On the non-commutative G-C. Rota dilation theorem

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Workshop on von Neumann algebras

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I Classical Markov processes

Definition

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It is defined by a transition kernel p :

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- $\Omega = X^{\mathbb{N}}$; $X_n : (x_k) \mapsto x_n$.
- ν Markov measure : $\nu(f_0 \circ X_0 f_1 \circ X_1 \cdots f_n \circ X_n)$

$$= \int_X d\mu(x_0) f_0(x_0) P\Big(\cdots P\Big(P\big(P(f_n)f_{n-1}\big)f_{n-2}\Big)\dots f_1\Big)(x_0)$$

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ν is such that the distribution of X_n is $\mu P^n : f \mapsto \int_X P^n(f)(x)d\mu(x)$.

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$$\int f(x_0, x_1) d\mu_1(x_0, x_1) = \int f(x_0, x_1) p(x_0, dx_1) d\mu_0(x_0)$$

Set $N_{1]} = L^\infty(X \times X, \mu_1)$,

$\alpha_1(f)(x_0, x_1) = f(x_0), \quad \beta_1(f)(x_0, x_1) = f(x_1).$

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• Second step : on $X \times X \times X$ defines μ_2 by

$$\int f(x_0, x_1, x_2) d\mu_2(x_0, x_1, x_2) = \int \Big(\int f(x_0, x_1, x_2) p(x_1, dx_2) \Big) d\mu_1(x_0, x_1).$$

Set $N_{2]} = L^\infty(X \times X \times X, \mu_2)$,

$\alpha_2(f)(x_0, x_1, x_3) = f(x_0, x_1), \quad \beta_2(f)(x_0, x_1, x_2) = f(x_1, x_2),$

and so on....

We have a commutative diagram :

$$\begin{array}{ccccccc}
N_{0]} & \xrightarrow{\alpha_1} & N_{1]} & \xrightarrow{\alpha_2} & N_{2]} & \cdots\cdots\cdots\to & M \\
\downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \beta_3} & & \downarrow{\scriptstyle \beta} \\
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(1) $\beta^p \circ \mathbb{E}_{q]} = \mathbb{E}_{(p+q)]} \circ \beta^p$ Covariance property

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Condition (1) for $q = 0$ gives $\mathbb{E}_{p]}(N_{[p}) = J_p(N)$ where $N_{[n} = \bigvee\{J_k(N), k \geq n\}$ is the future of time n, $J_n(N)$ being the present at time n.

II Quantum Markov processes

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A quantum Markov process with "state space N" is a vN algebra M with

- a time evolution endomorphism $\beta : M \to M$;
- a normal injective homomorphism J_0 from N into M :
- normal conditional expectations $\mathbb{E}_{n]}$ from M onto $N_{n]} = \bigvee\{J_k(N) : k \leq n\}$, $J_k = \beta^k \circ J_0$,

such that

$$\forall n, \quad \mathbb{E}_{n]}(N_{[n}) \subset J_n(N).$$

Markov dilation problem

Given a Markov operator $P : N \hookleftarrow$, construct a quantum Markov process such that

$$\forall n, \quad \mathbb{E}_{0]} \circ J_n = \mathbb{E}_{0]} \circ \beta^n \circ J_0 = J_0 \circ P^n.$$

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Theorem, Sauvageot (1986)

Let $P : N \hookleftarrow$ be a Markov operator. There exists a Markov quantum process $(M, \beta, J_0, (\mathbb{E}_{n]})_{n \geq 0})$ such that

(1) $\mathbb{E}_{n]} \circ \mathbb{E}_{q]} = \mathbb{E}_{n]}, \quad \text{for} \quad n \leq q\,;$

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The construction is not at all straightforward. It is a combination of the Stinespring dilation construction and of free products constructions.

In the commutative case the first step is indeed the Stinespring construction.

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In the non-commutative case, α_1 cannot be defined. The Stinespring construction gives $P = V^* \beta_1(\cdot) V$, where V is the canonical isometry from $L^2(N)$ into H_P.

Theorem, Evans (1978)

Let $P : N \hookleftarrow$ be a Markov operator. There exist a von Neumann algebra $M \supset N$, a conditional expectation $\mathbb{E} : M \to N$, and an injective endomorphism β of M such that for all $n \geq 1$,

$$P^n = \mathbb{E} \circ \beta^n \circ i$$

where i is the inclusion from N into M.

Definition

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$\leadsto$ If μ is P-stationary there exists a Markov map $P^\star : L^\infty(X,\mu) \hookleftarrow$ such that $\int_X P(f) g d\mu = \int f P^\star(g) d\mu$ for all $f, g \in L^\infty(X,\mu)$.

