Orbital Free Entropy and Its Dimension Counterpart

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Nov. 2007

Mutual Information

$$I(X,Y) = \int\int \log \frac{d\mu_{(X,Y)}}{d(\mu_X \otimes \mu_Y)} d\mu_{(X,Y)} \quad \text{for r.v.'s } X, Y$$

$$= \int\int \log \frac{d\mu_{(X,Y)}}{dxdy}\, d\mu_{(X,Y)}$$

$$- \int\int \log \frac{d\mu_X}{dx} d\mu_{(X,Y)} - \int\int \log \frac{d\mu_Y}{dy}\, d\mu_{(X,Y)}$$

$$= -H(X,Y) + H(X) + H(Y).$$

Properties.

- $I(X,Y) = 0$ if and only if X, Y are independent.
- $I(X,Y)$ depends only on $\sigma(X), \sigma(Y)$ inside $\sigma(X,Y)$.
- $H(X,Y) = -I(X,Y) + H(X) + H(Y)$.

Free Analog ?

- $H(X,Y), H(X), H(Y) \rightsquigarrow \chi(X,Y), \chi(X), \chi(Y)$
- $\mathrm{RelEnt}(\mu_{(X,Y)}, \mu_X \otimes \mu_Y) \rightsquigarrow$ No analog !
- "Free mutual information" should be determined by $W^*(X), W^*(Y) \subseteq W^*(X,Y)$, and "$= 0$" iff "$X, Y$ are freely independent."
- If $\chi(X), \chi(Y), \chi(X,Y)$ are finite, then the "free mutual information" should be:

$$-\chi(X,Y) + \chi(X) + \chi(Y)$$

- Non-microstate definition of "free mutual information" was already introduced by Voiculescu; where the process $\mathrm{Ad}U_t(\,\cdot\,)$ with free unitary BM U_t plays an important role. However no relation to χ (or χ^*) has been yet proven !

Setting & Notations

- a tracial W^*-prob. sp.
  = a vN alg. M with a tracial state τ.
- (non-comm.) r.v.'s (write X, Y, etc.)
  = self-adjoints in M.
- (non-comm.) multi-r.v.'s (write $\mathbb{X}, \mathbb{Y}$, etc.)
  = families of self-adjoints in M.
- $W^*(X)$ or $W^*(\mathbb{X})$
  = the vN alg. gen. by X or resp. $\mathbb{X}$ inside M.
- $M_N(\mathbf{C})^{sa}$ = the $N \times N$ self-adjoints;
  $\mathbf{R}^N_{\geq}$ = its diagonals with decreasing entries.

Voiculescu's Microstate Free Ent. χ

$\Gamma_R(X_1, \ldots, X_n; N, m, \gamma)$ (possibly $R = \infty$)
= all $(A_1, \ldots, A_n) \in (M_N(\mathbb{C})^{sa})^n$ satisfying

- $||A_k||_\infty \leq R$ (nothing when $R = \infty$),
- $\left|\frac{1}{N}\mathrm{Tr}_N(A_{i_1} \cdots A_{i_l}) - \tau(X_{i_1} \cdots X_{i_l})\right| < \gamma$ ($l \leq m$).

$\chi_R(X_1, \ldots, X_n)$ is the $\inf_{m=1,2,\ldots,\gamma>0} \limsup_{N\to\infty}$ of

$$\frac{1}{N^2} \log \mathrm{Vol}(\Gamma_R(X_1, \ldots, X_n; N, m, \gamma)) + \frac{n}{2} \log N.$$

Property & Def.

$$\chi := \chi_\infty \overset{\text{[B-B]}}{=} \sup_{R>0} \chi_R \ (\overset{\text{[V]}}{=} \chi_R \text{ for large } R > 0)$$

(Write $\chi_* = \chi_*(X_1, \ldots, X_n)$ for short.)

