Strong homotopy algebra categories via co-rings over operads

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Outline

Introduction

- Joint work with Kathryn Hess (EPFL).
- *Strong homotopy* algebras are complicated structures that are best defined by *operads*.
- Operads do not describe the desired *morphisms* of SH algebras.
- Co-rings over operads "free" the morphisms from the algebras.
- The *Koszul resolution* of an operad, if it exists, is a co-ring that describes SH morphisms.
- Applies to SH associative, Lie, Poisson, Gerstenhaber ... algebras. (Any algebra described by a "quadratic Koszul operad".)

SH (Co)algebras

Canonical example

- ΩX – space of based loops on pointed space X
- $C_*(\Omega X)$ – *strongly homotopy-associative* algebra
- Notice: $H_*(\Omega X)$ is a strictly associative algebra
- We will work for the moment with graded *coalgebras*.

The cobar construction

Let (C, ∂) be a coaugmented dg coalgebra over a commutative ring R.

- The *cobar construction on* C is the associative algebra:

$$\Omega C = (T(s^{-1}\bar{C}), d).$$

- $\bar{C}$ is the cokernel of the coaugmentation $R \to C$
- d is the sum of derivations d_1 and d_2

Cobar differential

Let $V = s^{-1}\bar{C}$.

- $d_1 : V \to V$ is the derivation defined by

$$d_1(s^{-1}c) = -s^{-1}\partial(c)$$

- $d_2 : V \to V^{\otimes 2}$ is the derivation defined by

$$d_2(s^{-1}c) = (s^{-1})^{\otimes 2}\bar{\Delta}(c)$$

where $\bar{\Delta}$ is the reduced diagonal on C.

Reversing the process

- Given $(TV, d_1 + d_2)$, one may define
  - $C = R \oplus sV$
  - $\partial = sd_1s^{-1}$
  - $\bar{\Delta} = s^{\otimes 2}d_2s^{-1}$
- $(d_1 + d_2)^2 = 0$ implies that
  - $d_1^2 = 0 \Rightarrow \partial$ is a differential
  - $d_1d_2 + d_2d_1 = 0 \Rightarrow \partial$ is a coderivation
  - $d_2^2 = 0 \Rightarrow \bar{\Delta}$ is associative.
- Therefore, (C, Δ, ∂) is a dg coalgebra.

- Consider (TV, d), where d is the sum of derivations,

$$d_1 + d_2 + d_3 + \cdots ,$$

  such that $d_i : V \to V^{\otimes i}$.
- Now, $d^2 = 0$ does not imply that $d_2^2 = 0$; instead,

$$d_2^2 + d_1 d_2 + d_2 d_1 = 0.$$

- Set $C = R \oplus sV$, and, for $i \geq 2$, define $\Delta_i = s^{\otimes i} d_i s^{-1}$.
- Δ_2 is a homotopy-associative diagonal, with homotopy Δ_3.

Definition 1. $(C, \partial, \{\Delta_i\})$ is a *strongly homotopy-associative coalgebra*.

Remarks

- $(H_*(C), H_*(\Delta_2))$ is an associative coalgebra.
- We write $\tilde{\Omega}C = (TV, d)$.

Example: Adams-Hilton model

Let X be a CW-complex.

- The *Adams-Hilton model* for $C_*(\Omega X)$ is a quasi-isomorphism of algebras of the form,

$$(TV, d) \xrightarrow{\simeq} C_*(\Omega X).$$

- The SH coalgebra $W(X) := s(V, d_1)$ may be identified with the cellular complex of X.
- $H_*(W(X)) \cong H_*(X)$ (in fact, as coalgebras, if $H_*(X; R)$ is torsion-free).

- We identify a dg Lie algebra (L, ∂) with its *Chevalley-Eilenberg complex*, $C(L, \partial)$.
- $C(L, \partial) = (\Gamma(sL), d_1 + d_2)$
  - $\Gamma(sL)$ is the co-free symmetric co-algebra over sL.
  - d_1 and d_2 are coderivations, and

$$d_2(sx \cdot sy) = \pm s[x, y] \ \forall x, y \in L.$$

- Given $(\Gamma(V), d_1 + d_2)$, we set $L = s^{-1}V$, and define

$$[x, y] = \pm s^{-1} d_2(sx \cdot sy).$$

- (L, ∂) is a dg Lie algebra; the bracket satisfies the Jacobi identity because $d_2^2 = 0$.

