Operator-valued Semicircular Elements: at the Intersection of Combinatorics, Random Matrix Theory and Complex Analysis

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Kingston

joint work with Reza Rashidi Far, Tamer Oraby, Wlodek Bryc, Bill Helton

- Rashidi Far, Oraby, Bryc, Speicher:
  On slow-fading MIMO systems with non-separable correlation.
  To appear in IEEE Transactions on Information Theory

- Helton, Rashidi Far, Speicher:
  Operator-valued Semicircular Elements: Solving A Quadratic Matrix Equation with Positivity Constraints.
  IMRN 2007

Consider **Gaussian $N \times N$-random matrices**

$$A_N = \left(a_{ij}\right)_{i,j=1}^{N}$$

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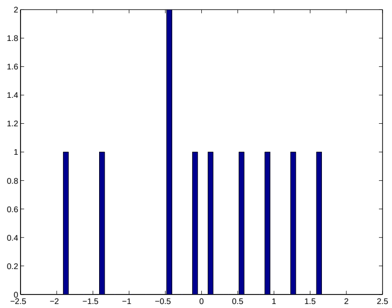
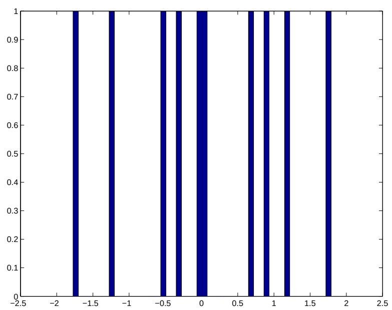
- A_N is selfadjoint, i.e., $a_{ji} = \bar{a}_{ij}$

- otherwise, all entries are independent and identically distributed with centered normal distribution of variance $1/N$


