$\mathbb{Z}^2$-actions on UHF algebras

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Cocycle conjugacy

Definition

Let $\alpha : G \curvearrowright A$ be an action of a countable discrete group G on a unital C^*-algebra A.
$\{u_g\}_{g\in G} \subset U(A)$ is called an α-cocycle, if

$$u_g \alpha_g(u_h) = u_{gh} \quad \forall g, h \in G.$$

Two actions $\alpha, \beta : G \curvearrowright A$ are said to be cocycle conjugate, if

$$\exists \gamma \in \mathrm{Aut}(A), \quad \exists \{u_g\}_g \ \alpha\text{-cocycle}$$

$$\mathrm{Ad}\, u_g \circ \alpha_g = \gamma \circ \beta_g \circ \gamma^{-1} \quad \forall g \in G.$$

Uniform outerness

Definition (Kishimoto 1996)

$\alpha \in \mathrm{Aut}(A)$ is said to be uniformly outer, if

$$\forall a \in A,\ \forall \text{non-zero projection } p \in A,\ \forall \varepsilon > 0$$

$$\exists \text{projections } p_1, p_2, \ldots, p_n \in A$$

$$p = \sum_{i=1}^{n} p_i, \quad \|p_i a \alpha(p_i)\| < \varepsilon \quad \forall i.$$

- If α is uniformly outer, then for any α-invariant tracial state τ, the extension $\bar{\alpha}$ on $\pi_\tau(A)''$ is outer.
- If A is purely infinite simple, then outerness implies uniform outerness.

Main theorem

$\alpha : G \curvearrowright A$ is said to be uniformly outer, if α_g is uniformly outer for all $g \in G \setminus \{e\}$.

Theorem (Katsura-M 2007)

Let A be a UHF algebra and let $\alpha, \beta : \mathbb{Z}^2 \curvearrowright A$ be uniformly outer actions.
Then, α and β are cocycle conjugate if and only if $[\alpha] = [\beta]$.

Corollary

If A is of infinite type, i.e. $A \cong A \otimes A$, then any uniformly outer actions of $\mathbb{Z}^2$ are cocycle conjugate to each other.

Background

- (Connes, Jones, Ocneanu 1980's)
  If $\alpha, \beta : G \curvearrowright R$ are outer actions of an amenable group G on the AFD II$_1$-factor R, then α is cocycle conjugate to β.
- (Herman-Ocneanu 1984)
  Rohlin property and stability of single automorphisms of AF-algebras.
- (Bratteli-Kishimoto-Rørdam-Størmer 1993)
  Rohlin property of the shift automorphism of UHF(2^∞).
- (Evans-Kishimoto 1997)
  EK intertwining argument and classification of automorphisms of AF algebras with the Rohlin property.

Classification of $\mathbb{Z}$-action

Theorem (Kishimoto 1998)

Let A be a unital simple AT algebra of real rank zero with unique tracial state τ and let $\alpha \in \mathrm{Aut}(A)$ be approximately inner. Then, the following are equivalent.

1. α *has the Rohlin property.*
2. α^m *is uniformly outer for all* $m \in \mathbb{N}$.
3. $\bar{\alpha}^m$ *on* $\pi_\tau(A)''$ *is outer for all* $m \in \mathbb{N}$.

Theorem (Kishimoto 1998)

Let A be a unital AT algebra of real rank zero. If $\alpha, \beta \in \mathrm{Aut}(A)$ have the Rohlin property and $\alpha\beta^{-1}$ is asymptotically inner, then α is cocycle conjugate to β.

Rohlin property for $\mathbb{Z}^2$-action (1)

$$\xi_1 = (1,0),\ \xi_2 = (0,1) \in \mathbb{Z}^2$$

We let $A^\infty = \ell^\infty(\mathbb{N}, A)/c_0(\mathbb{N}, A)$ and $A_\infty = A^\infty \cap A'$.

