Applications of the Elliott program

Huaxin Lin

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at the Fields

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This gives a counter-example of Powers and Sakai conjecture for AF-algebras
One of keys to the proof: $(\otimes_{\mathbb{Z}} A) \rtimes_\sigma \mathbb{Z}$ has tracial rank zero, where σ is the two-sided shift. So classification theory can be applied.

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Since $K_1(A_\theta \rtimes \mathbb{Z}/4\mathbb{Z}) = \{0\}$ and $K_0(A_\theta \rtimes \mathbb{Z}/4\mathbb{Z})$ is torsion free, applying classification program, one has the following:

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Corollary

(S. Walters) *For irrational numbers θ in a dense G_δ set, the C^*-algebra $A_\theta \rtimes \mathbb{Z}/4\mathbb{Z}$ is AF.*

(Existence Theorem) *Let A and B be two unital separable amenable C^*-algebras. If*

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(Uniqueness Theorem) Let $\phi, \psi : A \to B$ be two unital monomorphisms. If

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then there exists a unitary $u \in B$ such that

$$\mathrm{ad}\, u \circ \phi \approx \psi.$$

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for all $\tau \in T(A)$.

$\star$ A Kishimoto problem:
Let A be a unital simple A$\mathbb{T}$-algebra of real rank zero and let α be an approximately inner automorphism (or α^m be approximately inner for some $m \geq 1$). Suppose also that α has some Rokhlin property.

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Is $A \rtimes_\alpha \mathbb{Z}$ simple A$\mathbb{T}$-algebra ?

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Let A be a unital simple C^*-algebra and let $\alpha \in Aut(A)$. Then α has the *cyclic* tracial Rokhlin property if the following holds: For any $\epsilon > 0$, any finite subset $\mathcal{F}$, any $n \in \mathbb{N}$ and any non-zero element $a \in A_+$, there are mutually orthogonal projections $e_1, e_2, ..., e_n \in A$ (with $e_{n+1} = e_0$) such that

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Let A be a C^*-algebra and let $T(A)$ be the tracial state space. Denote by $\rho_A : K_0(A) \to Aff(T(A))$ be the positive homomorphism defined by

$$\rho_A([p]) = \tau \otimes Tr(p) \text{ for all } p \in M_\infty(A).$$

Theorem

(L-Osaka, and L-2006) *Let A be a unital separable simple C^*-algebra with tracial rank zero and let $\alpha \in Aut(A)$ be an automorphism such that $\alpha_{*0}^m|_G = \mathrm{id}|_G$, where $G \subset K_0(A)$ is a subgroup for which $\rho_A(G)$ is dense in $\rho_A(K_0(A))$ for some m and α has cyclic tracial Rokhlin property,*

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Corollary

Let A be a unital simple AH-algebra with real rank zero and with slow dimension growth and let α be an automorphism on A. Suppose that α has tracial Rokhlin property and suppose that that $\alpha_{*0}^m|_G = \mathrm{id}_G$ for some subgroup $G \subset K_0(A)$ for which $\rho_A(G)$ is dense in $\rho_A(K_0(A))$. Then A is again a unital simple AH-algebra (with real rank zero and slow dimension growth).

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AF-embedding

Let X be a compact metric space and α be a homeomorphism on X.

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(N. Brown –1998) *Let A be a unital AF algebra and let $\alpha \in Aut(A)$. Then TFAE:*

(1) *$A \rtimes_\alpha \mathbb{Z}$ can be embedded into an AF-algebra;*

(2) *$A \rtimes_\alpha \mathbb{Z}$ is quasidiagonal;*

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(3) *$A \rtimes_\alpha \mathbb{Z}$ is stably finite and*

(4) *if $x \in K_0(A)$, $\alpha_*(x) \le x$, then $\alpha_*(x) = x$.*

Theorem

(N. Brown) *Let A be a UHF-algebra and $\alpha : \mathbb{Z}^n \to Aut(A)$ be a homomorphism. Then there is an AF-algebra B and a monomorphism $\phi : A \rtimes_\alpha \mathbb{Z}^n \to B$.*

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Theorem

(H. Matui) *Let A be a unital simple $A\mathbb{T}$-algebra of real rank zero and let $\alpha \in Aut(A)$. Then there is always a unital simple AF-algebra B and a unital monomorphism $\phi : A \rtimes_\alpha \mathbb{Z} \to B$.*

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Proposition

Let C be a unital AH-algebra. Then C can be embedded into a unital simple AF-algebra if and only if C admits a faithful tracial state.