N=10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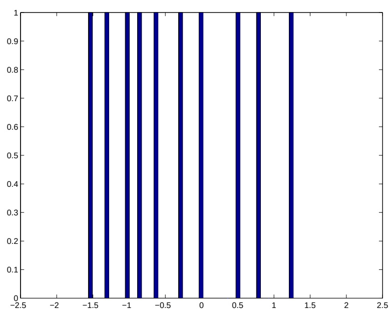
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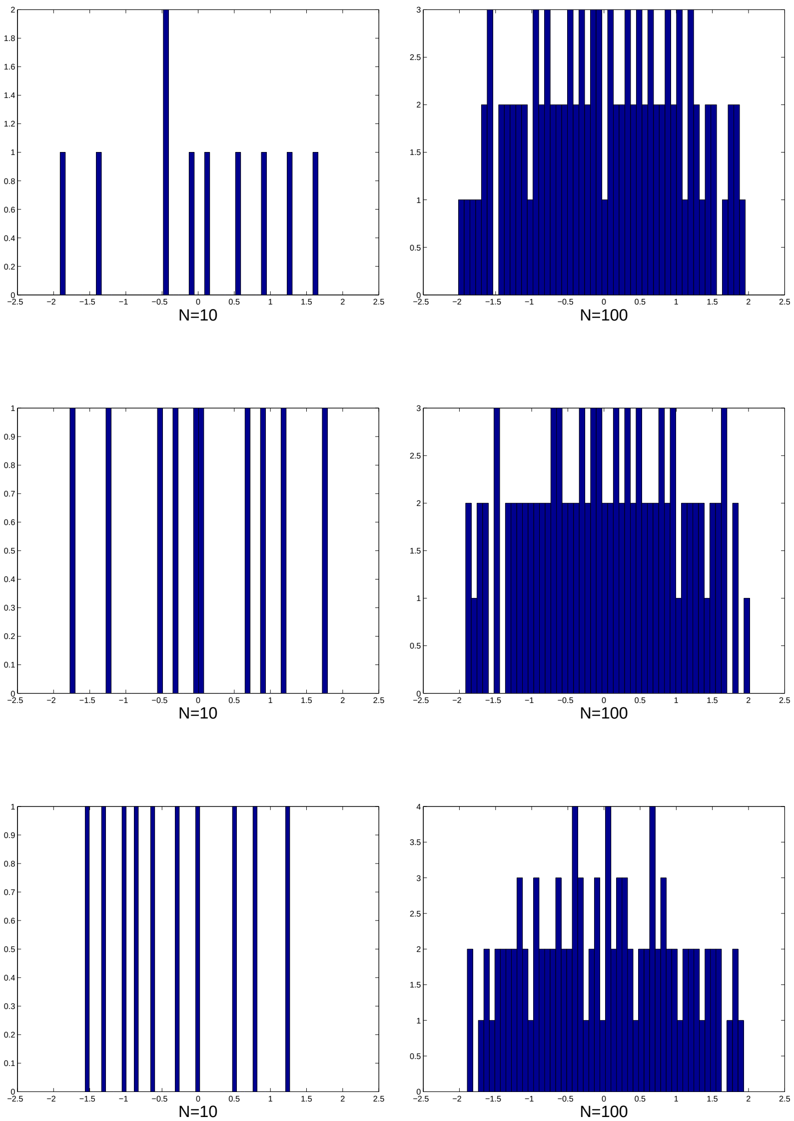
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N=100
N=10
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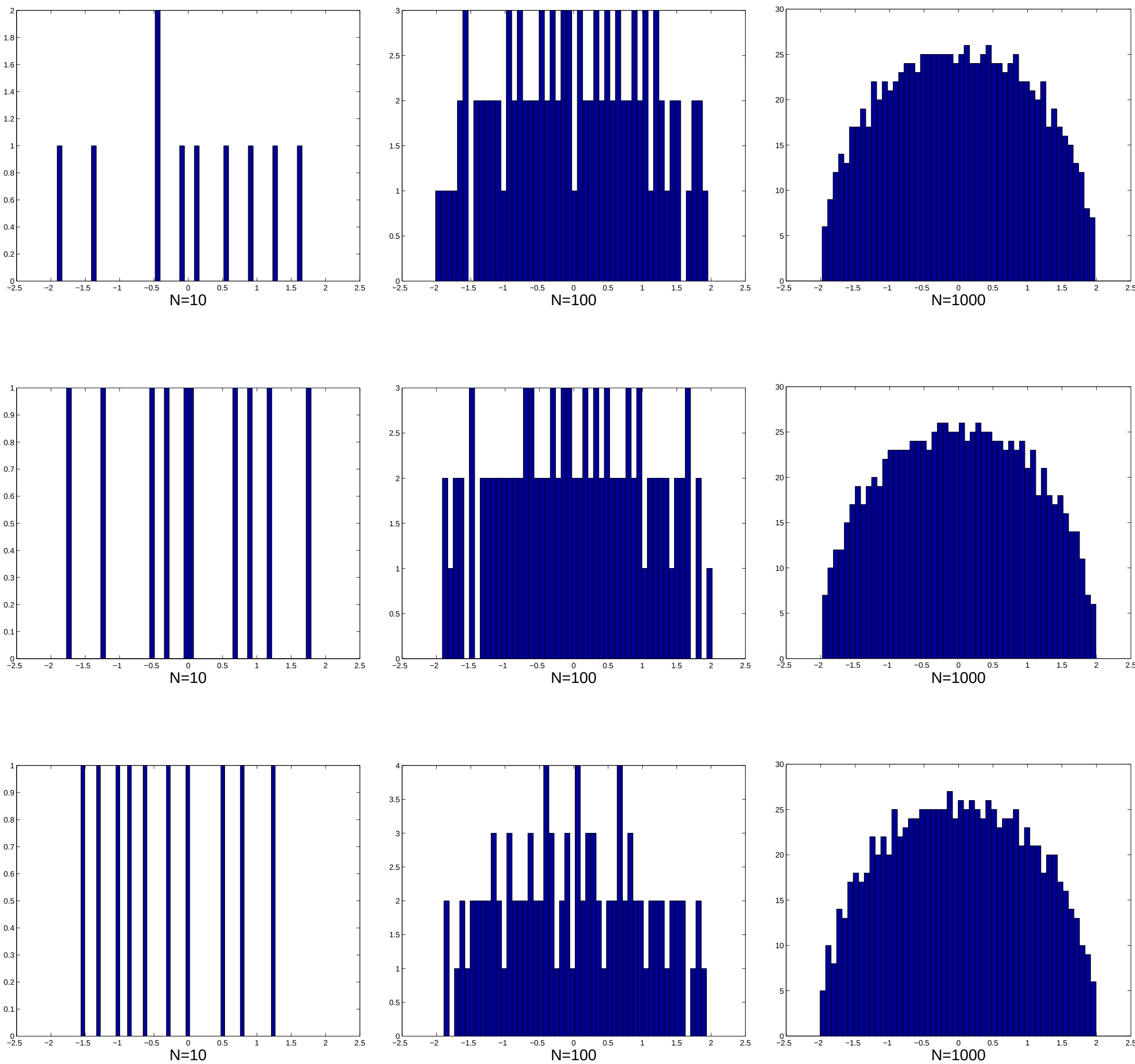
N=100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N=10
N=100
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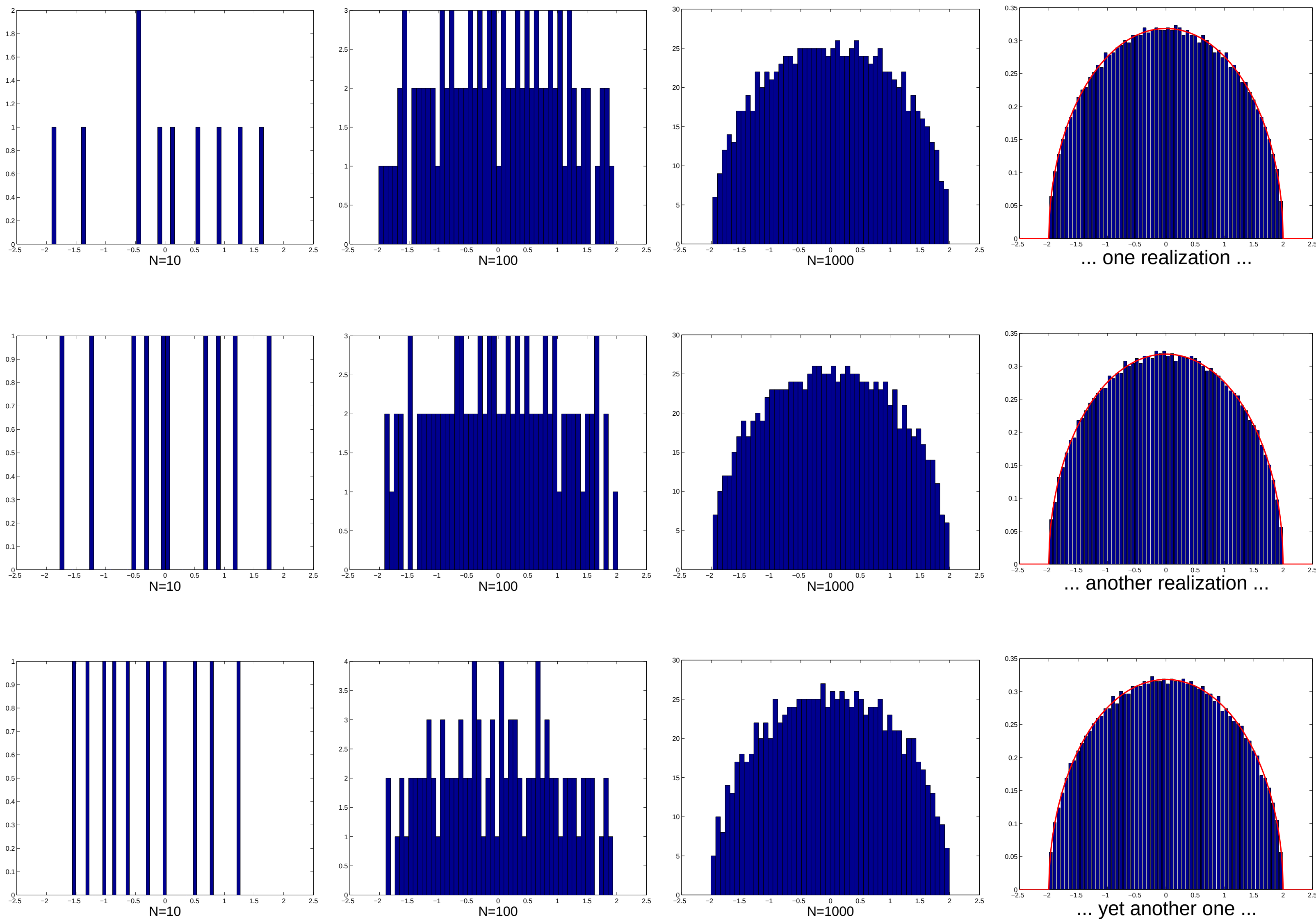
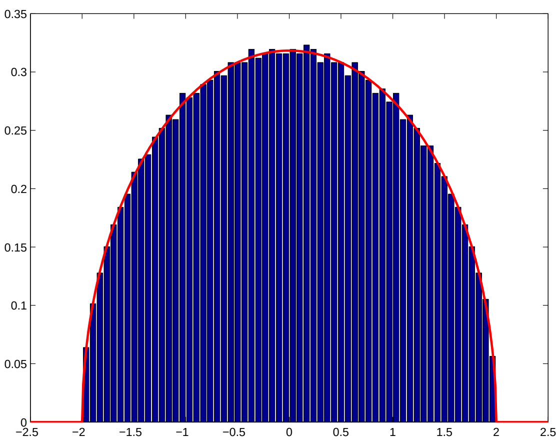
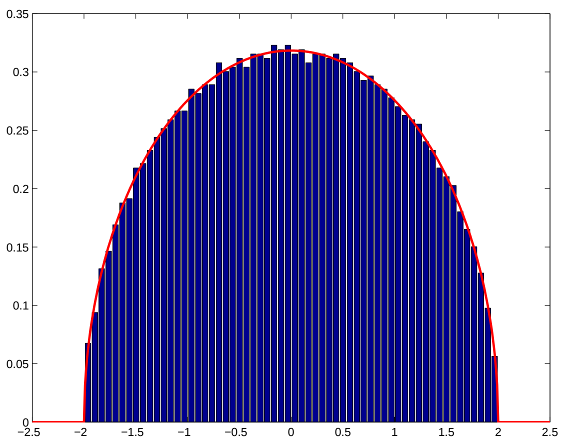

N=10
N=100
N=1000
... one realization ...
N=10
N=100
N=1000
... another realization ...
N=10
N=100
N=1000
... yet another one ...

Convergence of **typical eigenvalue distribution** of Gaussian $N \times N$ random matrices to **Wigner's semicircle**

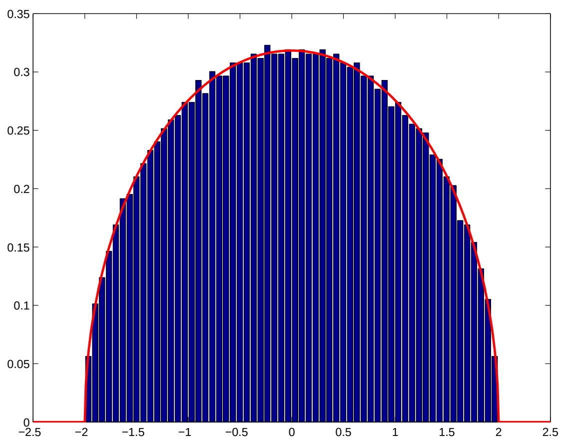
$N = 4000$



... one realization ...



... another realization ...



... yet another one ...