Orbital Microstates

$$
\begin{aligned}
&(M_N(\mathbb{C})^{sa}, \mathrm{Vol}) \\
&\qquad \cong \Big((\mathrm{U}(N)/\mathbb{T}^N) \times \mathbb{R}^N_{\geq},\ \text{LeftInvProb.} \otimes \nu_N \Big) \\
&\qquad \twoheadleftarrow \Big(\mathrm{U}(N) \times \mathbb{R}^N_{\geq},\ \text{HaarProb.} \otimes \nu_N \Big)
\end{aligned}
$$

Identify $(M_N(\mathbb{C})^{sa})^n$ with $(\mathrm{U}(N)/\mathbb{T}^N)^n \times \mathbb{R}^{Nn}_{\geq}$, and get

$$
\begin{aligned}
&\Gamma_R(X_1, \ldots, X_n; N, m, \gamma) \subseteq (\mathrm{U}(N)/\mathbb{T}^N)^n \times \mathbb{R}^{Nn}_{\geq} \\
&\qquad \rightsquigarrow \Gamma_{\mathrm{orb},R}(X_1, \ldots, X_n; N, m, \gamma) \subseteq \mathrm{U}(N)^n
\end{aligned}
$$

Orbital Microstates, continued

$\Delta_R(X_k; N, m, \gamma) := \Gamma_R(X_k; N, m, \gamma) \cap \mathbf{R}^N_{\geq}$,
$\mathbb{P}_u$:= HaarProb.

Fact:

$$\begin{aligned}
&\mathrm{Vol}(\Gamma_R(X_1, \ldots, X_n; N, m, \gamma)) \\
&\leq \mathbb{P}_u(\Gamma_{\mathrm{orb},R}(X_1, \ldots, X_n; N, m, \gamma)) \times \prod_{k=1}^{n} \nu_N(\Delta_R(X_k; N, m, \gamma)) \\
&= \mathbb{P}_u(\Gamma_{\mathrm{orb},R}(X_1, \ldots, X_n; N, m, \gamma)) \times \prod_{k=1}^{n} \mathrm{Vol}(\Gamma_R(X_k; N, m, \gamma))
\end{aligned}$$

Moreover, the converse inequality holds “in the limit” as $N \to \infty$, $m \to \infty$, $\gamma \searrow 0$!!!

Orbital Free Ent. – Def. & Rel. to χ

Def. [Hiai-Miyamoto-U]

- $\chi_{\mathrm{orb},R}(X_1,\dots,X_n)$ is the $\inf_{m=1,2,\dots,\gamma>0} \limsup_{N\to\infty}$ of $\frac{1}{N^2}\log \mathbb{P}_u(\Gamma_{\mathrm{orb},R}(X_1,\dots,X_n;N,m,\gamma))$.
- $\chi_{\mathrm{orb}} := \sup_{R>0}\chi_{\mathrm{orb},R}$ ($= \chi_{\mathrm{orb},R}$ for large $R>0$) (write $\chi_{\mathrm{orb},R} = \chi_{\mathrm{orb},R}(X_1,\dots,X_n)$, etc., for simplicity).

Thm 1. [H-M-U]

$$\chi(X_1,\dots,X_n) = \chi_{\mathrm{orb}}(X_1,\dots,X_n) + \sum_{k=1}^n \chi(X_k)$$

(possibly with $-\infty = -\infty$)

Orbital Free Ent. is determined by $W^*(X_k)$'s

Thm. 2 [H-M-U]
If $W^*(X_k) = W^*(Y_k)$ for all $k = 1, \ldots, n$, then

$$\chi_{\mathrm{orb}}(X_1, \ldots, X_n) = \chi_{\mathrm{orb}}(Y_1, \ldots, Y_n).$$

Therefore,

$$\chi_{\mathrm{orb}}(X_1, \ldots, X_n) = \text{“}\chi_{\mathrm{orb}}(W^*(X_1), \ldots, W^*(X_n)).\text{”}$$

Orbital Free Ent. – Rel. to Free Indep.

Prop. 3

If X_1, $\{X_2, \ldots, X_n\}$ are free, then

$$\chi_{\mathrm{orb}}(X_1, X_2, \ldots, X_n) = \chi_{\mathrm{orb}}(X_2, \ldots, X_n).$$

(Proof in the case when $X_1, \ldots, X_n$ are freely indep.)

- **Choose $\xi_k(N) \in \mathrm{R}^N_{\geq}$ s.t. $\xi_k(N) \to X_k$ in moments, i.e., $\forall m, \gamma$, $\exists N$, $\xi_k(N) \in \Delta_R(X_k; N, m, \gamma)$.**
- **Voiculescu's asymptotic freeness:**

  $\mathbb{P}_u\{(U_k)_{k=1}^n : (U_k \xi_k(N) U_k^*)_{k=1}^n$ **"almost" free$\}$** $\geq \exists \varepsilon > 0$ **for all large N**
- $0 \geq \frac{1}{N^2} \log \mathbb{P}_u(\cdots) \geq \frac{\log \varepsilon}{N^2} \to 0$ **as $N \to 0$. QED**

Orbital Microstates, revisited

Choose $\xi_k(N) \in \mathbb{R}^N_{\geq}$ (in $M_N(\mathbb{C})^{sa}$ O.K.) in such a way that $\xi_k(N) \longrightarrow X_k$ in moments as $N \to \infty$.