Theorem

(L–2007) *Let C be a unital AH-algebra and let $\alpha \in Aut(C)$. Then $C \rtimes_\alpha \mathbb{Z}$ can be embedded into a unital simple AF-algebra if and only if C admits a faithful α-invariant tracial state.*

Denote by $\mathcal{U}$ the universal UHF-algebra $\mathcal{U} = \otimes_{n\geq 1} M_n$.
Let $\{e_{i,j}^{(n)}\}$ be the canonical matrix units for M_n. Let $u_n \in M_n$ be the unitary matrix such that $\mathrm{ad}\, u_n(e_{i,i}^{(n)}) = e_{i+1,i+1}^{(n)}$ (modulo n). Let $\sigma = \otimes_{n\geq 1} \mathrm{ad}\, u_n \in Aut(\mathcal{U})$ be the shift.

Definition

Let A be a unital separable C^*-algebra and let B be a unital C^*-algebra. Suppose that $h : A \to B$ is a unital monomorphism. We say h satisfies property (H) for a positive number L, if the following holds: For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset A$, if there is a continuous path of unitaries $\{v_t : t \in [0,1]\}$ in B with $v(1) = 1$ such that

$$\|v_t \mathrm{ad}\, w \circ h(a) - \mathrm{ad}\, w \circ h(a) v_t\| < \delta \text{ for all } a \in \mathcal{G} \text{ and } t \in [0,1],$$

where $w \in B$ is a unitary, there is a continuous path of unitaries $\{u_t : t \in [0,1]\}$ such that

$$u_0 = v_0, \;\; v_1 = 1 \quad \text{and} \;\; \|u_t \mathrm{ad}\, w \circ h(a) - \mathrm{ad}\, w \circ h(a) u_t\| < \epsilon$$

for all $a \in \mathcal{F}$ and all $t \in [0,1]$. Moreover,

$$\|u_t - u_{t'}\| \le L|t - t'| \text{ for all } t, t' \in [0,1].$$

The following is a version of a lemma of H. Matui:

Lemma

Let A be a unital separable C^*-algebra, let B be a unital simple separable C^*-algebra of tracial rank zero, let $\phi : A \to B$ be a unital embedding, let $\alpha \in Aut(A)$ and let $\beta_0 \in Aut(B)$ be automorphisms. Suppose that $\phi \circ \alpha$ has the property (H) for a positive number $L > 0$ and suppose that there is a continuous path $\{v(t) : t \in [0, \infty)\}$ of unitaries in $B \otimes \mathcal{U}$ satisfying the following:

$$\lim_{t\to\infty} \|\phi \circ \alpha(a) - \mathrm{ad}\, v(t) \circ \beta \circ \phi(a)\| = 0$$

for all $a \in A$, where $\beta = \beta_0 \otimes \sigma$ and $\mathcal{U}$ and σ are defined before. Then there are unitaries w, $V_n \in B \otimes \mathcal{U}$ $(n = 1, 2, ...,)$ such that

$$\mathrm{ad}\, \phi' \circ \alpha = \mathrm{ad}\, w \circ \beta \circ \phi',$$

where $\phi'(a) = \lim_{n\to\infty} \mathrm{ad}\, (V_1 V_2 \cdots V_n) \circ \phi$.

Definition

Let C and B be two unital C^*-algebras. Suppose that $\phi, \psi : C \to B$ be two unital monomorphisms. We say that ϕ and ψ are asymptotically unitarily equivalent if there is a continuous path of unitaries $\{u_t : t \in [0, \infty)\} \subset B$ such that

$$\lim_{t\to\infty} \mathrm{ad}\, u_t \circ \psi(c) = \phi(c) \text{ for all } c \in C.$$

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Mapping torus. Let C and A be unital C^*-algebras and let $\phi_1, \phi_2 : C \to A$ be two unital monomorphisms. Set

$$M_{\phi_1,\phi_2} = \{f \in C([0,1], A) : f(0) = \phi_1(c), f(1) = \phi_2(c) \ \text{for some} \ c \in C\}.$$

We obtain a short exact sequence:

$$0 \longrightarrow SA \longrightarrow M_{\phi_1,\phi_2} \longrightarrow C \longrightarrow 0.$$

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Suppose that there exists a continuous path $\{u(t)\}$ so that $\lim_{t\to\infty} \mathrm{ad}\, u(t) \circ \phi_1(c) = \phi_2(c)$ for all $c \in C$. Define $\theta : C \to M_{\phi_1,\phi_2}$ by

$$\theta(c)(t) = u(t)^* \phi_1(c) u(t) \ \text{for all} \ t \in [0,1)$$

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Let $u \in M_{\phi_1, \phi_2}$ be a unitary such that $t \mapsto u(t)$ is piecewise C^1. For $\tau \in T(A)$, we define

$$\rho_\tau(u) = \frac{1}{2\pi} \int_0^1 \tau\left(\frac{du(t)}{dt} u(t)^*\right) dt.$$

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$$\rho_\tau(uv) = \rho_\tau(u) + \rho_\tau(v).$$

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$$\rho_\tau(u) = \int_0^1 \tau(\frac{du(t)}{dt})dt = \tau(h(1)) - \tau(h(0)) = 0,$$

since $\tau \circ \phi_1 = \tau \circ \phi_2$. From here one concludes that ρ is constant on each connected component of C^1-unitary group of M_{ϕ_1,ϕ_2}.