Consider **the empirical eigenvalue distribution of** A_N

$$\mu_{A_N}(\omega) = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i(\omega)}$$

$\lambda_i(\omega)$ are the N eigenvalues (counted with multiplicity) of $A_N(\omega)$

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Then **Wigner's semicircle law** says that

$$\boldsymbol{\mu_{A_N} \Longrightarrow \mu_W} \qquad \textbf{almost surely,}$$

i.e., for all continuous and bounded f

$$\lim_{N\to\infty} \int_{\mathbb{R}} f(t) d\mu_{A_N}(t) = \int_{\mathbb{R}} f(t) d\mu_W(t) = \frac{1}{2\pi} \int_{-2}^{2} f(t) \sqrt{4 - t^2} dt$$

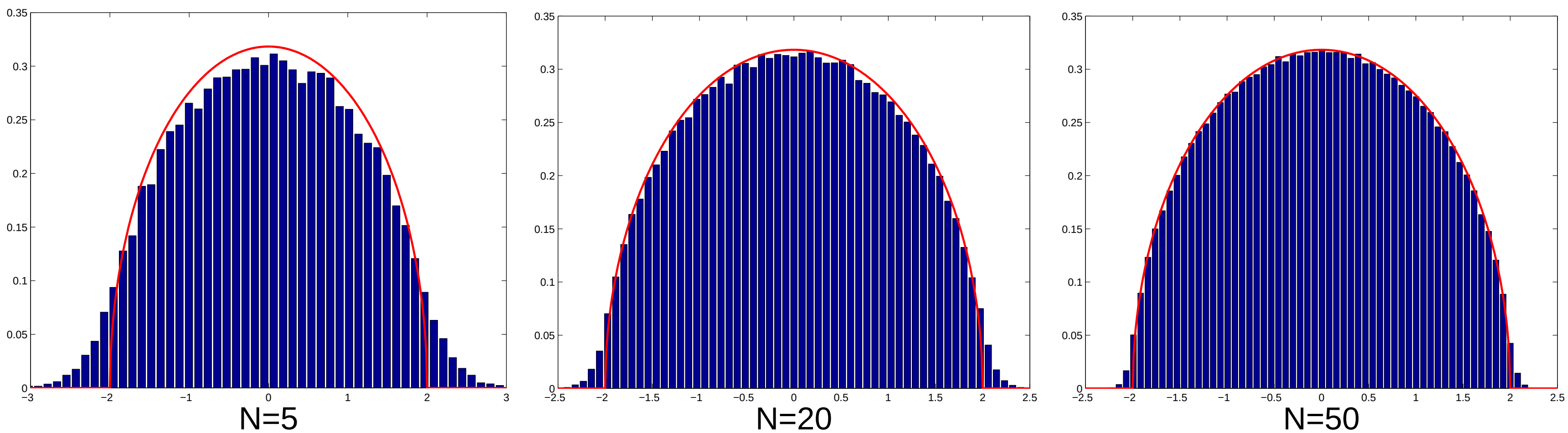
Show

$$\lim_{N\to\infty} \mu_{A_N}(f) = \mu_W(f) \qquad \text{almost surely}$$

in two steps:

- $\lim_{N\to\infty} E[\mu_{A_N}(f)] = \mu_W(f)$
- $\sum_N \mathrm{Var}[\mu_{A_N}(f)] < \infty$

Convergence of **averaged eigenvalue distribution** of Gaussian $N \times N$ random matrices to **Wigner's semicircle**



number of realizations = 10000

For

$$\lim_{N\to\infty} E[\mu_{A_N}(f)] = \mu_W(f)$$

it suffices to treat **convergence of all averaged moments**, i.e.,

$$\lim_{N\to\infty} E[\int t^n d\mu_{A_N}(t)] = \int t^n d\mu_W(t) \qquad \forall n \in \mathbb{N}$$

Note:

$$E[\int t^n d\mu_{A_N}(t)] = E[\frac{1}{N}\sum_{i=1}^{N} \lambda_i^n] = E[\mathrm{tr}(A_N^n)]$$

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$$E[\mathrm{tr}(A_N^n)] = \frac{1}{N} \sum_{i_1,\dots,i_n=1}^{N} E[a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_n i_1}]$$

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but

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Asymptotically, for $N \to \infty$, only **non-crossing pairings** survive:

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Define limiting **semicircle element** s by

$$\varphi(s^n) := \#NC_2(n).$$

($s \in \mathcal{A}$, where $\mathcal{A}$ is some unital algebra, $\varphi : \mathcal{A} \to \mathbb{C}$)

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Then we say that our Gaussian random matrices A_N converge in distribution to the semicircle element s,

$$\boldsymbol{A_N} \xrightarrow{\text{distr}} \boldsymbol{s}$$

What is distribution of s?

Claim:

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more concretely:

$$\# NC_2(n) = \frac{1}{2\pi} \int_{-2}^{+2} t^n \sqrt{4 - t^2} dt$$

What is distribution of s?

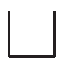
$n = 2$: $\varphi(s^2) =$

$n = 4$: $\varphi(s^4) =$

$n = 6$: $\varphi(s^6) =$

What is distribution of s?

$n = 2$: $\varphi(s^2) = 1$



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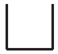
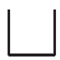


$n=4$: $\varphi(s^4)=2$

$n=6$: $\varphi(s^6)=$

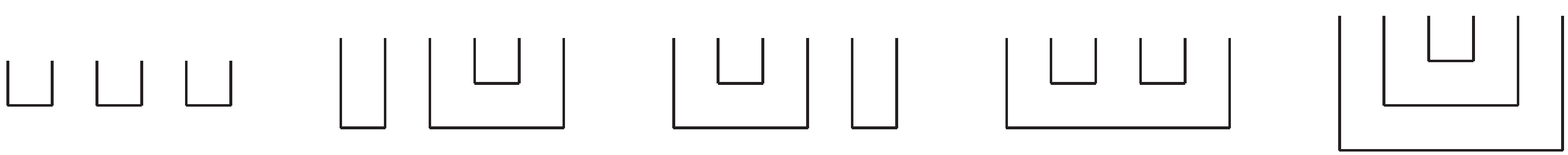
What is distribution of s?

$n = 2$: $\varphi(s^2) = 1$



$n = 4$: $\varphi(s^4) = 2$

$n = 6$: $\varphi(s^6) = 5$



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- $C_k = \frac{1}{k+1}\binom{2k}{k}$

- C_k is determined by $C_0 = C_1 = 1$ and the recurrence relation

$$C_k = \sum_{l=1}^{k} C_{l-1} C_{k-l}.$$

$$\varphi(s^{2k}) = \sum_{l=1}^{k} \varphi(s^{2l-2})\varphi(s^{2k-2l})$$

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$$M(z) := \sum_{n=0}^{\infty} \varphi(s^n) z^n = 1 + \sum_{k=1}^{\infty} \varphi(s^{2k}) z^{2k}$$

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Then

$$M(z) = 1 + z^2 \sum_{k=1}^{\infty} \sum_{l=1}^{k} \varphi(s^{2l-2})z^{2l-2}\varphi(s^{2k-2l})z^{2k-2l}$$

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$$= 1 + z^2 M(z) \cdot M(z)$$

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Instead of moment generating series $M(z)$ consider **Cauchy transform**

$$G(z) := \varphi(\frac{1}{z-s})$$

Note

$$G(z) = \sum_{n=0}^{\infty} \frac{\varphi(s^n)}{z^{n+1}} = \frac{1}{z} \sum_{n=0}^{\infty} \varphi(s^n) \Big(\frac{1}{z}\Big)^n = \frac{1}{z} M(1/z),$$

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thus

$$zG(z) = 1 + G(z)^2$$

For any probability measure μ on $\mathbb{R}$ corresponding Cauchy transform

$$G(z) := \int_{\mathbb{R}} \frac{1}{z-t} d\mu(t)$$

is analytic function on complex upper half plane $\mathbb{C}^+$ and allows to recover μ via **Stieltjes inversion formula**

$$d\mu(t) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \operatorname{Im} G(t+i\varepsilon)$$