$\Gamma_{\mathrm{orb}}(X_1, \dots, X_n : \xi_1(N), \dots, \xi_n(N); N, m, \gamma)$ is

$$\{(U_k)_{k=1}^n : (U_k \xi_k(N) U_k^*)_{k=1}^n \in \Gamma(X_1, \dots, X_n; N, m, \gamma)\}$$

$$\rightsquigarrow \chi_{\mathrm{orb}}(X_1, \dots, X_n : \xi) \text{ with } \xi = \{(\xi_1(N), \dots, \xi_n(N))\}_{N=1}^\infty$$

Prop. 4 $\quad \chi_{\mathrm{orb}}(\cdots) = \chi_{\mathrm{orb}}(\cdots : \xi)$.

Cor. $\quad \chi_{\mathrm{orb}}(P_1, \dots, P_n) = \chi_{\mathrm{proj}}(P_1, \dots, P_n)$.

$\chi_{\mathrm{orb}} = 0$ implies Free Indep.

Prop. 5 (A new variant of multivariable TCI)

$$W_{\mathrm{free},2}(\mu_{(X_1,\dots,X_n)}, \mu_{X_1} \star \cdots \star \mu_{X_n}) \le C\sqrt{-\chi_{\mathrm{orb}}(X_1,\dots,X_n)}$$

with a universal constant $C > 0$ depending only on the op.-norms of X_k's.

Key ingredient: "Orbital RM-model" $U_k \mapsto U_k \xi_k(N) U_k^*$ of X_k

Thm 6 [H-M-U]

$$\chi_{\mathrm{orb}}(X_1,\dots,X_n) = 0$$

$$\Longrightarrow \quad \mu_{(X_1,\dots,X_n)} = \mu_{X_1} \star \cdots \star \mu_{X_n}$$

$\Longleftrightarrow \quad X_1,\dots,X_n$ are freely indep.

Generalization: χ_{orb} of hyperfinite multivariables

The description of χ_{orb} by $\chi_{\mathrm{orb}}(\cdots) := \chi_{\mathrm{orb}}(\cdots : \xi)$ allows to define χ_{orb} for $\mathbb{X}_1, \dots, \mathbb{X}_n$ with all $W^*(\mathbb{X}_k)$'s hyperfinite

$\rightsquigarrow$ χ_{orb} of hyperfinite multivariables $\chi_{\mathrm{orb}}(\mathbb{X}_1, \dots, \mathbb{X}_n)$.

- Each approx. seq. $\xi_k(N)$ should be changed from one element to a set of microstates.
- All the properties of χ_{orb} still hold for hyperfinite multivariables, but

  $$\chi(\mathbb{X}_1 \sqcup \cdots \sqcup \mathbb{X}_n) = \chi_{\mathrm{orb}}(\mathbb{X}_1, \dots, \mathbb{X}_n) + \sum_{k=1}^n \chi(\mathbb{X}_k)$$

  is meaningless since it always becomes $-\infty = -\infty$.
- Hyperfiniteness is essential due to Jung's result of unitary equiv. on embeddings to R^ω.

Bonus $\mathbb{Y}_k \subset W^*(\mathbb{X}_k)$ for all k implies

$$\chi_{\mathrm{orb}}(\mathbb{X}_1, \dots, \mathbb{X}_n) \leq \chi_{\mathrm{orb}}(\mathbb{Y}_1, \dots, \mathbb{Y}_n)$$

Orbital Free Ent. Dim. [HMU]

Replacing

$$\chi \quad \rightsquigarrow \quad \chi_{\mathrm{orb}};$$

$$X + \sqrt{t}S \quad \rightsquigarrow \quad X \mapsto U_t X U_t^*$$

gives the new dimensions:

$\delta_{\mathrm{orb}}(\mathbb{X}_1, \dots, \mathbb{X}_n)$ (orbital free ent. dim.)

