SOME OPEN PROBLEMS ON $\mathbf{II}_1$ FACTORS OF GROUP ACTIONS

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Notations:

Γ, Λ countable (discrete infinite) groups.

$(X, \mu), (Y, \nu)$ probability measure spaces.

$\Gamma \curvearrowright X, \Lambda \curvearrowright Y$ measure preserving actions.

Given $\Gamma \curvearrowright X$, denote $M = L^\infty(X) \rtimes \Gamma$ its *group measure space* vN algebra. $\{u_g\}_g \subset M$ the *canonical unitaries*, implementing $\Gamma \curvearrowright L^\infty(X)$ by $u_g a u_g^* = g(a)$, $a \in L^\infty(X) \subset M$.

$\mathcal{L}(\Gamma) = \mathbb{C} \rtimes \Gamma$ the *group vN algebra* of Γ. Note: $\mathcal{L}(\Gamma) \simeq \{u_g\}'' \subset L^\infty(X) \rtimes \Gamma$.

$\mathcal{R}_\Gamma = \{(t, gt) \mid t \in X\}$ is the (countable) *equivalence relation* implemented by $\Gamma \curvearrowright X$ and $\mathcal{L}(\mathcal{R}_\Gamma)$ the *associated vN algebra*. Note: $A = L^\infty(X)$ is maximal abelian in $M = \mathcal{L}(\mathcal{R}_\Gamma)$ and $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) \mid uAu^* = A\}$ generates M, i.e. A is *Cartan subalgebra* in M. Also, $\mathcal{L}(\mathcal{R}_\Gamma) = L^\infty(X) \rtimes \Gamma$ when action free ergodic.

Fact: $\Gamma \curvearrowright X$ free ergodic $\Rightarrow L^\infty(X) \rtimes \Gamma$ II_1 factor; $\mathcal{L}(\Gamma)$ II_1 factor iff Γ is ICC; $\mathcal{L}(\mathcal{R}_\Gamma)$ II_1 factor iff $\Gamma \curvearrowright [0,1]$ ergodic.

Conjugacy of $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ means $\Delta : (X,\mu) \simeq (Y,\nu)$ and $\delta : \Gamma \simeq \Lambda$ with $\Delta(gt) = \delta(g)\Delta(t)$, $\forall g \in \Gamma, t \in X$.

Note: Conjugacy implements isomorphism $L^\infty X \rtimes \Gamma \simeq L^\infty Y \rtimes \Lambda$ by $\Sigma a_g u_g \mapsto \Sigma \Delta(a_g) v_{\delta(g)}$

Fact: $L^\infty(X) \rtimes \Gamma$ *can only "remember"* $\mathcal{R}_\Gamma$. More precisely: An iso $\Delta : (X,\mu) \simeq (Y,\nu)$ extends to $L^\infty(X) \rtimes \Gamma \simeq L^\infty(Y) \rtimes \Lambda$ iff Δ is an *orbit equivalence* (*OE*), i.e. $\Delta(\mathcal{R}_\Gamma) = \mathcal{R}_\Lambda$, or $\Delta(\Gamma t) = \Lambda\Delta(t)$, $\forall t$.

Obs: *Conjugacy* $\Rightarrow$ *OE* $\Rightarrow$ *iso of vN algebras* (W^*-equivalence)

A *deformation* of II_1 factor $M = L^\infty(X) \rtimes \Gamma$, $\mathcal{L}(\Gamma)$ is a sequence of *completely positive* (c.p.) maps $\phi_n : M \to M$ which are unital, trace preserving and satisfy $\lim_n \|\phi_n(x) - x\|_2 = 0$, $\forall x \in M$.

Typical examples of c.p. maps:

- Automorphisms of M;

- Maps of the form $\phi(\Sigma_g a_g u_g) = \Sigma_g \varphi(g) a_g u_g$, where $\varphi : \Gamma \to \mathbb{C}$ is positive definite.

- **Questions on relative property (T)**

A subalgebra $B \subset M$ has *relative property* (T) if any deformation ϕ_n of M is uniform on B: $\lim_n(\sup\{\|\phi_n(b) - b\|_2 \mid b \in (B)_1\}) = 0$.
(N.B. If $B = M$ this amounts to Connes-Jones prop T of M)

An action $\Gamma \curvearrowright (X, \mu)$ (resp. its eq. rel. $\mathcal{R}_\Gamma$) has *relative property* (T) if $L^\infty(X) \subset \mathcal{L}(\mathcal{R}_\Gamma)$ has relative property (T).