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implies

$$G(z) = \frac{z - \sqrt{z^2 - 4}}{2}$$

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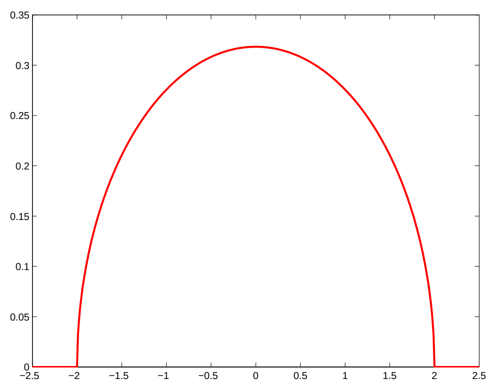
and thus

$$d\mu(t) = \frac{1}{2\pi}\sqrt{4 - t^2}dt \qquad \text{on } [-2, 2]$$



Consider now more general random matrices

- Keep the entries independent, but change distribution of entries, globally or depending on position of entry

  Arnold
  Bai und Silverstein
  Molchanov, Pastur, Khorunzhii (1996)
  Khorunzhy, Khoruzhenko, Pastur (1996)
  Shlyakhtenko (1996)
  Guionnet (2002)
  Anderson, Zeitouni (2006)

- Keep the distributions normal, but allow correlations between entries

  - for weak correlations one still gets semicircle

    Chatterjee (2006)
    Götze + Tikhomirov (2005)
    Schenker und Schulz-Baldes (2006)

  - for stronger correlations other distributions occur

    Boutet de Monvel, Khorunzhy, Vasilchuck (1996)
    Girko (2001)
    Hachem, Loubaton, Najim (2005)
    Anderson, Zeitouni (Preprint)
    Rashidi Far, Oraby, Bryc, Speicher (Preprint)

Consider **block matrix**

$$X_N = \begin{pmatrix} A_N & B_N & C_N \\ B_N & A_N & B_N \\ C_N & B_N & A_N \end{pmatrix},$$

where A_N, B_N, C_N are independent Gaussian $N \times N$-random matrices.

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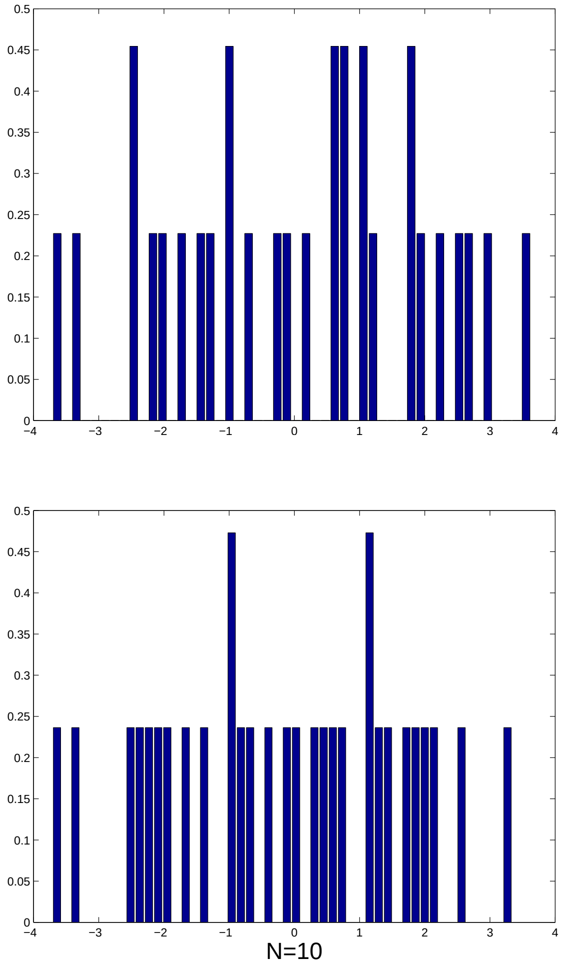
What is eigenvalue distribution of X_N for $N \to \infty$?

eigenvalue distribution for one realisation



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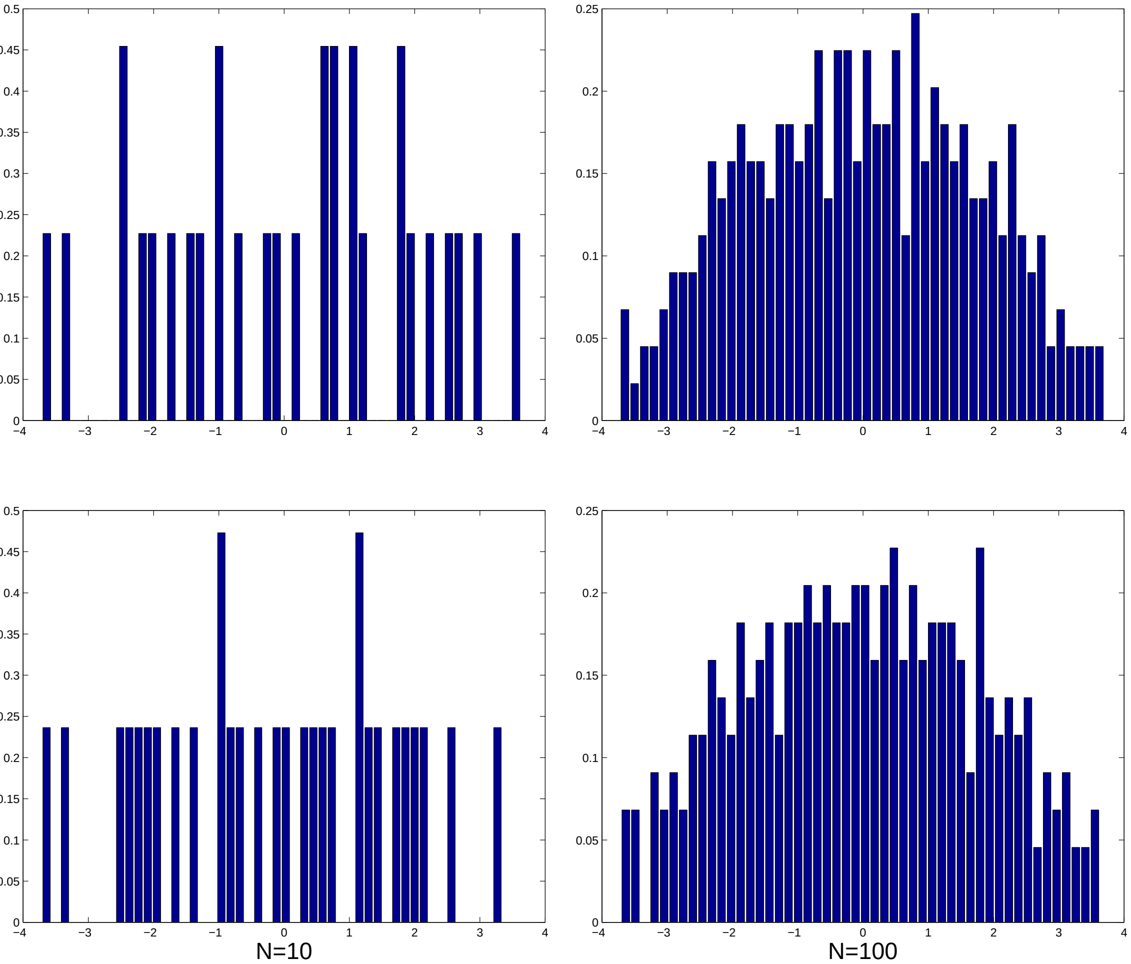