Definition (Nakamura 1999)

$\alpha : \mathbb{Z}^2 \curvearrowright A$ is said to have the Rohlin property, if for any $M \in \mathbb{N}$, there exist $R \in \mathbb{N}$, $m_r \in \mathbb{N}^2$ $(r = 1, 2, \ldots, R)$ and

projections $e_g^{(r)} \in A_\infty \quad (r = 1, 2, \ldots, R,\ g \in \mathbb{Z}^2/m_r\mathbb{Z}^2)$

such that

$$m_r \geq M, \quad \sum_{r,g} e_g^{(r)} = 1, \quad \alpha_{\xi_i}(e_g^{(r)}) = e_{g+\xi_i}^{(r)} \quad (i = 1, 2).$$

Rohlin property for $\mathbb{Z}^2$-action (2)

Theorem (Nakamura 1999)

Let A be a UHF algebra. For a $\mathbb{Z}^2$-action $\alpha : \mathbb{Z}^2 \curvearrowright A$, the following are equivalent.

- *α has the Rohlin property.*
- *α is uniformly outer.*

Theorem (Nakamura 1999)

Let A be a UHF algebra of infinite type. Any two uniformly outer $\mathbb{Z}^2$-actions of product type on A are cocycle conjugate to each other.

What is stability?

We would like to compare two actions $\alpha, \beta : G \curvearrowright A$. Suppose that there exists an α-cocycle $\{u_g\}_{g \in G}$ in A^∞ such that

$$\beta_g(a) = \mathrm{Ad}\, u_g \circ \alpha_g(a) \quad \forall a \in A,\ g \in G.$$

Stability of α implies there exists a unitary $v \in A^\infty$ such that

$$u_g = v\alpha_g(v^*) \quad \forall\ g \in G.$$

Then, we would have

$$\beta_g(a) = \mathrm{Ad}\, v \circ \alpha_g(a) \circ \mathrm{Ad}\, v^*(a) \quad \forall a \in A,\ g \in G,$$

which may induce 'conjugacy' between α and β.

Stability for $\mathbb{Z}$-action

An automorphism $\alpha \in \mathrm{Aut}(A)$ with the Rohlin property, a unitary $u \in A$ and $\varepsilon > 0$ are given.
We wish to find $v \in A$ such that $\|u - v\alpha(v^*)\| < \varepsilon$.

von-Neumann algebras setting: $v = \sum_{k=0}^{n} u_k e_k$,
where $e_0, e_1, \ldots, e_{n-1}$ are Rohlin projections for α and u_k is determined by $u_0 = 1$ and $u_k = u\alpha(u_{k-1})$.

In C^*-algebras setting, we need 'adjustment term'.

$$v = \sum_{k=0}^{n-1} u_k \alpha^k(w_k) e_k$$

w_k's are unitaries satisfying $w_k \approx w_{k-1}$, $w_0 \approx u_n$, $w_{n-1} \approx 1$,
They are obtained by a path from u_n to 1.

Stability for $\mathbb{Z}^2$-action (1)

We are given an action $\alpha : \mathbb{Z}^2 \curvearrowright A$ with the Rohlin property and an α-cocycle $\{u_n\}_{n\in\mathbb{Z}^2}$ in A (or A^∞).
Let $e_{(i,j)}$ $(0 \le i < k,\ 0 \le j < l)$ be Rohlin projections and put

$$v = \sum_{i=0}^{k-1}\sum_{j=0}^{l-1} u_{(i,j)}\alpha_{(i,j)}(w_{(i,j)})e_{(i,j)}.$$

