Asymptotic Real Differential Algebra

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Goal: to understand the model theory of the ordered differential field $\mathbb{T}$ of transseries

What does this mean? What is $\mathbb{T}$?

$\mathbb{T} \supseteq \mathbb{R}((x^{-1}))$, the elements of $\mathbb{T}$ are series like

$$f(x) = \underbrace{e^{xe^{x^2}} - 5e^{x\log x} + x + \log\log x}_{\text{infinite part}} + 3 \underbrace{- \frac{2}{x} + \frac{1}{x\log x} + \cdots}_{\text{infinitesimal part}}$$

$$f = f_{\succ} + f_{\asymp} + f_{\prec}$$

There are natural operations of <u>exponentiation</u> (exp), <u>differentiation</u> (∂), and <u>composition</u> on $\mathbb{T}$

$$f(x), g(x) \mapsto f(g(x))$$

for $g > \mathbb{R}$

For example, for $f = f_{\succ} + c + f_{\prec}$,

$$\exp(f) = e^{c} \exp(f_{\succ}) \left(1 + f_{\prec} + \frac{f_{\prec}^{2}}{2!} + \frac{f_{\prec}^{3}}{3!} + \cdots\right)$$

$$= e^{c} \exp(f_{\succ}) + e^{c} \exp(f_{\succ}) f_{\prec} + e^{c} \exp(f_{\succ}) \frac{f_{\prec}^{2}}{2!} + \cdots$$

$\mathbb{T}$ also admits <u>inverse</u> operations to the above:

$\exp(\mathbb{T}) = \mathbb{T}^{>0}$, $\partial \mathbb{T} = \mathbb{T}$ with $\partial^{-1} 0 = \mathbb{R}$,

for each $f > \mathbb{R}$ there is unique $g > \mathbb{R}$ such that

$$f(g(x)) = g(f(x)) = x.$$

$\mathbb{R}_{\exp} \preccurlyeq \mathbb{T}_{\exp}$, even

$\mathbb{R}_{an,\exp} \preccurlyeq \mathbb{T}_{an,\exp}$.

From now on consider $\mathbb{T}$ just as a differential field

$\mathbb{R} = \{f \in \mathbb{T} : f' = 0\}$ is definable in $\mathbb{T}$

Q Is $\mathbb{Z} \subseteq \mathbb{T}$ definable in $\mathbb{T}$?

A positive answer would destroy our project.
But: $\mathbb{Z}$ is definable in the differential subfield of purely exponential series.

We hope for and work toward results like:

(1) $\mathbb{T}$ is model complete

(2) $\mathbb{T}$ is asymptotically o-minimal

(1) would give a "theorem of the complement": for any differential polynomial P over $\mathbb{T}$ in $m+n$ variables

$$\mathbb{T}^m \setminus \{a \in \mathbb{T}^m : P(a,b)=0 \text{ for some } b \in \mathbb{T}^n\}$$

$$=$$

$$\{a \in \mathbb{T}^m : Q(a,c)=0 \text{ for some } c \in \mathbb{T}^{n'}\}$$

for some differential polynomial Q over $\mathbb{T}$ in $m+n'$ variables.

(2) means that for each definable set $X \subseteq \mathbb{T}$ there is $a \in \mathbb{T}$ such that either all $f > a$ in $\mathbb{T}$ lie in X, or all $f > a$ in $\mathbb{T}$ lie outside X.

(Known to be true for quantifier-free definable sets $X \subseteq \mathbb{T}$.)

The best evidence for (1) and (2) is the detailed analysis by JvdH in his book "Transseries and Real Differential Algebra", Springer Lecture Notes 1888, of the zeroset in $\mathbb{T}$ of any given differential polynomial in one variable over $\mathbb{T}$. In particular he proved the <u>Intermediate Value Property</u>:

Given any $P(Y) \in \mathbb{T}\{Y\}$ and $a, b \in \mathbb{T}$ with

$a < b$, $\quad P(a) < 0 < P(b)$,

there is $y \in \mathbb{T}$ with $a < y < b$, $P(y) = 0$.

(Here and later we let $K\{Y\} = K[Y, Y', Y'', \ldots]$ be the ring of differential polynomials in the indeterminate Y over a differential field K.)

A. Robinson has taught model theorists the significance of model completeness and tools to establish it in various indirect ways.

