Unit equations having few solutions

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§1 Unit equations

Let K be a number field with $[K : \mathbb{Q}] = d$.

Let S be a finite set of places of K containing all infinite places.

Put $s = \#S < \infty$.

S-integers $\mathfrak{O}_S := \{x \in K \,:\, v(x) \geq 0 \, for \, \forall v \notin S\}$

S-units $U_S := \{x \in K \,:\, v(x) = 0 \, for \, \forall v \notin S\}$.

For $\alpha_1, \alpha_2 \in K^* := K - \{0\}$, consider the Unit equation

$$\alpha_1 x + \alpha_2 y = 1 \quad \text{in} \quad x, y \in U_S \tag{1}$$

Theorem [Siegel, Mahler, Lang]

The Unit equation

$$\boldsymbol{\alpha_1 x + \alpha_2 y = 1} \quad \text{in} \quad \boldsymbol{x, y \in U_S}$$

has only finitely many solutions.

Proven by means of Diophantine approximation of

Thue-Siegel-Roth-Mahler in 2 variables' case and

by the Subspace Theorem of W. M. Schmidt in $\boldsymbol{n}$ variables' case.

Theorem [Roth's theorem]

Let α be an algebraic number of degree d (≥ 2).

For any $\varepsilon > 0$, there are only finitely many rational numbers $\dfrac{p}{q}$ with $q > 0$ satisfying the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\varepsilon}}$$

It is not easy to precise more; namely

Conjecture **[Still Open]**

Let α be an algebraic number of degree ≥ 3.

Then there exists a constant $\kappa_0(\alpha) > 0$

depending on α with the following property ;

if $\kappa > \kappa_0(\alpha)$,

then the rational numbers $\frac{p}{q}$ with $q > 0$ satisfying

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2(\log q)^\kappa}$$

are only finitely many.

Definition For $X \in \mathbb{P}^N(\overline{\mathbb{Q}})$ and a number field K such that $(x_0, \ldots, x_N) \in \mathbb{P}^N(K)$, define a logarithmic height of X by

$$h(X) := \frac{1}{[K:\mathbb{Q}]} \sum_v n_v \log(\max\{|x_0|_v, \ldots, |x_N|_v\})$$

with $n_v = [K_v : \mathbb{Q}_v]$ the local degree.

Definition

For $\alpha \in \overline{\mathbb{Q}}$ and $K = \mathbb{Q}(\alpha)$, define $h(\alpha) := h(1 : a)$.

For $X = (x_1, \cdots, x_N) \in \mathbb{A}^N(\overline{\mathbb{Q}})$,

define an exponential height of X by $H(X) := \exp(h(1, x_1, \cdots, x_N))$.

Let $L_i(X) = a_{i1}X_1 + \cdots + a_{iN}X_N$ $(i = 1, \ldots, N)$ be N linearly independent linear forms with coefficients in K. Let $\delta > 0$.

Consider the inequality

$$|L_1(x) \cdots L_N(x)| \leq H(x)^{-\delta} \;\; \text{in } x \in \mathbb{Z}^N.$$

Theorem **[The Subspace Theorem, W. M. Schmidt]**

The set of the solutions $x \in \mathbb{Z}^N$ to the above inequality

is contained in a finite union

$T_1 \cup \cdots \cup T_t$ of proper linear subspaces $T_1, \cdots, T_t$ of $\mathbb{Q}^N$.

§2 General Unit equations

It is shown the finiteness of the solutions to the Unit equations by van der Poorten-Schlickewei in 2 variables' case, and by Evertse-Schlickeweri-Schmidt, in the following general setting.

Let K a field of characteristic 0.

Let Γ be a subgroup of $(K^*)^n$ of finite rank $r < \infty$.

Namely, there are $u_1, u_2, \cdots, u_r \in \Gamma$ satisfying that

for every $x \in \Gamma \subset (K^*)^n$ there exist integers

$z, z_1, \cdots, z_r$ such that $x^z = u_1^{z_1} \cdots u_r^{z_r}$.

For $\alpha_1, \cdots, \alpha_n \in K^*$, consider the General Unit equation

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = 1 \text{ in } x := (x_1, \cdots, x_n) \in \Gamma \quad \text{with} \sum_{i \in I} \alpha_i x_i \neq 0 \tag{2}$$

for every non-empty subset I of $\{1 \cdots n\}$.

We call such solutions "non-degenerate" solutions to the General Unit equation.

Otherwise we call "degenerate" solutions those with vanishing subsum $\sum_{i \in I} \alpha_i x_i = 0$.