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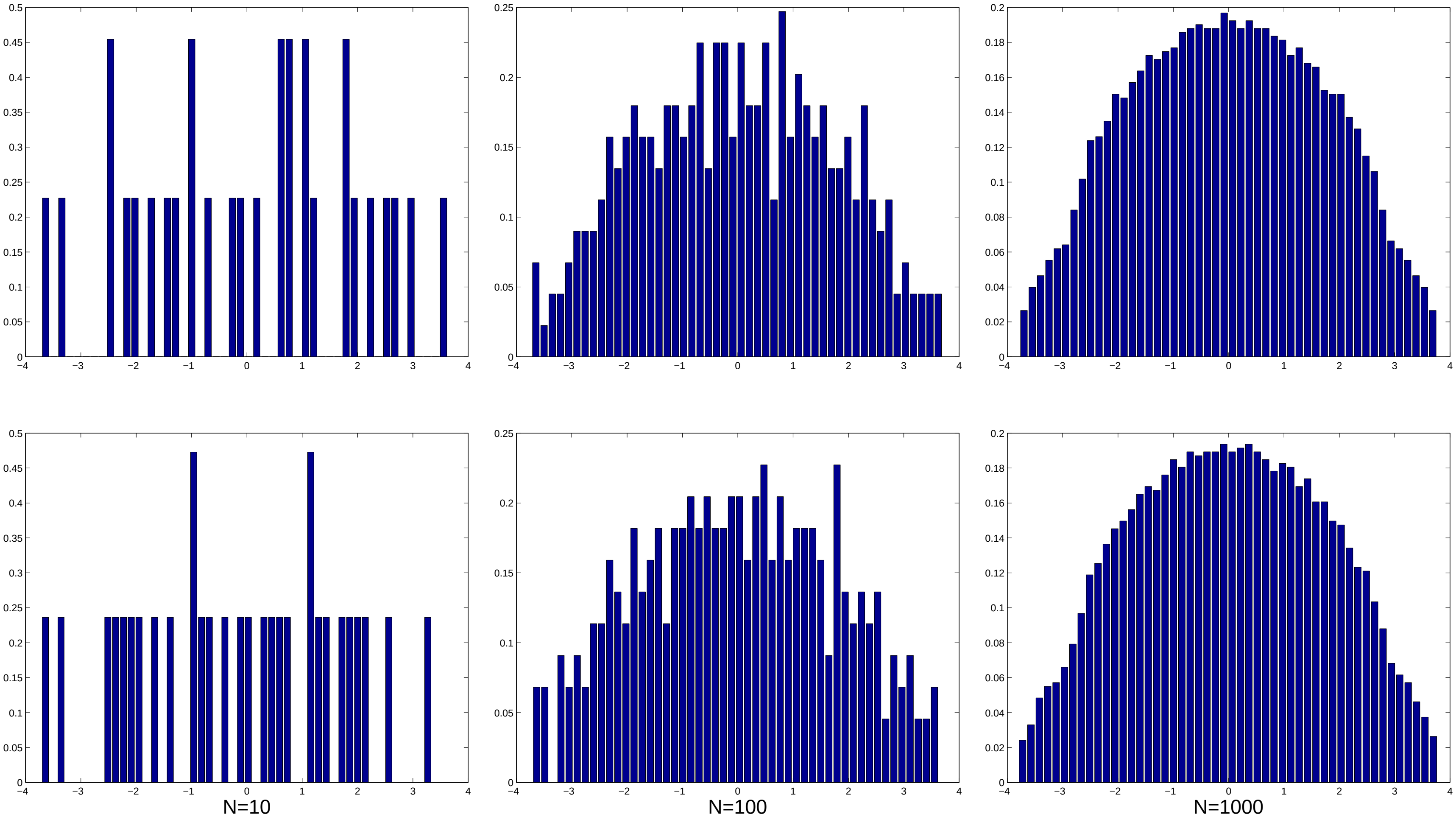
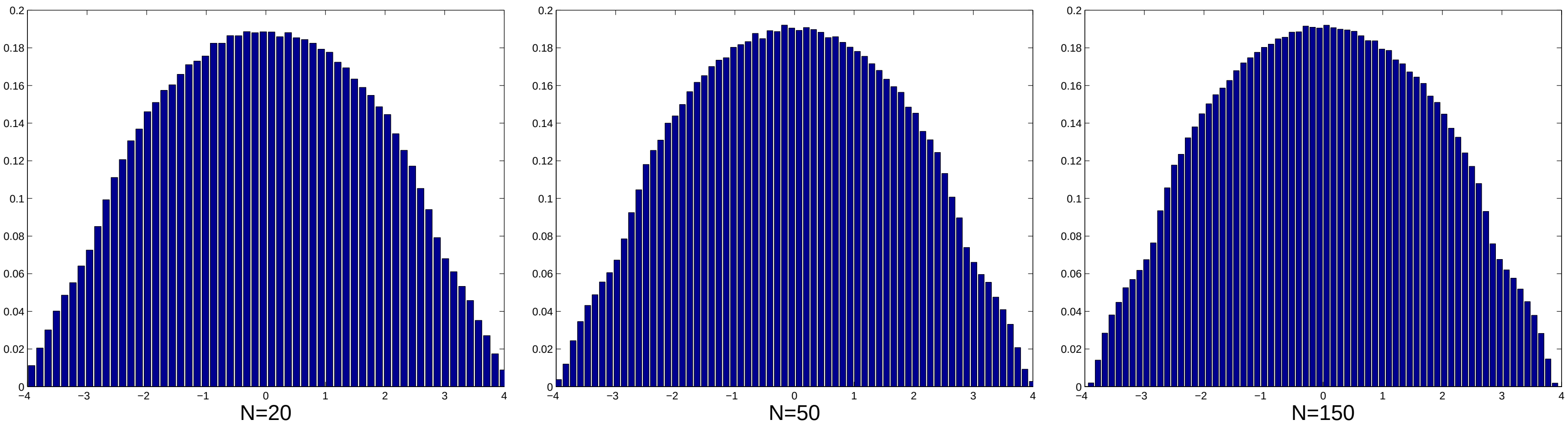


averaged eigenvalue distributions



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However, it fits well into the frame of

operator-valued free probability theory!

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scalars	$\longrightarrow$	operator-valued scalars
$\mathbb{C}$		$\mathcal{B}$

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state	$\longrightarrow$	conditional expectation
$\varphi : \mathcal{A} \to \mathbb{C}$		$E : \mathcal{A} \to \mathcal{B}$
		$E[b_1 a b_2] = b_1 E[a] b_2$

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		$E[b_1 a b_2] = b_1 E[a] b_2$
moments	$\longrightarrow$	operator-valued moments
$\varphi(a^n)$		$E[a b_1 a b_2 a \cdots a b_{n-1} a]$

$\mathcal{B}$ can be a general algebra of bounded operators on a Hilbert space; in the following just consider

$$\mathcal{B} \quad \hat{=} \quad \text{matrix algebra}$$

What is an **operator-valued semicircular element**?