SH Lie algebras

- Dualize: $C^*(L) = (\wedge V, d_1 + d_2)$, $V = (sL)^\sharp$.
- $\wedge V$ is the free graded commutative algebra on V.
- By considering cdga's of the form,

$$(\wedge V, d_1 + d_2 + d_3 + \cdots)$$

  we get *strong homotopy Lie algebras*.
- d_2 dualizes to define an anti-commutative bracket that satisfies the Jacobi identity up to a null homotopy, provided by d_3.
- We write $\tilde{C}^*(L) = (\wedge V, d)$.
- Data: $(L, \partial, \{[\,]_i\}_{i\geq 2})$, where $[\,]_i$ is an n-ary bracket of degree $i - 2$.

Let X be a simply connected space.

- Let $A_{PL}(X)$ be the commutative dga of polynomial forms on X.
- The Sullivan model is a quasi-isomorphism,

$$(\wedge V, d) \xrightarrow{\cong} A_{PL}(X)$$

  where d is nilpotent in a certain sense.
- $L = (s^{-1}V)^{\sharp}$ is a SH Lie algebra.
- $H_*(L) \cong \pi_*(\Omega X) \otimes \boldsymbol{Q}$. (Cartan-Serre, Milnor-Moore, Sullivan)

SH Morphisms

The SH coalgebra category

Let $\mathbf{SHC}$ be the *full* subcategory of $\mathbf{DGA}$ consisting of all objects of the form (TV, d).

- $C_1 \xrightarrow{SH} C_2$ is *really* $\tilde{\Omega}(C_1) \xrightarrow{DGA} \tilde{\Omega}(C_2)$.
- Get a sequence of morphisms $\varphi_i : C_1 \to C_2^{\otimes i}$ of degree $i - 1$.

- In particular, $\varphi_2 : \Delta\varphi_1 \simeq (\varphi_1 \otimes \varphi_1)\Delta$.

Examples

1. Co-unit of cobar-bar adjunction yields
$$\Omega(B\tilde{\Omega}C) \xrightarrow{\simeq} \tilde{\Omega}C.$$
$\Rightarrow$ every SH coalgebra is weakly SH equivalent to a strict coalgebra (Stasheff 1963).
2. Let C be a strict coalgebra. The unit of the cobar-bar adjunction,
$$C \xrightarrow{\simeq} B\Omega C,$$
is a coalgebra morphism with an SH splitting:
$$\Omega B\Omega C \to \Omega C.$$

Another example: Adams-Hilton and Adams cobar

Lifting (up to homotopy) the Adams cobar equivalence

$$\Omega C_*(X) \xrightarrow{\simeq} C_*(\Omega X)$$

through the Adams-Hilton equivalence

$$(TV, d) \xrightarrow{\simeq} C_*(\Omega X)$$

yields an SH quasi-isomorphism,

$$C_*(X) \xrightarrow[SH]{\simeq} W_*(X).$$

The SH Lie algebra category

Let $\mathbf{SHL}$ be the *full* subcategory of $\mathbf{CDGA}^{\text{op}}$ consisting of all objects of the form $(\wedge V, d)$.

- $L_1 \xrightarrow{SH} L_2$ is *really* $\tilde{C}^*(L_2) \xrightarrow{CDGA} \tilde{C}^*(L_1)$.
- Get a sequence of morphisms $\varphi_i : L_1^{\otimes i} \to L_2$ of degree $i-1$.
- In particular,

$$\varphi_2 : [,](\varphi_1 \otimes \varphi_1) \simeq \varphi_1[,].$$

Example

Let X be simply connected, finite type.

- Let (L_X, ∂) be the *Quillen dg Lie algebra model* for X.
- $C^*(L_X)$ is a model for $A_{PL}(X)$.
- By lifting, we obtain

$$C^*(L_X) \xrightarrow{\simeq} (\wedge V, d),$$

  where $(\wedge V, d)$ is any Sullivan model for $A_{PL}(X)$.
- Set $L = (s^{-1}V)^\sharp$; recall that L is an SH Lie algebra. So $L \xrightarrow[SH]{\simeq} L_X$.

Operads and Algebras

Why operads?

- SH coalgebras and SH Lie algebras are easy to describe via their "standard" resolutions (cobar and Chevalley-Eilenberg, respectively).
- other types of algebras are of interest: for example, SH Gerstenhaber or SH Poisson; they often have standard resolutions, but these get complicated.
- need a way of parametrizing (and keeping books on) sequences of n-ary operations.
- operads are (one) solution: they are "analytic monads".

Symmetric sequences

- Let $\mathbf{Sym}$ be the category of symmetric sequences of dg modules.
- Objects are sequences of chain complexes, $\mathcal{X} = \mathcal{X}(0), \mathcal{X}(1), \mathcal{X}(2), \ldots$, such that $\mathcal{X}(n)$ is a right Σ_n-module.
- n is the *arity* of $\mathcal{X}(n)$.
- Morphisms are sequences of Σ_n-equivariant chain maps.