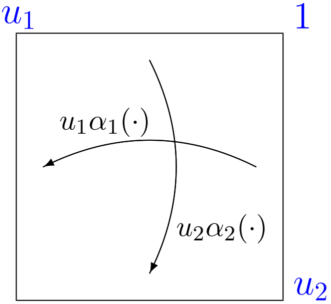
In order to obtain $\|u_{\xi_i} - v\alpha_{\xi_i}(v^*)\| < \varepsilon$ $(i = 1, 2)$, we need

$$\|w_{(i,j)} - w_{(i+1,j)}\| < \varepsilon, \quad \|w_{(i,j)} - w_{(i,j+1)}\| < \varepsilon,$$

$$\|w_{(0,j)} - u_{(k,0)}\alpha_{(k,0)}(w_{(k-1,j)})\| < \varepsilon,$$
$$\|w_{(i,0)} - u_{(0,l)}\alpha_{(0,l)}(w_{(i,l-1)})\| < \varepsilon.$$

Stability for $\mathbb{Z}^2$-action (2)

To simplify notation, we denote $\alpha_{(k,0)}$, $\alpha_{(0,l)}$, $u_{(k,0)}$, $u_{(0,l)}$ by α_1, α_2, u_1, u_2. Note that we have $u_1\alpha_1(u_2) = u_2\alpha_2(u_1)$.



How do we define a continuous map from $[0,1] \times [0,1]$ to $U(A)$?

Can we extend unitaries on the boundary to the whole square ?

— $\kappa \in K_1(SA) \cong K_0(A)$.

— Lipschitz continuity.

Homotopy lemma

Lemma

For any $C > 0$ *and* $\varepsilon > 0$, *there exists* $C' > 0$ *such that: For any* $n \in \mathbb{N}$ *and* $u : [0,1] \to U(M_n)$ *with*

- $u(0) = u(1) = 1$,
- $\mathrm{Lip}(u) \leq C$,
- $[u] \in K_1(C_0(0,1) \otimes M_n) \cong K_0(M_n)$ *is zero,*

there exists a path of self-adjoint elements $a : [0,1] \to M_n$ *satisfying the following.*

1. $\|u(t) - e^{2\pi\sqrt{-1}a(t)}\| < \varepsilon$ *for all* $t \in [0,1]$.
2. $a(0) = a(1) = 0$.
3. $\mathrm{Lip}(a) \leq C'$.

Preliminary

Let A be a UHF algebra. For each prime number p, we put

$$\begin{aligned}\zeta(p) &= \sup\{k \in \mathbb{N} \cup \{0\} \mid [1] \text{ is divisible by } p^k \text{ in } K_0(A)\} \\ &\in \{0, 1, 2, \ldots, \infty\}.\end{aligned}$$

For a $\mathbb{Z}^2$-action α on A, the invariant $[\alpha]$ will be defined as an element in

$$\prod_{p \in P(A)} \mathbb{Z}/p^{\zeta(p)}\mathbb{Z},$$

where $P(A) = \{p \mid 1 \leq \zeta(p) < \infty\}$.

Note that A is of infinite type, i.e. $A \cong A \otimes A$ if and only if $P(A)$ is empty.

Invariant (1)

Let α, β be two $\mathbb{Z}^2$-actions on A.
Take $p \in P(A)$ and let $A_0 \subset A$ be a unital subalgebra which is isomorphic to $M_{p^{\zeta(p)}}(\mathbb{C})$.
Choose a finite dimensional subalgebra $A_1 \subset A$ sufficiently large.
For each $i = 1, 2$, we can find $u_i \in U(A)$ such that

$$\beta_i(a) = \mathrm{Ad}\, u_i \circ \alpha_i(a) \qquad \forall a \in A_1.$$

Then, $x = u_1\alpha_1(u_2)(u_2\alpha_2(u_1))^*$ almost commutes with A_0.
Take a path of unitaries $h : [0, 1] \to U(A)$ from 1 to x, which is almost contained in $A \cap A_0'$. We may assume h is piecewise smooth.

Invariant (2)

Compute

$$\delta = \frac{1}{2\pi\sqrt{-1}} \int_0^1 \tau(\dot{h}(t)h(t)^*)\, dt,$$

where τ is the unique tracial state on A.