Consider an operator-valued probability space

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  $$E[sbs] = \eta(b)$$

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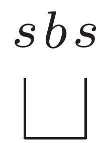
- higher moments of s are given in terms of second moments by summing over non-crossing pairings

$$E[sbs] = \eta(b)$$



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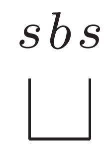
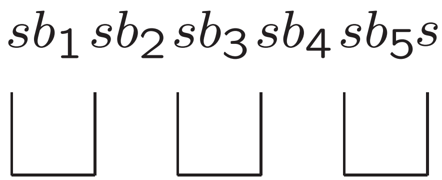
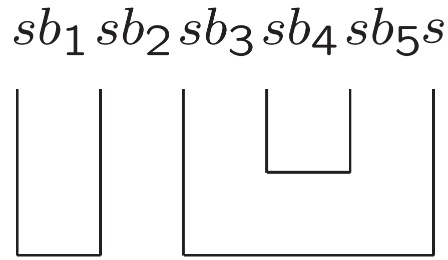


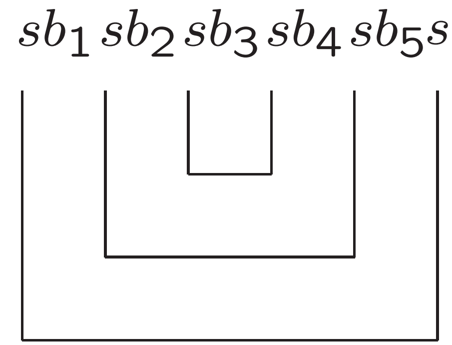
$$E[sb_1sb_2s\cdots sb_{n-1}s] = \sum_{\pi\in NC_2(n)} \Big(\text{iterated application of } \eta \text{ according to } \pi\Big)$$



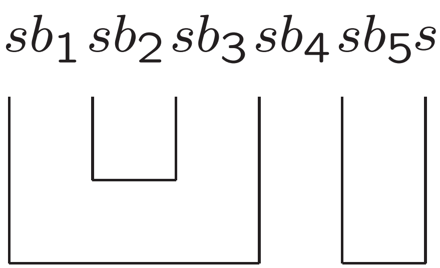
$$\eta(b_1)\cdot b_2\cdot\eta(b_3)\cdot b_4\cdot\eta(b_5)$$



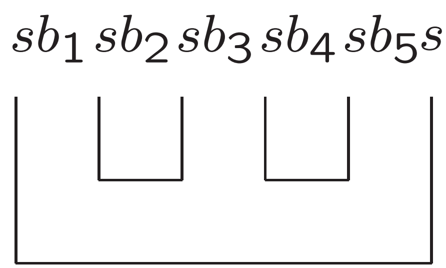
$$\eta(b_1)\cdot b_2\cdot\eta\Big(b_3\cdot\eta(b_4)\cdot b_5\Big)$$



$$\eta\Bigg(b_1\cdot\eta\Big(b_2\cdot\eta(b_3)\cdot b_4\Big)\cdot b_5\Bigg)$$

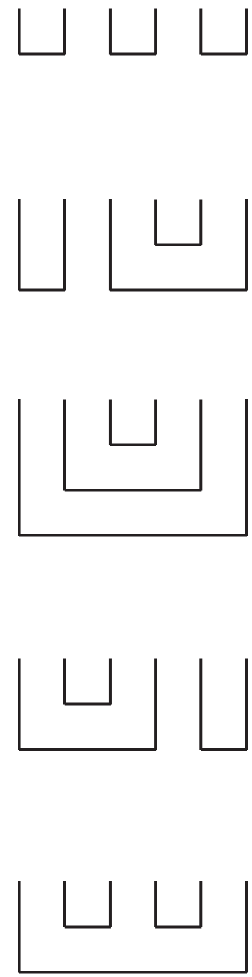


$$\eta\Big(b_1 \cdot \eta(b_2) \cdot b_3\Big) \cdot b_4 \cdot \eta(b_5)$$



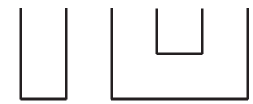
$$\eta\Big(b_1 \cdot \eta(b_2) \cdot b_3 \cdot \eta(b_4) \cdot b_5\Big)$$

$$
\begin{aligned}
E[sb_1sb_2sb_3sb_4sb_5s] =& \eta(b_1) \cdot b_2 \cdot \eta(b_3) \cdot b_4 \cdot \eta(b_5) \\
&+ \eta(b_1) \cdot b_2 \cdot \eta\Big(b_3 \cdot \eta(b_4) \cdot b_5\Big) \\
&+ \eta\bigg(b_1 \cdot \eta\Big(b_2 \cdot \eta(b_3) \cdot b_4\Big) \cdot b_5\bigg) \\
&+ \eta\Big(b_1 \cdot \eta(b_2) \cdot b_3\Big) \cdot b_4 \cdot \eta(b_5) \\
&+ \eta\Big(b_1 \cdot \eta(b_2) \cdot b_3 \cdot \eta(b_4) \cdot b_5\Big)
\end{aligned}
$$

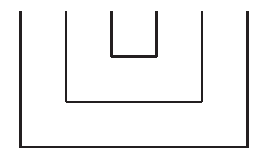


$$E[ssssss] = \eta(1) \cdot \eta(1) \cdot \eta(1)$$

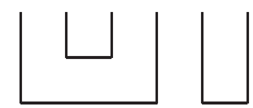
$$+ \eta(1) \cdot \eta\Big(\eta(1)\Big)$$



$$+ \eta\bigg(\eta\Big(\eta(1)\Big)\bigg)$$



$$+ \eta\Big(\eta(1)\Big) \cdot \eta(1)$$



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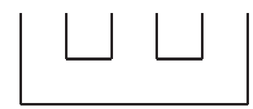

We have the recurrence relation

$$E[s^{2k}] = \sum_{l=1}^{k} \eta\Big(E[s^{2l-2}]\Big) \cdot E[s^{2k-2l}].$$

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Put

$$M(z) := \sum_{n=0}^{\infty} E[s^n] z^n = 1 + \sum_{k=1}^{\infty} E[s^{2k}] z^{2k},$$

thus

$$M(z) = 1 + z\eta\Big(M(z)\Big) \cdot M(z)$$

Consider the **operator-valued Cauchy transform**

$$G(z) := E[\frac{1}{z-s}]$$

for $z \in \mathbb{C}^+$.

Note

$$G(z) = E[\frac{1}{z} \cdot \frac{1}{1 - sz^{-1}}] = \frac{1}{z} M(sz^{-1}),$$

thus

$$zG(z) = 1 + \eta\Big(G(z)\Big) \cdot G(z)$$

Thus: operator-valued Cauchy-transform of s

$$G : \mathbb{C}^+ \to \mathcal{B}$$

satisfies

- G analytic

- G solution of

$$zG(z) = 1 + \eta\Big(G(z)\Big) \cdot G(z)$$

- $G(z) \sim \frac{1}{z}1$ for $z \to \infty$

back to random matrices

special classes of random matrices are asymptotically described by operator-valued semicircular elements, e.g.

- band matrices (Shlyakhtenko 1996)
- block matrices (Rashidi Far, Oraby, Bryc, Speicher 2006)

Example:

$$X_N = \begin{pmatrix} A_N & B_N & C_N \\ B_N & A_N & B_N \\ C_N & B_N & A_N \end{pmatrix},$$

where A_N, B_N, C_N are independent Gaussian $N \times N$ random matrices

For $N \to \infty$, X_N converges to

$$s = \begin{pmatrix} s_1 & s_2 & s_3 \\ s_2 & s_1 & s_2 \\ s_3 & s_2 & s_1 \end{pmatrix},$$

where s_1, s_2, s_3 is free semicircular family.

$$s = \begin{pmatrix} s_1 & s_2 & s_3 \\ s_2 & s_1 & s_2 \\ s_3 & s_2 & s_1 \end{pmatrix}, \qquad s_1, s_2, s_3 \in (\tilde{\mathcal{A}}, \varphi)$$

s is an operator-valued semicircular element over $M_3(\mathbb{C})$ with respect to

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- $E = \mathrm{id} \otimes \varphi : M_3(\tilde{\mathcal{A}}) \to M_3(\mathbb{C}), \quad \Big(a_{ij}\Big)_{i,j=1}^{3} \mapsto \Big(\varphi(a_{ij})\Big)_{i,j=1}^{3}$

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- $\eta : M_3(\mathbb{C}) \to M_3(\mathbb{C})$ given by $\eta(D) = E[sDs]$

$$s = \begin{pmatrix} s_1 & s_2 & s_3 \\ s_2 & s_1 & s_2 \\ s_3 & s_2 & s_1 \end{pmatrix}, \qquad s_1, s_2, s_3 \in (\tilde{\mathcal{A}}, \varphi)$$