Monoidal structures I

- The *graded tensor product* of symmetric sequences $\mathcal{X}$ and $\mathcal{Y}$ is defined by

$$(\mathcal{X} \odot \mathcal{Y})(n) = \coprod_{i+j=n} (\mathcal{X}(i) \otimes \mathcal{Y}(j)) \otimes_{\Sigma_i \times \Sigma_j} R[\Sigma_n]$$

  with action induced by natural right action of Σ_n on $R[\Sigma_n]$.
- $\mathcal{Y}^{\odot m}(n)$ is a left Σ_m, right Σ_n module.

Monoidal structures II

- The *composition product* (Joyal) is defined by

$$(\mathcal{X} \circ \mathcal{Y})(n) = \coprod_{m \geq 0} \mathcal{X}(m) \otimes_{\Sigma_m} \left(\mathcal{Y}^{\odot m}(n)\right)$$

  with unit $\mathcal{J}$ given by

$$\mathcal{J}(n) = \begin{cases} R & \text{if } n = 1, \\ O & \text{otherwise.} \end{cases}$$

- This is the formula that you get for the coefficients in the composition of two formal power series.

Operads as monoids

An *operad* is a monoid in $(\mathbf{Sym}, \circ, \mathcal{J})$.
So, an operad is a symmetric sequence $\mathcal{P}$, equipped with a unit

$$\eta : \mathcal{J} \to \mathcal{P}$$

and an associative, unital multiplication

$$\gamma : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}.$$

Let $\mathcal{P}$ be an operad. A *left $\mathcal{P}$-module* is a symmetric sequence, $\mathcal{M}$, along with an associative, unital structure morphism,

$$\lambda : \mathcal{P} \circ \mathcal{M} \to \mathcal{M}.$$

This is equivalent to a collection of associative, equivariant, unital morphisms,

$$\mathcal{P}(k) \otimes \mathcal{M}(n_1) \otimes \cdots \otimes \mathcal{M}(n_k) \to \mathcal{M}(n)$$

where $n = \sum_j n_j$.

A $\mathcal{P}$-*algebra* is a left $\mathcal{P}$-module, concentrated in arity zero, that is, a dg module A along with a sequence of associative, equivariant, unital structure maps,

$$\lambda_n : \mathcal{P}(n) \otimes A^{\otimes n} \to A.$$

We denote the category of $\mathcal{P}$-algebras and their morphisms by $\mathcal{P}$-$\mathbf{Alg}$.

Morphisms

Let A_1, A_2 be $\mathcal{P}$-algebras. A map $f : A_1 \to A_2$ is a morphism of $\mathcal{P}$-algebras if

$$\begin{array}{ccc} \mathcal{P}(n) \otimes A_1^{\otimes n} & \xrightarrow{\lambda} & A_1 \\ \downarrow{\scriptstyle 1 \otimes f^{\otimes n}} & & \downarrow{\scriptstyle f} \\ \mathcal{P}(n) \otimes A_2^{\otimes n} & \xrightarrow[\lambda]{} & A_2 \end{array}$$

commutes for all n.
This means that f must commute *strictly* with the operations parametrized by $\mathcal{P}$.

Examples

- $\mathcal{I}$; algebras are dg modules
- $\mathcal{C}$; algebras are dg commutative, associative algebras
- $\mathcal{A}$; algebras are dg associative algebras
- $\mathcal{L}$; algebras are dg Lie algebras

From now on, R is a field of characteristic zero.
Ginzburg and Kapranov (1994) provide a construction on a *quadratic* operad $\mathcal{P}$ that yields an operad for SH $\mathcal{P}$-algebras:

$$\mathbf{\Omega}(\mathcal{P}^{\perp})$$

where $\mathcal{P}^{\perp}$ is the *quadratic dual cooperad* to $\mathcal{P}$, and $\mathbf{\Omega}$ is the operadic cobar construction.
If

$$\mathbf{\Omega}(\mathcal{P}^{\perp}) \xrightarrow{\simeq} \mathcal{P}$$

then we say that $\mathcal{P}$ is *Koszul*.
Again, morphisms must commute with operations on the nose.