- $\Delta_\tau(x) = \Delta_\tau(u_1\alpha_1(u_2)(u_2\alpha_2(u_1))^*) = 0$, where $\Delta_\tau : U(A) \to \mathbb{R}/\tau_*(K_0(A))$ is the de la Harpe-Skandalis determinant.
- So, this value δ belongs to $\tau_*(K_0(A))$.
- If we take another path k from 1 to x, then by connecting h and k, we obtain a closed path which is almost contained in $A \cap A_0'$.
- $K_0(A)/K_0(A \cap A_0')$ is isomorphic to $\mathbb{Z}/p^{\zeta(p)}\mathbb{Z}$.

We let $[\beta, \alpha](p) \in \mathbb{Z}/p^{\zeta(p)}\mathbb{Z}$ be this value mod $\tau_*(K_0(A \cap A_0'))$.

Invariant (3)

We let

$$[\beta, \alpha] \in \prod_{p \in P(A)} \mathbb{Z}/p^{\zeta(p)}\mathbb{Z}$$

be the collection of all $[\beta, \alpha](p)$.

Lemma

In the setting above,

- $[\beta, \alpha](p)$ *does not depend on the choice of* A_0, A_1, $u_i \in U(A)$ *and* $h : [0, 1] \to U(A)$,
- $[\gamma, \alpha] = [\gamma, \beta] + [\beta, \alpha]$.

Define $[\alpha] = [\alpha, \mathrm{id}]$.
Our main theorem claims that $[\alpha]$ is the complete invariant for cocycle conjugacy of uniformly outer $\mathbb{Z}^2$-actions on the UHF algebra A.

Example

Let $A = M_2 \otimes M_3 \otimes M_5 \otimes M_7 \otimes M_{11} \otimes \dots$ and let $\lambda \in \prod_p \mathbb{Z}/p\mathbb{Z}$.
For each prime number p, choose unitaries u_p, v_p in M_p satisfying

$$u_p v_p = e^{2\pi\sqrt{-1}\lambda(p)/p} v_p u_p.$$

Define $\alpha : \mathbb{Z}^2 \curvearrowright A$ by

$$\alpha_{\xi_1} = \bigotimes_p \mathrm{Ad}\, u_p \quad \text{and} \quad \alpha_{\xi_2} = \bigotimes_p \mathrm{Ad}\, v_p.$$

Then we have $[\alpha] = \lambda$.

If $\{p \mid \lambda(p) \neq 0\}$ is an infinite set, then the action α possesses the Rohlin property automatically (Nakamura 1999).
If $\lambda(p) = 0$ for all but finitely many p, then we have to choose u_p, v_p carefully in order to make α possess the Rohlin property.

Intertwining argument (1)

Theorem (Katsura-M 2007)

Let A be a UHF algebra and let $\alpha, \beta : \mathbb{Z}^2 \curvearrowright A$ be uniformly outer actions. Then, α and β are cocycle conjugate if and only if $[\alpha] = [\beta]$.

We wish to show that $[\alpha] = [\beta]$ implies cocycle conjugacy, by using the Evans-Kishimoto intertwining argument.

Choose an increasing family of finite subsets $F_1, F_2, \dots$ of A whose union is dense in A. Let $\delta_1 > \delta_2 > \dots$ be a decreasing sequence of positive real numbers with $\delta_n \to 0$.

We will construct $\mathbb{Z}^2$-actions $\alpha^{(0)}, \beta^{(1)}, \alpha^{(2)}, \beta^{(3)}, \dots$ on A, cocycles $u^{(0)}, u^{(1)}, u^{(2)}, \dots$ and unitaries $v_0, v_1, v_2, \dots$.

Intertwining argument (2)

Put $\alpha^{(0)} = \alpha$ and $\beta^{(1)} = \beta$. We can find an $\alpha^{(0)}$-cocycle $u^{(0)}$ such that

$$\|\beta_i^{(1)}(x) - \mathrm{Ad}\, u_i^{(0)} \circ \alpha_i^{(0)}(x)\| < \delta_1 \quad \forall x \in F_1,\ \forall i = 1, 2.$$

Let $\alpha^{(2)}$ be the perturbed action of $\alpha^{(0)}$ by $u^{(0)}$, that is, $\alpha^{(2)} = \mathrm{Ad}\, u^{(0)} \circ \alpha^{(0)}$.