Thus, we obtain a homomorphism $\rho_\tau : K_1(M_{\phi_1,\phi_2}) \to \mathbb{R}$.

Consequently, we obtain a homomorphism $R_{\phi_1,\phi_2} : K_1(M_{\phi_1,\phi_2}) \to Aff(T(A))$.

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It splits. If $p \in A$ is a projection, $\imath_{*0}([p]) = [u]$ can be defined by

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We have the following commutative diagram:

$$\begin{tikzcd} K_0(A) \arrow[rr, "\imath_*"] \arrow[dr, "\rho_A"'] & & K_1(M_{\phi_1,\phi_2}) \arrow[dl, "R_{\phi_1,\phi_2}"] \\ & Aff(T(A)), & \end{tikzcd}$$

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where $\rho_A([p])(\tau) = \tau(p)$ for each $\tau \in T(A)$.

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where $\rho_A([p])(\tau) = \tau(p)$ for each $\tau \in T(A)$. Moreover, R_{ϕ_1,ϕ_2} extends ρ_A.

When $[\phi_1] = [\phi_2]$ in $KK(C, A)$, there exists a θ so that the following splits

$$0 \to \underline{K}(SA) \to \underline{K}(M_{\phi_1,\phi_2}) \to \underline{K}(C) \to 0.$$

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$$0 \to \underline{K}(SA) \to \underline{K}(M_{\phi_1,\phi_2}) \to \underline{K}(C) \to 0.$$

We write

$$\tilde{\eta}_{\phi_1,\phi_2} = 0$$

if θ maps $K_1(C)$ into $\ker R_{\phi_1,\phi_2}$, or we say a rotation map vanishes. When $K_1(C)$ is free, if

$$R_{\phi_1,\phi_2}(K_1(C)) \subset \rho_A(K_0(A)),$$

then $\tilde{\eta}_{\phi_1,\phi_2} = 0$.
If $[\phi_1] = [\phi_2]$ in $KK(C, A)$, $R_{\phi_1,\phi_2} \circ \theta(K_1(C)) \subset \rho_A(K_0(A))$ and

$$0 \to \ker \rho_A \to G \to R_{\phi_1,\phi_2} \circ \theta(K_1(C)) \to 0$$

splits (where $G = \rho_A^{-1}(R_{\phi,\psi} \circ \theta(K_1(C))$). then $\tilde{\eta}_{\phi_1,\phi_2} = 0$.

Proposition

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$$\lim_{\to\infty} \mathrm{ad}\, u(t) \circ \phi_1(a) = \phi_2(a) \ \text{ for all } a \in A$$

for some continuous and piecewise smooth path of unitaries $\{u(t) : t \in [0, \infty)\} \subset B$.

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for some continuous and piecewise smooth path of unitaries $\{u(t) : t \in [0, \infty)\} \subset B$. Then

$$[\phi_1] = [\phi_2], \ \ \tau \circ \phi_1 = \tau \circ \phi_2 \ \textit{for all } \tau \in T(A)$$
$$\textit{and } \tilde{\eta}_{\phi_1,\phi_2} = 0.$$

Theorem

(Kishimoto and Kumjian) *Let A be a unital simple $A\mathbb{T}$-algebra with real rank zero and let α, $\beta \in Aut(A)$. Then α and β are asymptotically unitarily equivalent if and only if*

$$[\alpha] = [\beta] \text{ in } KK(A, A) \text{ and } \tilde{\eta}_{\alpha,\beta} = \{0\}.$$

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This is a uniqueness theorem.

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Then $\phi_1, \phi_2 : C \to A$ are asymptotically unitarily equivalent if and only if

$$\begin{aligned} [\phi_1] &= [\phi_2] \text{ in } KK(C, A), \\ \tau \circ \phi_1 &= \tau \circ \phi_2 \text{ and} \\ \tilde{\eta}_{\phi_1,\phi_2} &= 0. \end{aligned}$$

This is a uniqueness theorem.
It is used a previously mentioned uniqueness theorem, an existence theorem as well as the Basic Homotopy Lemma.