Co-rings over operads

Composition over an operad

Let $\mathcal{L}$ be a left $\mathcal{P}$-module with structure morphism

$$\lambda : \mathcal{P} \circ \mathcal{L} \to \mathcal{L},$$

and let $\mathcal{R}$ be a right $\mathcal{P}$-module with structure morphism

$$\rho : \mathcal{R} \circ \mathcal{P} \to \mathcal{R}.$$

Define $\mathcal{R} \circ_{\mathcal{P}} \mathcal{L}$ to be the coequalizer of $\rho \circ 1$ and $1 \circ \lambda$:

$$\mathcal{R} \circ \mathcal{P} \circ \mathcal{L} \rightrightarrows \mathcal{R} \circ \mathcal{L} \to \mathcal{R} \circ_{\mathcal{P}} \mathcal{L}.$$

Then $- \circ_{\mathcal{P}} -$ makes $\mathcal{P}$-$\mathbf{Mod}$-$\mathcal{P}$ into a monoidal category with unit $\mathcal{P}$.

A *co-ring* over $\mathcal{P}$ is a comonoid in the monoidal category $(\mathcal{P}\text{-}\mathbf{Mod}\text{-}\mathcal{P}, \circ_{\mathcal{P}}, \mathcal{P})$.
Thus, a $\mathcal{P}$-co-ring consists of a triple $(\mathcal{K}, \psi, \varepsilon)$, where

$$\psi : \mathcal{K} \to \mathcal{K} \circ_{\mathcal{P}} \mathcal{K}$$

and

$$\varepsilon : \mathcal{K} \to \mathcal{P}$$

are morphisms of $\mathcal{P}$-bimodules, and ψ is coassociative and counital with respect to ε.

$\mathcal{K}$-morphisms

Let $\mathcal{K}$ be a $\mathcal{P}$-co-ring. Let $A_1, A_2 \in \mathcal{P}$-$\mathbf{Alg}$.
A $\mathcal{K}$-*morphism* is a morphism of left $\mathcal{P}$-modules,

$$f : \mathcal{K} \circ_{\mathcal{P}} A_1 \to A_2$$

Composition of $\mathcal{K}$-morphisms

Composition of $\mathcal{K}$-morphisms is accomplished via the diagonal in $\mathcal{K}$.
Let $f : \mathcal{K} \circ_{\mathcal{P}} A_1 \to A_2$, $g : \mathcal{K} \circ_{\mathcal{P}} A_2 \to A_3$ be $\mathcal{K}$-morphisms.
gf is the composite:

$$\mathcal{K} \circ_{\mathcal{P}} A_1 \to \mathcal{K} \circ_{\mathcal{P}} \mathcal{K} \circ_{\mathcal{P}} A_1 \xrightarrow{1 \circ f} \mathcal{K} \circ_{\mathcal{P}} A_2 \xrightarrow{g} A_3.$$

The category of $\mathcal{P}$-algebras and $\mathcal{K}$-morphisms is denoted $\mathcal{P}$-$\mathbf{Alg}_{\mathcal{K}}$.

Let $\mathcal{P}$ be a quadratic Koszul operad. Let $\mathcal{P}' = \mathbf{\Omega}(\mathcal{P}^\perp)$.
Since $\mathcal{P}$ is Koszul, $\mathcal{P}' \xrightarrow{\simeq} \mathcal{P}$ is an *operad* resolution, hence controls SH $\mathcal{P}$-algebras.
To describe SH $\mathcal{P}$-*morphisms*, we want a *co-ring* resolution of $\mathcal{P}'$.

Let $\mathcal{P}$ be a quadratic operad.

- The Koszul construction for $\mathcal{P}$ is "well-known" (as a bimodule).
- The Koszul construction for $\mathcal{P}' = \mathbf{\Omega}(\mathcal{P}^{\perp})$ is defined as follows:

$$K(\mathcal{P}') = \mathcal{P}' \circ \mathcal{P}^{\perp} \circ \mathcal{P}'$$

- differential: $d_{\Omega} + d_L + d_R$,
- diagonal:

$$\mathcal{P}^{\perp} \to \mathcal{P}^{\perp} \circ \mathcal{P}^{\perp} \cong \mathcal{P}^{\perp} \circ \mathcal{J} \circ \mathcal{P}^{\perp} \to \mathcal{P}^{\perp} \circ \mathcal{P}' \circ \mathcal{P}^{\perp} \to K(\mathcal{P}') \circ_{\mathcal{P}'} K(\mathcal{P}').$$

Proposition 2. *$K(\mathcal{P}')$ is a $\mathcal{P}'$-co-ring resolution of $\mathcal{P}'$.*

Theorem 3. *(Hess-S.) Let $\mathcal{P}$ be a quadratic Koszul operad. Then the category of SH $\mathcal{P}$-algebras and SH morphisms is equivalent to the category of $\mathbf{\Omega}(\mathcal{P}^{\perp})$-algebras and $K(\mathbf{\Omega}(\mathcal{P}^{\perp}))$-morphisms.*

Remark 4. The result holds for coalgebras, as well as categories of strict coalgebras and SH morphisms.