We can find a $\beta^{(1)}$-cocycle $u^{(1)}$ such that

$$\|\alpha_i^{(2)}(x) - \mathrm{Ad}\, u_i^{(1)} \circ \beta_i^{(1)}(x)\| < \delta_2 \quad \forall x \in F_2,\ \forall i = 1, 2.$$

Let $\beta^{(3)}$ be the perturbed action of $\beta^{(1)}$ by $u^{(1)}$, that is, $\beta^{(3)} = \mathrm{Ad}\, u^{(1)} \circ \beta^{(1)}$.

Note $\|[u_i^{(1)}, \beta_i^{(1)}(x)]\| < \delta_1 + \delta_2$ for $x \in F_1$ and $i = 1, 2$.

Intertwining argument (3)

We can find an $\alpha^{(2)}$-cocycle $u^{(2)}$ such that

$$\|\beta_i^{(3)}(x) - \mathrm{Ad}\, u_i^{(2)} \circ \alpha_i^{(2)}(x)\| < \delta_3 \quad \forall x \in F_3,\ \forall i = 1, 2.$$

Let $\alpha^{(4)}$ be the perturbed action of $\alpha^{(2)}$ by $u^{(2)}$, that is, $\alpha^{(4)} = \mathrm{Ad}\, u^{(2)} \circ \alpha^{(2)}$.
Note $\|[u_i^{(2)}, \alpha_i^{(2)}(x)]\| < \delta_2 + \delta_3$ for $x \in F_2$ and $i = 1, 2$.

In such a way, we obtain

$$\mathrm{Ad}(u^{(2k+1)} u^{(2k-1)} \dots u^{(1)}) \circ \beta \approx \mathrm{Ad}(u^{(2k)} u^{(2k-2)} \dots u^{(0)}) \circ \alpha.$$

In each step, we apply the stability to the cocycles $u^{(k)}$ and get

$$\|u_i^{(k)} - v_k \beta_i^{(k)}(v_k^*)\| < 2^{-k} \text{ and } \|u_i^{(k)} - v_k \alpha_i^{(k)}(v_k^*)\| < 2^{-k}.$$

Intertwining argument (4)

'Centrality' of $u^{(k)}$ implies 'centrality' of the unitaries v_k. Hence

$$\gamma_1 = \lim_{k\to\infty} \mathrm{Ad}(v_{2k+1}v_{2k-1}\dots v_1)$$

and

$$\gamma_0 = \lim_{k\to\infty} \mathrm{Ad}(v_{2k}v_{2k-2}\dots v_0)$$

exist in $\mathrm{Aut}(A)$.

Consequently we obtain

$$\mathrm{Ad}\, w_i^{(1)} \circ \gamma_1 \circ \beta_i \circ \gamma_1^{-1} = \mathrm{Ad}\, w_i^{(0)} \circ \gamma_0 \circ \alpha_i \circ \gamma_0^{-1} \quad \forall i = 1, 2$$

for some cocycles $w^{(0)}, w^{(1)}$, thereby completing the proof.

Remark

- Two uniformly outer $\mathbb{Z}^2$-actions α, β on a UHF algebra A are outer conjugate if and only if $[\alpha](p) = [\beta](p)$ for all but finitely many $p \in P(A)$.
- In a similar fashion, we can prove the following: Any outer actions of $\mathbb{Z}^N$ on the Cuntz algebra $\mathcal{O}_2$ are cocycle conjugate to each other.

Problem

Let A be a UHF algebra of infinite type and let $\alpha, \beta : \mathbb{Z}^N \curvearrowright A$ be uniformly outer actions. Are they cocycle conjugate ?