Along these lines we propose to establish (1) as follows

- introduce <u>H-fields</u> as ordered differential fields whose ordering and derivation interact "as in $\mathbb{T}$"

- find a set Σ of elementary properties of $\mathbb{T}$ such that the H-fields satisfying Σ are exactly the existentially closed H-fields

<u>Def</u> An H-field is an ordered differential field K with constant field C such that

(H_1) $\mathcal{O} = C + \mathfrak{m}(\mathcal{O})$, $\mathcal{O}$ = convex hull of C
$\mathfrak{m}(\mathcal{O})$ = maximal ideal of $\mathcal{O}$

(H_2) $a > C \Rightarrow \partial a > 0$

(H_3) $\partial\,\mathfrak{m}(\mathcal{O}) \subseteq \mathfrak{m}(\mathcal{O})$

<u>Examples</u>: Hardy fields, $\mathbb{T}$

An H-field K is <u>existentially</u> <u>closed</u> if for every $P \in K\{Y_1, \dots, Y_n\}$ such that the equation $P(y_1, \dots, y_n) = 0$ has a solution in an H-field extension of K, the equation also has a solution in K.

We know many important elementary properties of $\mathbb{T}$, and part of our program must be to show that each H-field can be extended to one that has those same elementary properties.

<u>Simple example</u>: $\mathbb{T}$ is real closed

Indeed, the real closure of an H-field is again an H-field.

<u>Other example</u>: $\mathbb{T}$ is Liouville closed: any differential eq'n $ay' + by + c = 0$ $(a \neq 0)$ has a solution.

<u>Likewise</u>: any H-field can be extended to a Liouville closed H-field

Unfortunately, it can happen that an H-field K has up to K-isomorphism <u>two</u> Liouville closures. This is related to the fact that $\mathbb{T}$ does <u>not</u> admit quantifier elimination in the language of ordered valued differential fields.

From now on K, L are real closed H-fields

$a' = \partial a$, $\quad a^\dagger = \frac{a'}{a} \quad (a \neq 0)$

$C = C_K$, $\quad \mathcal{O} = \mathcal{O}_K$ = convex hull of C in K

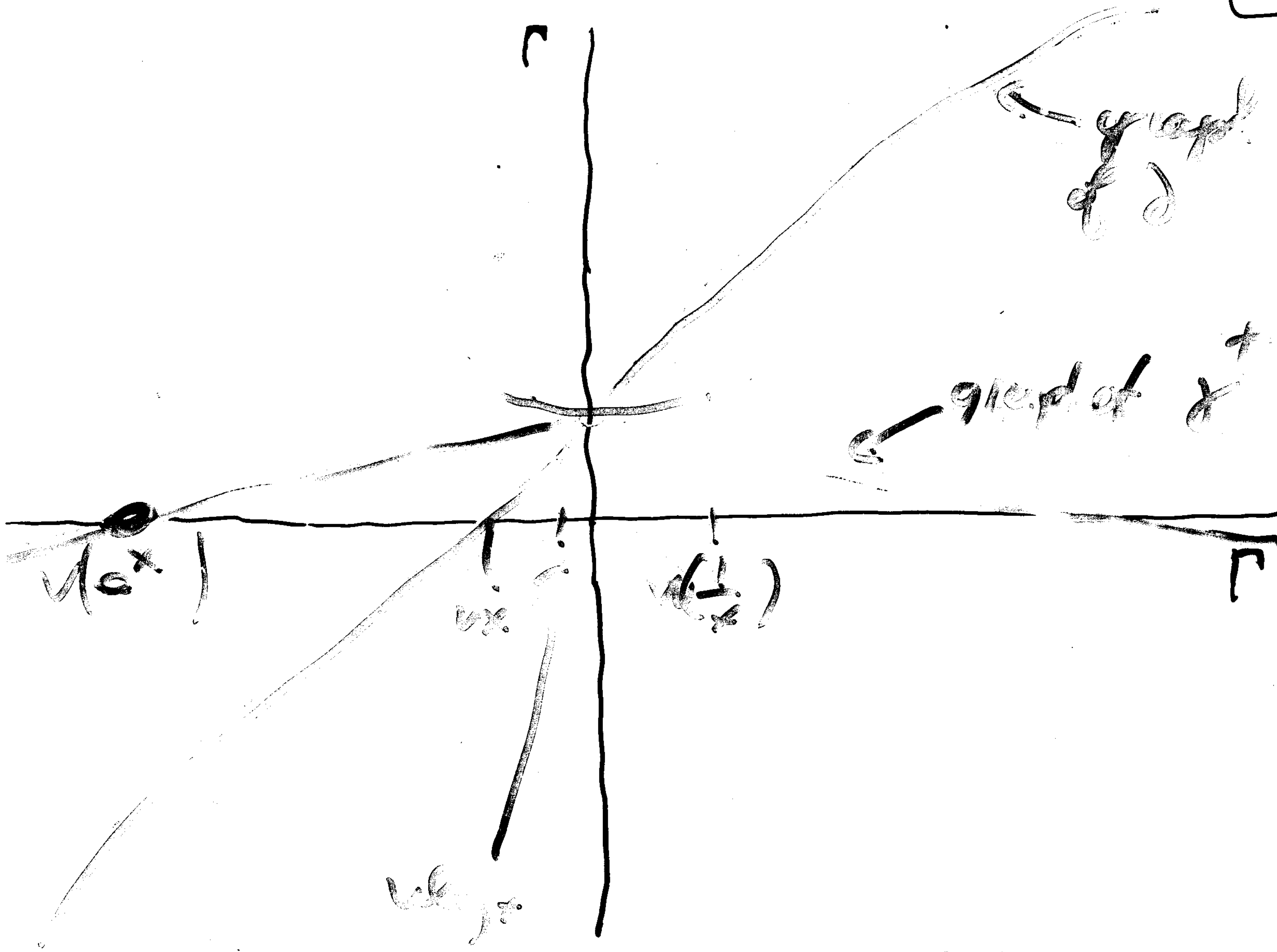
$\mathcal{O}$ is the valuation ring of a valuation $v : K^\times \to \Gamma$, and if $va = vb \neq 0$ then $va' = vb'$