Asymptotic eigenvalue distribution μ of X_N is given by distribution of s with respect to $\text{tr}_3 \otimes \varphi$:

$$H(z) = \int \frac{1}{z-t} d\mu(t) = \text{tr}_3 \otimes \varphi\Big(\frac{1}{z-s}\Big)$$

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and $G(z) = E[\frac{1}{z-s}]$ is solution of

$$zG(z) = 1 + \eta\Big(G(z)\Big) \cdot G(z)$$

$$X = \begin{pmatrix} A & B & C \\ B & A & B \\ C & B & A \end{pmatrix} :$$

$$X = \begin{pmatrix} A & B & C \\ B & A & B \\ C & B & A \end{pmatrix} : \qquad G(z) = \begin{pmatrix} f(z) & 0 & h(z) \\ 0 & g(z) & 0 \\ h(z) & 0 & f(z) \end{pmatrix}$$

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$$\eta\Big(G(z)\Big) = \frac{1}{3}\begin{pmatrix} 2\,f(z)+g(z) & 0 & g(z)+2\,h(z) \\ 0 & 2\,f(z)+g(z)+2\,h(z) & 0 \\ g(z)+2\,h(z) & 0 & 2\,f(z)+g(z) \end{pmatrix},$$

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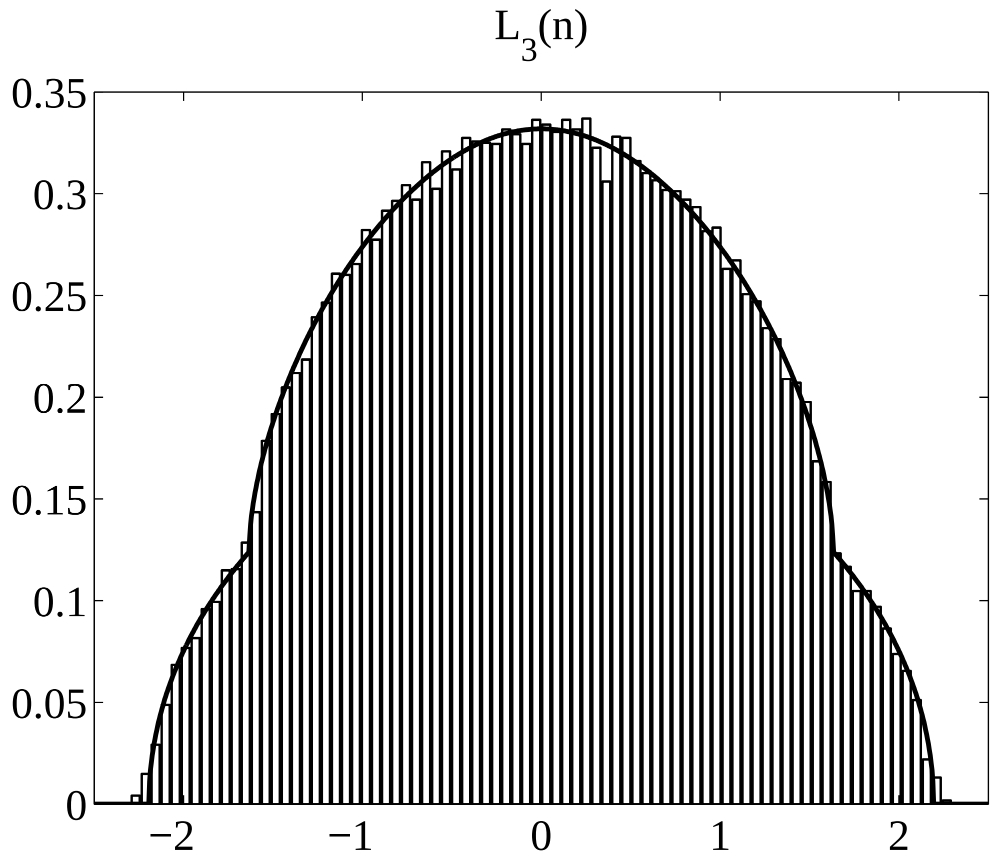
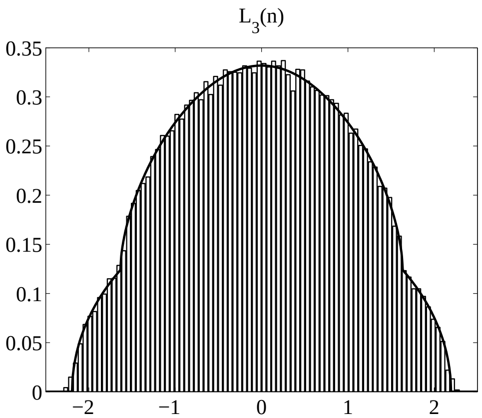
$$zG(z) = 1 + \eta\Big(G(z)\Big) \cdot G(z)$$

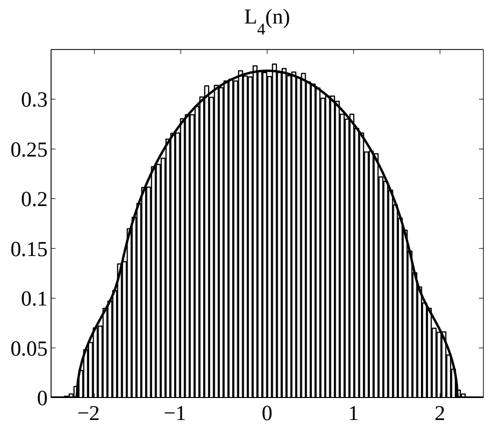
$$H(z) = \mathrm{tr}_3(G(z)) = \frac{1}{3}(2f(z) + g(z))$$

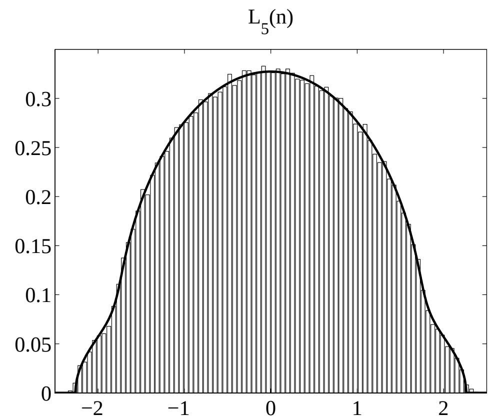
Comparison of this solution with simulations



some more examples







$$\begin{pmatrix} A & B & C \\ B & A & B \\ C & B & A \end{pmatrix} \qquad \begin{pmatrix} A & B & C & D \\ B & A & B & C \\ C & B & A & B \\ D & C & B & A \end{pmatrix} \qquad \begin{pmatrix} A & B & C & D & E \\ B & A & B & C & D \\ C & B & A & B & C \\ D & C & B & A & B \\ E & D & C & B & A \end{pmatrix},$$

So we have solved the problem of finding the eigenvalue distribution of block matrices by recognizing a block matrix as an operator-valued semicircular element ...

So we have solved the problem of finding the eigenvalue distribution of block matrices by recognizing a block matrix as an operator-valued semicircular element ...

... however, we are now actually facing another problem:

How to deal with our special

matrix-valued quadratic equations?

Problem: We have to solve

$$zG(z) = 1 + \eta\Big(G(z)\Big) \cdot G(z) \qquad (*)$$

for z close to real axis.

Recall: Stieltjes inversion formula

$$d\mu(t) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \operatorname{Im} H(t + i\varepsilon)$$

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For fixed z,

$$\begin{aligned} (*) \quad &= \quad \text{quadratic matrix equation} \\ &\widehat{=} \quad \text{system of quadratic equations} \end{aligned}$$

has many solutions!