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Definition

Let A and C be two unital C^*-algebras and let $\phi_1, \phi_2 : C \longrightarrow A$ be unital homomorphisms. We say that ϕ_1 and ϕ_2 are *strongly* asymptotically unitarily equivalent if there exists a continuous path of unitaries $\{u_t : t \in [0, \infty)\} \subset A$ such that

$$u_0 = 1_A \text{ and } \lim_{t\to\infty} \mathrm{ad}\, u_t \circ \phi_1(c) = \phi_2(c) \text{ for all } c \in C.$$

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Let A and C be unital C^*-algebras. Define

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$$[u] \in H_1(K_0(A), K_1(B)).$$

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$$\begin{aligned} [\phi_1] &= [\phi_2] \ \textit{in} \ KK(C, B), \ \tilde{\eta}_{\phi_1,\phi_2} = 0 \ \textit{and} \\ \tau \circ \phi_1 &= \tau \circ \phi_2 \ \text{for all} \ \tau \in T(B). \end{aligned}$$

Recall Voiculescu's problem

Let X be a compact metric space and let $\Gamma : \mathbb{Z}^2 \to Aut(C(X))$ be a homomorphism. When can $C(X) \rtimes_\Gamma \mathbb{Z}^2$ be embedded into an AF-algebra?

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There is a unital monomorphism $j : \mathcal{U} \times_\sigma \mathbb{Z} \to \mathcal{U}$. However, in this case, more is true. First $[\sigma] = [\mathrm{id}|_U]$ in $KK(\mathcal{U}, \mathcal{U})$ and $\tau = \tau \circ \sigma$. Since $K_1(\mathcal{U}) = \{0\}$, $K_1(M_{\mathrm{id}_{\mathcal{U}}, \sigma}) = K_0(\mathcal{U})$. In particular, there exists a continuous path of unitaries $\{v(t) : t \in [0, \infty)\}$ of $\mathcal{U}$ such that

$$\lim_{t \to \infty} v(t)^* a v(t) = \sigma(a) \text{ for all } a \in \mathcal{U}. \tag{e 0}$$

Therefore, there is a unital embedding $\phi : \mathcal{U} \rtimes_\sigma \mathbb{Z} \to \mathcal{U}$ such that

$$\tau \circ \phi = \tau. \tag{e 0}$$

Define $\psi : \mathcal{U} \rtimes_\sigma \mathbb{Z} \to \mathcal{U} \otimes \mathcal{U}$ by $\psi(a) = \phi(a) \otimes 1_{\mathcal{U}}$ for all $a \in \mathcal{U}$ and $\psi(u_\sigma) = \phi(u_\sigma) \otimes \phi(u_\sigma^*)$. Then ψ is a unital monomorphism. Denote by $s : \mathcal{U} \otimes \mathcal{U} \to \mathcal{U}$ an isomorphism with $s_{*0} = \mathrm{id}_{K_0(\mathcal{U} \otimes \mathcal{U})}$. We define $\imath : \mathcal{U} \rtimes_\sigma \mathbb{Z} \to \mathcal{U}$ by $s \circ \psi$.

Theorem

Let C be a unital AH-algebra, let $\alpha \in Aut(C)$ be an automorphism and let $A \cong A \otimes \mathcal{U}$ be a unital simple AF-algebra with a unique tracial state τ and $K_0(A) = \rho_A(K_0(A))$.

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Suppose also that

$$R_{\phi_1 \circ j_0, \phi_2 \circ j_0}(K_1(C)) \subset \rho_A(K_0(A)),$$

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where $j_0 : C \to C \rtimes_\alpha \mathbb{Z}$ is the embedding. Then there exists a sequence of unitaries $\{w_n\} \subset U(A \otimes \mathcal{U})$ such that

$$\lim_{n\to\infty} \mathrm{ad}\, w_n \circ \phi_1^{(1)}(a) = \phi_2^{(1)}(a) \ \text{for all}\ a \in C \rtimes_\alpha \mathbb{Z}.$$

where $\phi_i^{(1)} : C \rtimes_\alpha \mathbb{Z} \to A \otimes \mathcal{U}$ by $\phi_i^{(1)}(c) = \phi_i(c) \otimes 1$ for all $c \in C$ and $\phi_i^{(1)}(u_\alpha) = \phi_i(u_\alpha) \otimes \imath(u_\sigma)$, $i = 1, 2$.

Corollary

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Let C be a unital AH-algebra and let G be a finitely generated abelian group. Suppose that $\Lambda : G \longrightarrow Aut(C)$ is a homomorphism.

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Dynamical systems

Let X be a compact metric space with finite covering dimension, let $\alpha, \beta : X \to X$ be two minimal homeomorphisms.

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Let X be a compact metric space with finite covering dimension, let $\alpha, \beta : X \to X$ be two minimal homeomorphisms. Put $A_\alpha = C(X) \rtimes_\alpha \mathbb{Z}$ and $C(X) \rtimes_\alpha \mathbb{Z}$.

$$\begin{array}{ccc} K_{**}(A_\alpha) & \to & K_{**}(A_\beta) \\ \uparrow_{j_\alpha} & & \uparrow_{j_\beta} \\ K_{**}(C(X)) & \to & K_{**}(C(X)) \end{array}$$

Are α and β approximately (K-) conjugate?