Semiprojectivity of non-commutative CW-complexes

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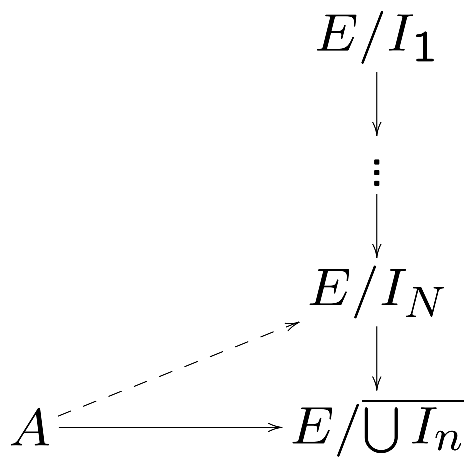
S. Eilers, T.A. Loring and G.K. Pedersen: Stability of anticommutation relations. An application of noncommutative CW complexes. Journal für die reine und angewandte Mathematik **499** *(1998), 101–143.*

Definition [Blackadar 1985]

A is **semiprojective** whenever



B. Blackadar: Semiprojectivity in simple C^*-algebras. Proceedings of the US-Japan Seminar held at Kyushu University, Fukuoka, June 7–11, 1999.

Semiprojective examples

$M_n(\mathbb{C})$	$e_{ij}e_{kl} = \delta_{jk}e_{il}, e_{ij}^* = e_{ji}$	$\surd$
$C([0,1])$	$x = x^*$	$\surd$
$C(S^1)$	$uu^* = u^*u = 1$	$\surd$
$\mathbb{I}_{p,q}$	$\{f : [0,1] \to \mathbf{M}_{pq}(\mathbb{C}) \mid$ $f(0) \in \mathbf{M}_p(\mathbb{C}), f(1) \in \mathbf{M}_q(\mathbb{C})\}$	$\surd$
$q\mathbb{C}$	$\{f : (0,1] \to \mathbf{M}_2(\mathbb{C}) \mid f(1)$ is diagonal$\}$	$\surd$
$\mathcal{O}_n$	$\sum_{i=1}^n s_i s_i^* = 1, s_i s_i^* s_i = s_i$	
$\mathcal{T}$	$s^*s = 1$	

Stable relations

$C^*(\mathcal{G} \mid \mathcal{R})$ is semiprojective when $\forall \epsilon \exists \delta$:

- (g_i) δ-satisfies $\mathcal{R}$ in A
- $(\pi(g_i))$ satisfies $\mathcal{R}$ in A/I

there exist h_i, $\|h_i - g_i\| < \epsilon$ such that

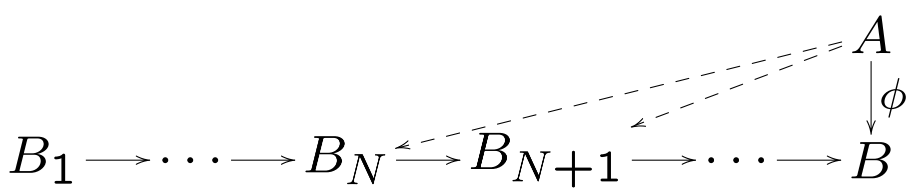
- (h_i) satisfies $\mathcal{R}$ in A
- $\pi(g_i) = \pi(h_i)$.

Inductive limits

When A is semiprojective then for any

$$\phi : A \to B = \varinjlim B_n$$

we get



with all maps from A to B homotopic — and pointwise converging — to ϕ.

NB: $\mathbb{K}$ is not semiprojective.

Classification seems to require an "ample supply" of semiprojective C^*-algebras.

Theorem [Spielberg 01]

Any purely infinite C^*-algebra A with $K_*(A)$ finitely generated and with $\operatorname{tor} K_1(A) = 0$ is semiprojective.

Theorem [E-Loring-Pedersen 98]

Any pullback

$$\begin{array}{ccc} * & \longrightarrow & F_0 \\ \downarrow & & \downarrow \\ C([0,1],F_1) & \xrightarrow[\partial]{} & F_1 \oplus F_1 \end{array}$$

is semiprojective when $\dim F_0, \dim F_1 < \infty$.

(-: *Closure properties* :-)

A, B semiprojective $\iff A \oplus B$ semiprojective.

A semiprojective $\iff A^{\sim}$ semiprojective.

A semiprojective $\Rightarrow \mathbf{M}_n(A)$ semiprojective. [$\Leftarrow$ when A is unital]

)-: *Closure properties* :-(

$$0 \longrightarrow A \longrightarrow X \longrightarrow B \longrightarrow 0$$

X semiprojective $\not\Rightarrow$ A semiprojective.

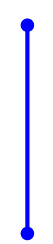
X semiprojective $\not\Rightarrow$ B semiprojective.

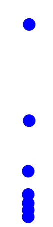
A and B semiprojective $\not\Rightarrow$ X semiprojective.