Obs: If H discrete abelian group and $\Gamma \curvearrowright H$ then $H \subset H \rtimes \Gamma$ has rel. prop. (T) iff $\Gamma \curvearrowright \hat{H}$ has rel. prop. (T);

Examples $SL(n, \mathbb{Z}) \curvearrowright \mathbb{Z}^n$, $n \geq 2$ (Kazhdan); $\Gamma \curvearrowright \mathbb{Z}^2$ for $\Gamma \subset SL(2, \mathbb{Z})$ non-amenable (Burger); more examples by Shalom, Fernos, Valette.

Q 1 Give a "non-vNAlgebra" def. of relative property (T) for actions.

Q 2 Denote $\mathcal{R}$ the OE relation of $SL(2,\mathbb{Z}) \curvearrowright \mathbb{T}^2$. $\forall \mathcal{R}_0 \subset \mathcal{R}$ non-amenable has rel. (T)? $\forall$ "quotient" (...) of $\mathcal{R}$ has rel. (T)?

Q 3 What are the groups Γ for which $\exists\ \Gamma \curvearrowright X$ free ergodic with rel. (T)? (Ioana...)

- **Related Questions:**

Fact ([P01]) If $\Gamma \curvearrowright X$ ergodic has rel prop (T) then $\mathrm{Out}(\mathcal{L}(\mathcal{R}_\Gamma))$, $\mathrm{Out}(\mathcal{R}_\Gamma)$ countable. Also, $\mathcal{R}_\Gamma$ has only countably many quotients of finite index.

Q 1 Calculate $\mathrm{Out}(\mathcal{R}_{\Gamma \curvearrowright \mathbb{T}^2})$, $\mathrm{Out}(\mathcal{L}(\mathcal{R}_\Gamma))$, for $\Gamma = SL(2,\mathbb{Z})$, $\Gamma = \mathbb{F}_2 \subset SL(2,\mathbb{Z})$. Calculate all finite index subeq. rel. and quotients of $SL(2,\mathbb{Z}) \curvearrowright \mathbb{T}^2$ (Vaes...)

Q 2 Show $\exists \mathbb{F}_2 \curvearrowright X$ free ergodic with $\mathrm{Out}(\mathcal{R}_{\mathbb{F}_2})$ $= 1$.

- **Questions on the fundamental group**

Fact ([P05]) $\forall \mathcal{S} \subset \mathbb{R}_+^*$ countable subgroup, $\exists \mathbb{F}_\infty \curvearrowright X$ ergodic (but not free) with its OE relation $\mathcal{R}$ satisfying $\mathcal{F}(\mathcal{R}) = \mathcal{F}(\mathcal{L}(\mathcal{R})) = \mathcal{S}$. Moreover, $\mathcal{R}^t$ cannot be implemented by a free action $\Gamma \curvearrowright X$, $\forall t > 0$.

Q 1 Is any fund. group $\mathcal{F}(\mathcal{R}_\Gamma)$, $\mathcal{F}(M)$, either countable or $\mathbb{R}_+^*$? ($\forall \mathcal{R}$ OE rel, $\forall M$ separable II_1 factor)

Q 2 Give examples of free ergodic $\Gamma \curvearrowright X$ with $\mathcal{F}(\mathcal{R}_\Gamma) \neq 1, \mathbb{R}_+^*$. Can $\mathcal{F}(\mathcal{R}_\Gamma)$ contain irrationals (when $\neq \mathbb{R}_+^*$) if $\Gamma \curvearrowright X$ free?

Q 3 $\exists \mathbb{F}_\infty \curvearrowright X$ free ergodic with $\mathcal{F}(\mathcal{R}_{\mathbb{F}_\infty}) = 1$, resp. with $\mathcal{F}(\mathcal{R}_{\mathbb{F}_\infty}) = \mathbb{R}_+^*$? (By Gaboriau, $\mathcal{F}(\mathcal{R}_{\mathbb{F}_n}) = 1$, $\forall \mathbb{F}_n \curvearrowright X$ free ergodic , $n < \infty$).

- **On Bernoulli actions & OE superrigidity**

Fact ([P05, P06]) Bernoulli actions $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^\Gamma$ of prop. (T) groups are $\mathcal{U}_{fin}$-Cocycle Superrigid. In particular OE Superrigid, i.e. $\forall$ $\Gamma \curvearrowright X \sim_{OE} \Lambda \curvearrowright^{free} Y$, "comes" from a conjugacy (...). Same true for $\Gamma \curvearrowright X$ *sub-malleable mixing* (e.g. quotients of Bernoulli) with Γ satisfying one of the following:

- $\exists$ $H \subset \Gamma$ infinite w-normal with rel prop (T)

- $\exists H \subset \Gamma$ infinite w-normal with non-amenable commutant (e.g. $\Gamma = H \times H'$, H' non-amen)

Q 1 Find the class $\mathcal{CS}$ (resp. $\mathcal{OES}$) of groups Γ such that any Bernoulli Γ-action is $\mathcal{U}_{fin}$-Cocycle (resp OE) Superrigid.