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Indeed, since μ is P-stationary, the predual P_* satisfies $P_*(1) = 1$, and $P_*(L^\infty(X)) \subset L^\infty(X)$. $P^\star = P_*$ restricted to $L^\infty(X)$

Theorem, G-C Rota (1962)

Let P be a μ-preserving Markov operator. Then for $p > 1$ and $f \in L^p(X)$,

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Compare with the von Neumann-Birkoff ergodic theorem : For $f \in L^1(X)$,

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Ornstein counterexample (1969) : Stein's result is not true for $p = 1$.

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The equalities (1) are called Rota dilations. When $P = P^\star$, one has

$$J_0 \circ P^{2n} = \mathbb{E}_{0]} \circ \mathbb{E}_{[n} \circ J_0.$$

Compare with Markov dilations : $J_0 \circ P^n = \mathbb{E}_{0]} \circ \beta^n \circ J_0$.

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Proposition

Let $P : N \to N$ be a Markov operator. The following conditions are equivalent :

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$$\varphi(P^\star(a)b) = \varphi(aP(b)).$$

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Example : Normal φ-preserving injective endomorphisms α such that $\alpha(N)$ is invariant under the modular automorphism group of φ. Then $\alpha^\star = \alpha^{-1} \circ \mathbb{E}$, where $\mathbb{E}$ is the φ-preserving conditional expectation onto $\alpha(N)$.

Non-commutative Rota dilation problem :

Given a φ-markovian operator P, find a von Neumann algebra M with a faithful normal state Φ, a time evolution $\beta : M \hookleftarrow$ and a normal injective homomorphism $J_0 : N \to M$ such that

- β is Φ-markovian, J_0 is (φ, Φ)-markovian ;
- If $\mathbb{E}_{n]}$ and $\mathbb{E}_{[n}$ are respectively the canonical expectations on $N_{n]}$ and $N_{[n}$ where

$$N_{n]} = \bigvee\{J_k(N), k \leq n\}, N_{[n} = \bigvee\{J_k(N), k \geq n\}, J_k = \beta^k \circ J_0,$$

then $(M, \beta, J_0, (\mathbb{E}_{n]})_{n\geq 0})$ is a quantum Markov process with

$$\mathbb{E}_{n]} \circ J_q = J_n \circ P^{q-n}$$
$$\mathbb{E}_{[n} \circ J_0 = J_n \circ (P^{\star})^n$$

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A φ-markovian operator admits a Rota dilation if and only if it is factorizable in the following sense :

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There exist a von Neumann algebra $\mathcal{N}$ with a faithful normal state ψ, two (φ, ψ)-markovian injective homomorphisms $J_0, J_1 : (N, \varphi) \to (\mathcal{N}, \psi)$ such that

$$P = J_0^\star \circ J_1$$

or equivalently, for all $a, b \in N$:

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In one direction : obvious

$$J_0^\star = J_0^{-1} \circ \mathbb{E}_{0]} \quad \text{and} \quad \mathbb{E}_{0]} \circ J_1 = J_0 \circ P \Rightarrow P = J_0^\star \circ J_1$$

The other direction : Assume the existence of $\beta_1, \alpha_1 : N_0 \to N_1$ with $P = \alpha_1^\star \circ \beta_1$. Set $N_2 = N_1 *_{N_0} N_1$, $\alpha_2, \beta_2 : N_1 \to N_2$ the canonical embeddings, and inductively :

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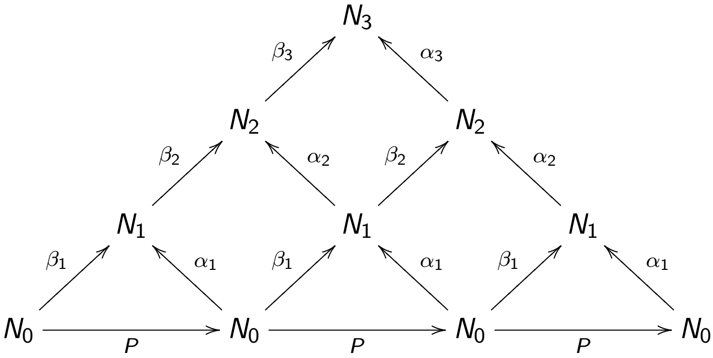

Set $(M, \Phi) = \varinjlim(N_n, \varphi_n)$ with respect to the embeddings α_n. The time evolution is constructed from the β_n's, J_0 is the embedding of $N = N_0$ into the inductive limit. Crucial relations : $\beta_n \circ \alpha_n^\star = \alpha_{n+1}^\star \circ \beta_{n+1}$.