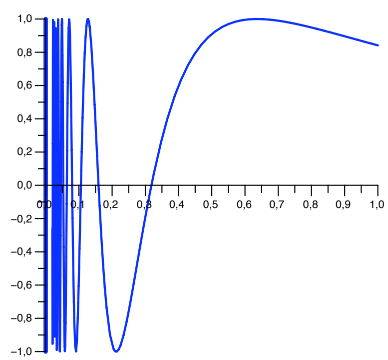
Counterexamples

$C(X)$ semiprojective $\Rightarrow X$ is an absolute neighborhood retract.

$C(X)$ semiprojective $\Rightarrow X$ has no closed set homeomorphic to $[0,1]^k$, $k \geq 2$.







|-: *Probable closure results* :-|

Say A, B and C are semiprojective. Then so is X when

(1)

$$0 \longrightarrow A \longrightarrow X \longrightarrow B \longrightarrow 0 \qquad\qquad \dim B < \infty$$

(1') [Blackadar]

$$0 \longrightarrow A \longrightarrow X \underset{\longrightarrow}{\overset{\dashleftarrow}{}} \mathbb{C} \longrightarrow 0$$

(2) Pullback over proper maps α,β:

$$\begin{array}{ccc} X & \longrightarrow & A \\ \downarrow & & \downarrow \alpha \\ B & \xrightarrow[\beta]{} & C \end{array}$$

Theorem [ELP98]

Any pullback

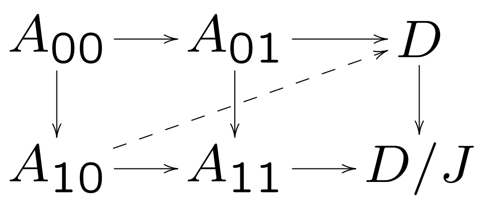
$$C([0,1],F_1)\oplus_{F_1^2} F_0$$

is semiprojective when $\dim F_0, \dim F_1 < \infty$.

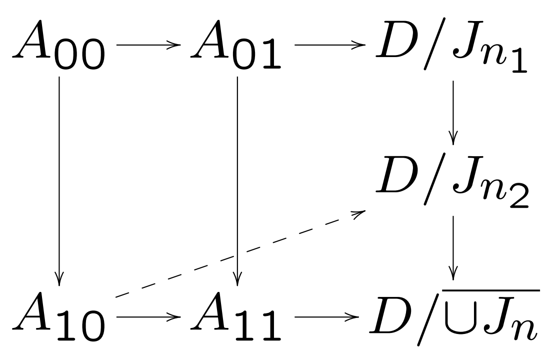
Key technical concepts in the proof:

- Conditionally (semi)projective diagram
- Corona extendible map

Conditionally projective diagram:



Conditionally semiprojective diagram:



Observation A is semiprojective when the diagram

$$\begin{array}{ccc} 0 & = & 0 \\ \downarrow & & \downarrow \\ A & = & A \end{array}$$

is conditionally semiprojective.

Lemma [ELP98]
For $F_1 \subseteq F_2$ with $\dim F_2 < \infty$ consider the canonical map

$$\phi : F_1 \to \widehat{\mathbf{T}}(F_1, F_2) = \{f \in C([0,1], F_2) \mid f(0) \in F_1\}.$$

The diagram

$$\begin{array}{ccc} F_1 & = & F_1 \\ \phi\downarrow & & \downarrow\phi \\ \widehat{\mathbf{T}}(F_1, F_2) & = & \widehat{\mathbf{T}}(F_1, F_2) \end{array}$$

is conditionally projective.

Theorem [ELP98]

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & X & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \chi} & & \| & & \\ 0 & \longrightarrow & A_1 & \longrightarrow & X_1 & \longrightarrow & F & \longrightarrow & 0 \end{array}$$

Suppose $\dim(F) < \infty$ and $\alpha(A)$ is an ideal of A_1 having a unit there. Then

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A_1 \\ \downarrow & & \downarrow \\ X & \xrightarrow[\chi]{} & X_1 \end{array}$$

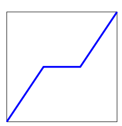
is conditionally semiprojective.

Definition [Loring-Pedersen]

A morphism $\theta : A \to B$ (necessarily injective) is **corona extendible** when

$$\begin{array}{ccc} A & \longrightarrow & M(E)/E \\ {\scriptstyle\theta}\downarrow & \nearrow & \\ B & & \end{array}$$

Observation Kasparov's technical theorem essentially amounts to corona extendibility of any map of the form



$$\otimes\, \mathrm{id} : C[0,1] \otimes D \to C[0,1] \otimes D$$

Proposition [ELP98]

Given a commutative diagram of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & X & \longrightarrow & B & \longrightarrow & 0 \\ & & \| & & \downarrow \chi & & \downarrow \beta & & \\ 0 & \longrightarrow & A & \longrightarrow & X_1 & \longrightarrow & B_1 & \longrightarrow & 0 \end{array}$$

If

$$\begin{array}{ccc} B & = & B \\ \beta \downarrow & & \downarrow \beta \\ B_1 & = & B_1 \end{array}$$

is conditionally projective, then χ is corona extendible.