Q 2 Find larger classes $\mathcal{U}$ of "target" groups with the property that any Bernoulli action of a Kazhdan (or other) group is $\mathcal{U}$-Cocycle Superrigid.

Q 3 Calculate $\mathrm{H}^2(\mathcal{R}_\Gamma)$ more generally $\mathrm{H}^n(\mathcal{R}_\Gamma)$ for some $\Gamma \curvearrowright X$. (No such calculations exist for $n \geq 2$! For Γ Kazhdan and action Bernoulli, one expects $\mathrm{H}^n(\mathcal{R}_\Gamma) = \mathrm{H}^n(\Gamma)$.)

Q 4 Is it true that $\forall \Gamma, \Lambda$ non-amenable, any OE of Bernoulli actions $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ comes from a conjugacy ? For free groups ?

Q 5 Let Γ be non-amenable ICC. Is any automorphism of the probability space $(X, \mu) = (X_0, \mu_0)^\Gamma$ that commutes with the Bernoulli action $\Gamma \curvearrowright X$ the product of a diagonal automorphism and a "right" Bernoulli shift by an element of the group?

- **Cartan decomposition & W*-superrigidity**

Fact ([P05, P06]) Γ, Λ ICC groups, with Γ either: Kazhdan; or $\exists H \subset \Gamma$ w-normal with rel prop (T); or $\exists H \subset \Gamma$, $|H| = \infty$, $H' \cap \Gamma$ non-amenable. Assume $\Gamma \curvearrowright X$ free mixing and $\Lambda \curvearrowright Y$ Bernoulli. Then any $\theta : L^\infty(X) \rtimes \Gamma \simeq L^\infty(Y) \rtimes \Lambda$ comes from a conjugacy (*Strong W*-Rigidity* result).

Q 1 Find group actions $\Gamma \curvearrowright X$ that are W*-Superrigid, i.e. given any other free ergodic action $\Lambda \curvearrowright Y$, any isomorphism $L^\infty(X) \rtimes \Gamma \simeq L^\infty(Y) \rtimes \Lambda$ comes from a conjugacy.

Note: If one could show that Γ Kazhdan (or product group) and $\Gamma \curvearrowright X$ Bernoulli implies $L^\infty(X) \rtimes \Gamma$ has unique Cartan, then $\Gamma \curvearrowright X$ follows W*-Superrigid (by [P05, P06])

Q 2 If Γ Kazhdan and $\Gamma \curvearrowright X$ Bernoulli, does $L^\infty(X) \rtimes \Gamma \simeq L^\infty(Y) \rtimes \Lambda$ imply Λ Kazhdan? **Obs**: If so, then Bernoulli actions of Kazhdan groups follow W*-Superrigid.

Related Obs: If $PSL(n, \mathbb{Z}) \curvearrowright \mathbb{T}^n$ gives a factor with unique Cartan, then this action would follow W*-Superrigid (by Furman 99).

Fact (Ozawa-Popa 07): If $\Gamma = \mathbb{F}_{n_1} \times ... \times \mathbb{F}_{n_k}$ $\curvearrowright X$ ergodic profinite (...), then $L^\infty(X) \rtimes \Gamma$ has Cartan iff $\Gamma \curvearrowright X$ free. And if free, then its Cartan is unique, up to unitary conjugacy.

Q 3 Show some $\Gamma \curvearrowright X$ as above is OE-superrigid. (Then $\Gamma \curvearrowright X$ follows W*-superrigid.)

Q 4 Generalize [OP07] to arbitrary $\Gamma \curvearrowright Q$ (not nec. profinite).

NB: This would imply:
(1) The Bernoulli $\mathbb{F}_n \times \mathbb{F}_m \curvearrowright X$ is W*-superrigid (since its factor would have unique Cartan).
(2) $Q \otimes \mathcal{L}(\mathbb{F}_n)$ would have no Cartan, $\forall Q$.

Q 5 Find (other) classes of factors $L^\infty(X) \rtimes \Gamma$ with unique Cartan.