Theorem [Evertse-Schlickeweri-Schmidt, 2002]

Consider the General Unit equation

$\alpha_1 x_1 + \cdots + \alpha_n x_n = 1$ **in** $x = (x_1, \cdots, x_n) \in \Gamma$.

Then the number of non-degenerate solutions is

at most $c(n)^{r+1}$ **with** $c(n) = \exp\left((6n)^{3n}\right)$.

Different proof by G. Rémond.

An earlier estimate of Beukers-Schlickewei is better if $n = 2$ **;**

the number of the solutions is at most $2^{8(r+2)}$.

The non-degenerate solutions to the Unit equation are only finitely many, but we can also construct Unit equations having many solutions.

For example, we have a result due to Erdős, Stewart, Tijdeman, and improved one by Konyagin and Soudararajan.

Let us denote $N(\alpha_1, \cdots, \alpha_n)$ the number of the non-degenerate solutions to the Unit equation $\alpha_1 x_1 + \cdots + \alpha_n x_n = 1$.

Theorem [Erdős-Stewart-Tijdeman] Let $K = \mathbb{Q}$. There exist an absolute constant $c > 0$ and a set of places S containing rational primes of arbitrary large cardinality s with

$$N(1,1) \geq \exp\left(c(s/\log s)^{1/2}\right).$$

Theorem [Konyagin-Soudararajan] There exists a constant

$\gamma < 2 - \sqrt{2}$ and a set of places S containing rational primes of arbitrary large cardinality s with

$$N(1,1) \geq \exp\left(s^{\gamma}\right).$$

This means that there is no uniform upper bound which is independent of S for the number of the solutions.

We are now interested in Unit equations having few solutions:

Definition **[Γ-equivalence class]**

Let Γ be a subgroup of $(K^*)^n$ of finite rank $r < \infty$.

We say that $\alpha := (\alpha_1, \alpha_2, \cdots, \alpha_n) \in (K^*)^n$

and $\beta := (\beta_1, \beta_2, \cdots, \beta_n) \in (K^*)^n$ are Γ-equivalent

if there exists an element $\sigma \in \Gamma$ such that

$\beta = \sigma \cdot \alpha$ (here the multiplication is coordinatewise).

We write $\alpha \sim \beta$ when they are Γ-equivalent.

§3 Unit equations having few solutions

Let us consider $n \geq 3$.

Instead of seeking an upper bound of the number of the non-degenerate solutions, we may consider the minimal number m such that the set of the solutions to General Unit equations may be contained in the union of m proper linear subspaces of $(K^*)^n$, outside of some finitely many Γ-equivalence classes.

Theorem [Evertse 2003]

Let $n \geq 3$. Let Γ be a subgroup of $(K^*)^n$ of finite rank.

There exists a finitely many Γ-equivalence classes $C_1, \cdots, C_t$ of tuples $(K^*)^n$ with the following property ;

For any $(\alpha_1, \cdots, \alpha_n) \in (K^*)^n - (C_1 \cup \cdots \cup C_t)$ the set of non-degenerate solutions to $\alpha_1 x_1 + \cdots + \alpha_n x_n = 1$ in $x = (x_1, \cdots, x_n) \in \Gamma$ is contained in the union of not more than 2^n proper linear subspaces of $(K)^n$.

Theorem 1

Let us restrict $n = 2$.

Let Γ be a subgroup of $(K^*)^2$ of finite rank r.

Consider the Unit equation

$\alpha_1 x_1 + \alpha_2 x_2 = 1$ **in** $x = (x_1, x_2) \in \Gamma$.

Then the number of the solutions is at most $2 \cdot 5^{2r+3}$.

Beukers-Schlickewei : $2^{8(r+2)}$.

Theorem 2 Let $n = 2$. Let Γ be a subgroup of $(K^*)^2$ of finite rank r. Consider the General Unit equation

$\alpha_1 x_1 + \alpha_2 x_2 = 1$ in $x = (x_1, x_2) \in \Gamma$.

Then there exists a finite set $B \subset (K^*)^2$ with the following property:

For any $(\alpha_1, \alpha_2) \in (K^*)^2$ which is not Γ-equivalent to any of $(\beta_1, \beta_2) \in B$, we have $N(\alpha_1, \alpha_2) \leq 2$

and $\#B \leq \exp(50^2(r + 2))$.

Tools of the refinements :

Hypergepmetric method (essentielly due to M. Bennett)

and Geometry of numbers.