$$:= \limsup_{\varepsilon \searrow 0} \frac{\chi_{\mathrm{orb}}(U_\varepsilon^{(1)}\mathbb{X}_1 U_\varepsilon^{(1)*}, \dots, U_\varepsilon^{(n)}\mathbb{X}_n U_\varepsilon^{(n)*})}{|\log \sqrt{\varepsilon}|},$$

$\delta_{0,\mathrm{orb}}(\mathbb{X}_1, \dots, \mathbb{X}_n)$ (modified orbital free ent. dim.)

$$:= \limsup_{\varepsilon \searrow 0} \frac{\chi_{\mathrm{orb}}(U_\varepsilon^{(1)}\mathbb{X}_1 U_\varepsilon^{(1)*}, \dots, U_\varepsilon^{(n)}\mathbb{X}_n U_\varepsilon^{(n)*} : U_\varepsilon^{(1)}, \dots, U_\varepsilon^{(n)})}{|\log \sqrt{\varepsilon}|}$$

Orbital Free Ent. Dim., continued

$$\delta_{\mathrm{orb}}(X_1,\ldots,X_n)$$

$$= \limsup_{\varepsilon\searrow 0} \frac{\chi_{\mathrm{orb}}(U_\varepsilon^{(1)}X_1U_\varepsilon^{(1)*},\ldots,U_\varepsilon^{(n)}X_nU_\varepsilon^{(n)*})}{|\log\sqrt{\varepsilon}|}$$

$$= \limsup_{\varepsilon\searrow 0} \frac{\chi_{(}U_\varepsilon^{(1)}X_1U_\varepsilon^{(1)*},\ldots,U_\varepsilon^{(n)}X_nU_\varepsilon^{(n)*}) - \sum_{k=1}^{n}\chi(U_\varepsilon^{(k)}X_kU_\varepsilon^{(k)*})}{|\log\sqrt{\varepsilon}|}$$

$$= \limsup_{\varepsilon\searrow 0} \frac{\chi(U_\varepsilon^{(1)}X_1U_\varepsilon^{(1)*},\ldots,U_\varepsilon^{(n)}X_nU_\varepsilon^{(n)*})}{|\log\sqrt{\varepsilon}|}.$$

Orbital Free Ent. Dim., continued

- $\delta_{0,\mathrm{orb}} \leq 0$ and Free Indep. implies $\delta_{0,\mathrm{orb}} = 0$.
- Reverse monotonicity, i.e., $\mathbb{Y}_k \subset W^*(\mathbb{X}_k)$ for all k implies
$$\delta_{0,\mathrm{orb}}(\mathbb{X}_1, \ldots, \mathbb{X}_n) \leq \delta_{0,\mathrm{orb}}(\mathbb{Y}_1, \ldots, \mathbb{Y}_n).$$
- Jung's packing formulation of δ_0 still works well for $\delta_{0,\mathrm{orb}}$.
- Rel. to δ_0:
$$\delta_0(\mathbb{X}_1, \ldots, \mathbb{X}_n) \leq \delta_{0,\mathrm{orb}}(\mathbb{X}_1, \ldots, \mathbb{X}_n) + \sum_{k=1}^{n} \delta_0(W^*(\mathbb{X}_k)).$$
- Equality can be proven when all $W^*(\mathbb{X}_k)$ are finite dim. We conjecture it is true in general !

Recent progress

- More Precise Upper bdd:

  $\delta_{0,\mathrm{orb}}(\mathbb{X}_1,\dots,\mathbb{X}_n) \leq -(n-1)\,\delta_0(W^*(\mathbb{X}_1)\cap\cdots\cap W^*(\mathbb{X}_n))$,

  $\rightsquigarrow$ Jung's hyperfinite ineq. for δ_0 when every "components" are hyperfinite.

- Equality holds when $\mathbb{X}_1,\dots,\mathbb{X}_n$ are free with amal. over $W^*(\mathbb{X}_1)\cap\cdots\cap W^*(\mathbb{X}_n)$ (use Brown-Dykema-Jung).

Recent Progress, continued

- (joint with Masaki Izumi and Fumio Hiai)
  In the case of $\tau(P) = \tau(Q) = 1/2$ we could prove
  $$-\chi_{\mathrm{orb}}(P, Q) = -\chi_{\mathrm{proj}}(P, Q) = i^*(W^*(P); W^*(Q))$$
  under a very reasonable assumption – a certain "weighted" L^3-condition for PQP + something.