- **Connes' Rigidity Conjecture (CRC)**

If Γ, Λ ICC groups with property (T), does $\mathcal{L}(\Gamma) \simeq \mathcal{L}(\Lambda)$ imply $\Gamma \simeq \Lambda$? At least for $PSL(n, \mathbb{Z})$, $n \geq 3$? (Known for $\Gamma_n \subset Sp(n,1)$ by Cowling-Haagerup.)

CRC Strong Version: *If Γ ICC with prop (T) and Λ ICC, then any $\theta : L(\Gamma) \simeq \mathcal{L}(\Lambda)^t$ forces $t = 1$ and $\exists \delta : \Gamma \to \Lambda$, $\gamma \in$ Hom$(\Gamma, \mathbb{T})$ such that $\theta(\Sigma_g c_g u_g) = \Sigma_g \gamma(g) c_g u_{\delta(g)}$?*

- **Free Group Factor Problems**

The Non-isomorphism Problem: $\mathcal{L}(\mathbb{F}_n) \simeq \mathcal{L}(\mathbb{F}_m) \Rightarrow n = m$? Sufficient to prove: $\mathcal{L}(\mathbb{F}_\infty) \neq \mathcal{L}(\mathbb{F}_n)$ for some n (cf. Voiculescu, Radulescu, Dykema). Related to this:

Finite Generation Problem Can $\mathcal{L}(\mathbb{F}_\infty)$ be fin gen as vN Alg ? Do there exist $\mathcal{L}(\Gamma)$ which cannot be fin gen ? (Obs: Any factor $\mathcal{L}(\mathcal{R}_\Gamma)$ can be generated by two unitaries)

Known Indecomposition Properties of $\mathcal{L}(\mathbb{F}_n)$
It has no Cartan (Voiculescu 94) and is prime (Ge 96), in fact $P' \cap \mathcal{L}(\mathbb{F}_n)$ amenable $\forall P \subset \mathcal{L}(\mathbb{F}_n)$ diffuse (Ozawa 03).

Stronger still (Ozawa-Popa 07): If $P \subset \mathcal{F}(\mathbb{F}_n)$ amenable diffuse then $\mathcal{N}(P)''$ amenable ($\mathcal{L}(\mathbb{F}_2)$ is *strongly-solid*). Also, $Q \otimes \mathcal{L}(\mathbb{F}_2)$ has no Cartan, $\forall Q$ with $\Lambda_{cb}(Q) = 1$, e.g. $Q = R$, $Q \subset \mathcal{L}(\Gamma_1) \otimes \mathcal{L}(\Gamma_2) \otimes ...$, where $\Gamma_i \subset SO(n,1)$, $SU(n,1)$. Moreover, if $M \subset Q \otimes \mathcal{L}(\mathbb{F}_2)$ with finite Jones index then M has no Cartan (Connes' examples of $M \neq M^{op}$).

Absence of Cartan Problem: $Q \otimes \mathcal{L}(\mathbb{F}_n)$ has no Cartan $\forall Q$ factor (Peterson's L^2-rigidity; Jung, Shlyakhtenko for amalg free products).

Abstract Characterization of $\mathcal{L}(\mathbb{F}_n)$: If M II_1 factor is s-solid and $\Lambda_{cb}(M) = 1$ then $M \simeq \mathcal{L}(\mathbb{F}_n)^t$? Is any non-amenable $M \subset \mathcal{L}(\mathbb{F}_n)$ iso to some $\mathcal{L}(\mathbb{F}_n)^t$?

Obs By [OP07], if $\Gamma = (\mathbb{F}_{n_1} \rtimes \mathbb{F}_{n_2}) \rtimes ...) \rtimes \mathbb{F}_{n_k}$ satisfies $\Lambda_{cb}(\Gamma) = 1$ then $\mathcal{L}(\Gamma)$ has no Cartan. Can it be s-solid for some Γ? Is $\Lambda_{cb}(\mathrm{Aut}(\mathbb{F}_2))$ equal to 1 ? Does $\Lambda_{cb}(\Gamma) = 1$ imply Γ has Haagerup property (Cowling).

W*-equiv vs orbit equiv of groups Γ, Λ are called *orbit equivalent* if $\exists$ free ergodic $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ that are (stably) OE.

Q (Shlyakhtenko): Does OE of ICC groups Γ, Λ imply (or is implied by) $\mathcal{L}(\Gamma) \simeq \mathcal{L}(\Lambda)^t$, for some $t > 0$?

- **Connes' approx. embedding problem**

Is $L^\infty(X) \rtimes \Gamma$ embeddable into R^ω,
$\forall$ countable group Γ, $\forall$ $\Gamma \curvearrowright X$?
Or at least for all Bernoulli Γ-actions?

References

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