \mathcal{T}_2 \to D = D_1 \oplus_B D_2$$

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$\rightsquigarrow$ If A unital continuous $C(X)$-algebra with prop. inf. fibres, $\exists\, n \geq 1$ s.t. $M_{2^n}(A)$ prop. inf.

K_1-injectivity

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$\mathcal{U}(B), \mathcal{U}_n(B) = \mathcal{U}(M_n(B)),$ $\qquad$ $\mathcal{U}^0(B), \mathcal{U}^0_n(B)$

$$\begin{array}{ccc} \mathcal{U}_n(B) & \rightarrow & \mathcal{U}_{n+1}(B) \\ u & \mapsto & u \oplus 1 \end{array} \qquad \text{and} \qquad K_1(B) = \lim_n \mathcal{U}_n(B)/\mathcal{U}^0_n(B)$$

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- In diagramme (♣), B K_1-injective $\Rightarrow$ D prop.inf.
  Hence **Q3** $\Rightarrow$ **Q2**

Equivalence (1)

Proposition. Let B be a unital **prop. inf.** C*-algebra and $v \in \mathcal{U}(B)$ a unitary s.t. $\begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in $\mathcal{U}_2(B)$

i.e. $\exists\, u \in C([0,1], \mathcal{U}_2(B))$ with $u(0) = \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}$, $u(1) = 1_2$.

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Proof. Put $w_t = u_t \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad (0 \le t \le 1)$ $\square$

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But are these projections equivalent in $C(\mathbb{T}; M_2(B))$?

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d) All unital prop. inf. C*-alg. are K_1-injective.

e) $\mathcal{O}_\infty * \mathcal{O}_\infty$ is K_1-injective

Concluding remarks

Proposition. [Cuntz]

Every $\boxed{\text{purely infinite simple unital}}$ C*-algebra is K_1-injective.

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Let
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Then p is prop. inf. $\Leftrightarrow$ p_{x_0} is prop. inf.