How do we find the right one?

$$X = \begin{pmatrix} A & B & C \\ B & A & B \\ C & B & A \end{pmatrix}, \qquad G(z) = \begin{pmatrix} f(z) & 0 & h(z) \\ 0 & g(z) & 0 \\ h(z) & 0 & f(z) \end{pmatrix}$$

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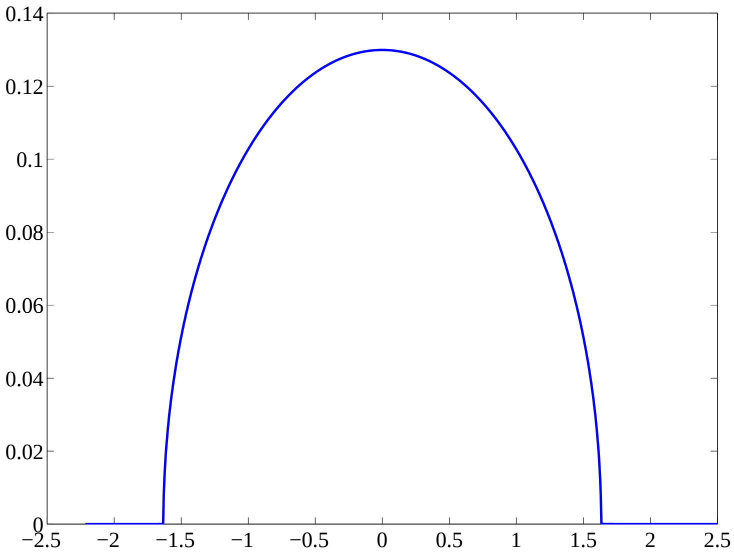
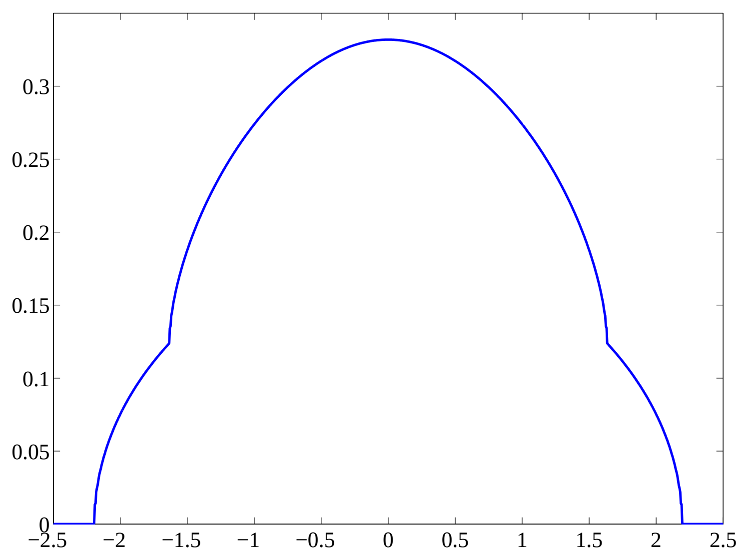
$$zf = 1 + \frac{g\ (f + h) + 2\ \left(f^2 + h^2\right)}{3}$$

$$zg = 1 + \frac{g\ (g + 2\ (f + h))}{3}$$

$$zh = \frac{4\,f\,h + g\ (f + h)}{3}.$$



solving the system
by Newton's method
with 'bad' initial values



correct solution

We are interested in a solution with a special **positivity property**:
For $\operatorname{Im} z > 0$ we have

$$\operatorname{Im} G(z) = \frac{1}{2i} E[\frac{1}{z-s} - \frac{1}{\bar{z}-s}] \leq -\frac{\operatorname{Im} z}{|z| + \|s\|^2} 1$$

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Thus: we look for

- a solution of $\quad zG(z) = 1 + \eta\Big(G(z)\Big) \cdot G(z)$

- with $\quad \operatorname{Im} G(z) := \frac{G(z) - G(z)^*}{2i} < 0$

Set

$$G(z) := -iW(z)$$

Then

- $\operatorname{Re} W(z) := \frac{W(z)+W(z)^*}{2} > 0$

- $-izW(z) + \eta\Big(W(z)\Big) \cdot W(z) = 1$

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Claim: For each z with $\operatorname{Im} z > 0$ there is exactly one solution with this property!

$$-izW(z)+\eta\Big(W(z)\Big)\cdot W(z)=1 \qquad \Longleftrightarrow \qquad \mathcal{F}_z(W)=W \qquad (**)$$

for

$$\mathcal{F}_z(W)=\frac{1}{-iz1+\eta(W)}$$

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Put

$$\mathcal{B}_+ := \{W \in \mathcal{B} \mid \operatorname{Re} W > 0\}$$

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Put

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Theorem [Helton, Rashidi Far, Speicher 2007]: For $\operatorname{Im} z>0$ there exists exactly one solution $W\in\mathcal{B}_+$ to $(**)$; this W is the limit of iterates

$$W_n=\mathcal{F}_z^n(W_0)$$

for any $W_0\in\mathcal{B}_+$.

Note: $\mathcal{F}_z$ is **not** a contraction in operator norm.

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However, is contraction in some other, canonical metric!

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Theorem [Earle, Hamilton 1968]: Consider X complex Banach space, $\emptyset \neq \mathcal{D}$ domain, $h : \mathcal{D} \to \mathcal{D}$ bounded holomorphic function.

If $h(\mathcal{D})$ lies strictly$^{(*)}$ inside $\mathcal{D}$, then h is a **strict contraction in the Caratheodory metric ρ**, thus has a unique fixed point in $\mathcal{D}$.

Furthermore, $\exists m > 0 : \rho(x, y) \geq m\|x - y\|$ for all $x, y \in \mathcal{D}$ and thus $\left(h^n(x_0)\right)_n$ converges in norm to fixed point, for all $x_0 \in \mathcal{D}$

$^{(*)}$ $\exists \varepsilon > 0 \forall x \in \mathcal{D} : B_\varepsilon(h(x)) \subset \mathcal{D}$

in our case:

- $h = \mathcal{F}_z$ analytic
- $\mathcal{D} = R_b := \{W \in \mathcal{B}_+ \mid \|W\| < b\}$

Check that $\mathcal{F}_z(R_b)$ lies strictly in R_b.

Note: $\mathcal{B}_+ = \bigcup_{b>0} R_b$

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Note: $\mathcal{B}_+ = \bigcup_{b>0} R_b$

Remark: Same arguments work for

$$ZG(Z) = 1 + \eta(G(Z)) \cdot G(Z)$$

where $Z \in \mathcal{B}$ with $\operatorname{Im} Z = \frac{Z - Z^*}{2i} > 0$ and invertible.

Theorem [Helton, Rashidi Far, Speicher 2007]: For fixed $V \in \mathcal{B}_+$ there exists exactly one solution $W \in \mathcal{B}_+$ to

$$VW + \eta(W)W = 1;$$

this W is the limit of iterates $W_n = \mathcal{F}_V^n(W_0)$ for any $W_0 \in \mathcal{A}_+$, where

$$\mathcal{F}_V(W) := \frac{1}{V + \eta(W)}.$$

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Furthermore,

$$\|W\| \leq \|(\mathrm{Re}\, V)^{-1}\|, \qquad \mathrm{Re}\, W \geq \frac{1}{m^2 \cdot \|(\mathrm{Re}\, V)^{-1}\|}$$

where

$$m := \|V\| + \|\eta\| \cdot \|(\mathrm{Re}\, V)^{-1}\|$$

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- interesting classes of random matrices with correlations between entries can be described by **operator-valued semicircular elements**

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- operator-valued semicircular elements have an accessible operator-valued structure, in terms of combinatorics of non-crossing pairings

- corresponding equation for operator-valued Cauchy transform (**quadratic matrix equation**) has exactly one solution with the right positivity properties

Literature on the combinatorial side of free probability

- A. Nica and R. Speicher: Lectures on the Combinatorics of Free Probability.
  London Mathematical Society Lecture Note Series, vol. 335, Cambridge University Press, 2006

- R. Speicher: Combinatorial theory of the free product with amalgamation and operator-valued free probability theory. Memoir of the AMS 627 (1998)