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- Convex combinations
- Pointwise weak* limits

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$Markov_\varphi(N) :=$ the convex compact space of φ-markovian maps

$FMarkov_\varphi(N) :=$ the subset of factorizable maps. It is stable under

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- When N is abelian, every φ-markovian map is factorizable.

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It is no longer true for $n \geq 3$ that the extreme points of $Markov_\tau(M_n(\mathbb{C}))$ are the automorphisms, and even in the simple case $n = 3$, it is an open question whether every τ-markovian map is factorizable.

E. Ricard has recently given a positive solution to the following particular case :

- $N = \mathcal{B}(\ell^2(I))$ with the state φ of diagonal density $D = \sum \lambda_i \theta_{e_i,e_i}$. We have $\lambda_i > 0$ and $\sum \lambda_i = 1$.

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- $P = m_T$ is the Schur multiplier associated with a positive type kernel $T = (t_{i,j})$:
$$m_T([x_{i,j}]) = [t_{i,j} x_{i,j}].$$
Immediate verification : m_T is φ-markovian.

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Theorem, Ricard (2007)

m_T is factorizable.

Proof : Define a new scalar product on the real linear span of the e_i's :

$$\langle \sum a_i e_i, \sum b_i e_i \rangle_T = \sum a_i b_j t_{i,j}.$$

Let ℓ^2_T be the real Hilbert space obtained after separation and completion.

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Let ℓ^2_T be the real Hilbert space obtained after separation and completion. The receptacle $\mathcal{N}$ for the embeddings J_0, J_1 is $\mathcal{N} = \mathcal{B}(\ell^2(I)) \otimes \Gamma_{-1}(\ell^2_T)$ where $\Gamma_{-1}(\ell^2_T)$ is the Fermion algebra constructed on $K = \ell^2_T$.
For $e \in K$ let $\ell(e)$ be the creation operator on the antisymmetric Fock space

$$\mathcal{F}(K) = \mathbb{C}\Omega \oplus \bigoplus K_{\mathbb{C}}^{\wedge n}$$

given by

$$\ell(e)(k_1 \otimes \cdots \otimes k_n) = e \otimes k_1 \otimes \cdots \otimes k_n.$$

$$\Gamma_{-1}(K) = \{\omega(e) = \ell(e) + \ell(e)^*, e \in K\}''.$$

If e is a norm one vector in K, then $\omega(e)$ is self adjoint with $\omega(e)^2 = 1$ and

$$\forall e, f \in K, \quad \tau(\omega(e)\omega(f)) = \langle e, f \rangle_K.$$

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$\mathcal{N} = \mathcal{B}(\ell^2(I)) \otimes \Gamma_{-1}(\ell^2_T)$ is equipped with the state $\psi = \varphi \otimes \tau$.
Let u be the diagonal unitary symmetry in $\mathcal{N}$ with $\omega(e_i)$ as i-th entry. We define

$$J_1 : N = \mathcal{B}(\ell^2(I)) \to \mathcal{N}, \quad \text{by} \quad J_1 = \mathrm{Id}_N \otimes 1$$
$$J_0 : N \to \mathcal{N}, \text{by} \quad J_0(a) = uJ_1(a)u.$$

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Verifications are straightforward. For instance

$$\begin{aligned}
\psi(J_1(a)J_0(b)) &= \varphi \otimes \tau\big((a \otimes 1)u(b \otimes 1)u\big) \\
&= \varphi \otimes \tau\big((a_{i,j}1)(b_{i,j}\omega(e_i)\omega(e_j))\big) \\
&= \sum_{i,j} \lambda_i a_{i,j} b_{j,i} \tau(\omega(e_i)\omega(e_j)) \\
&= \sum_{i,j} \lambda_i a_{i,j} b_{j,i} t_{i,j} = \varphi(m_T(a)b).
\end{aligned}$$

Related case :

$N = L(G)$, (G discrete group), equipped with its trace.

$P = m_\varphi$, Fourier multiplier, where φ is a real-valued positive type function :

$$m_\varphi(\lambda(g)) = \varphi(g)\lambda(g).$$

Related case :

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Then
$\mathcal{N} = \Gamma_{-1}(\ell^2_T) \rtimes_\alpha G$ where $t_{g,h} = \varphi(g^{-1}h)$ and the action α is such that $\alpha_g(\omega(h)) = \omega(gh)$.

$\mathcal{N}$ is equipped with its canonical trace and

$$J_1 : N \to \mathcal{N}, \quad \text{is the canonical embedding}$$
$$J_0 : N \to \mathcal{N}, \quad \text{is defined by} \quad J_0(a) = \omega(\delta_e)J_1(a)\omega(\delta_e).$$