Thus the derivation ∂ induces a function

$$\partial = \partial_v : \underset{\Gamma \setminus \{0\}}{\overset{\Gamma^*}{\shortparallel}} \to \Gamma, \quad \partial(va) = v(\partial a)$$

Likewise the logarithmic-derivative, $a \mapsto a^\dagger : K^\times \to K$ induces $\alpha \mapsto \alpha^\dagger : \Gamma^* \to \Gamma$ with $\alpha^\dagger = v(a^\dagger)$ for $\alpha = v(a)$

These functions on Γ have some very useful properties, for example, $\partial : \Gamma^* \to \Gamma$ is strictly increasing and <u>either</u> $\partial(\Gamma^*) = \Gamma$ <u>or</u> $\partial(\Gamma^*) = \Gamma \setminus \{\gamma\}$ for some γ.



x $\quad g^x$ $\quad x \log \log x$ $\quad \log \log \log x$

$\partial(\gamma) = \gamma + \gamma^\dagger$ and it is often useful to view $\partial : \Gamma^* \to \Gamma$ as the perturbation of the identity by the function $\gamma \mapsto \gamma^\dagger : \Gamma^* \to \Gamma$, since this last function varies only <u>very</u> slowly

After Rosenlicht we call the ordered abelian group Γ equipped with $\gamma \mapsto \gamma^\dagger : \Gamma^* \to \Gamma$ the <u>asymptotic couple</u> of K

<u>Basic Properties of</u> $\gamma \mapsto \gamma^\dagger$ $(0^\dagger := \infty)$:

- $(\alpha+\beta)^\dagger \geqslant \min(\alpha^\dagger, \beta^\dagger)$ ("valuation on Γ")
- $\alpha^\dagger < \partial(\beta)$ for all $\alpha, \beta > 0$
- $0 < \alpha \leq \beta \implies \alpha^\dagger \geqslant \beta^\dagger$

The model theory of these asymptotic couples is fairly well-understood. For example, the existentially closed asymptotic couples are those that satisfy

- $\partial(\Gamma^{*}) = \Gamma$ (asymptotic integrability)
- $\Gamma^{\dagger} := \{\gamma^{\dagger} : \gamma \in \Gamma^{*}\}$ is closed downward.

They are the models of a complete theory that admits QE in a natural language. The asymptotic couple of $\mathbb{T}$ (and of any Liouville closed H-field) is existentially closed.

Arbitrary asymptotic couples fall into three disjoint classes:

(1) $\Gamma^{\dagger} < \gamma < \partial(\Gamma^{>0})$ for some (unique) γ

(2) $\max(\Gamma^{\dagger})$ exists

(3) $\partial(\Gamma^{*}) = \Gamma$

called the gap

Case (1) <u>does not</u> occur in H-subfields of $\mathbb{T}$ but it <u>does</u> happen in H-subfields of elementary extensions of $\mathbb{T}$.

Case (1) is a <u>fork</u> in the road: roughly speaking we can either fill the gap with the derivative of an infinitesimal, or with the derivative of an infinitely large element, but not both. If we fill the gap in one of these two possible ways, we land in case (2).

<u>Once we are in case (2) we cannot get back in case (1) in a differentially algebraic extension</u>

$\therefore$ We are most comfortable with case (2).

Typically, H-fields satisfying (3) are obtained as directed unions of H-subfields satisfying (2).

For example, if $K \models (2)$, then the Liouville closure of K satisfies (3) and is an inductive union of H-subfields satisfying (2).

If $K \models (2)$ we have systematic ways to solve any linear differential equation in suitable H-field extensions and their algebraic closures, and even construct its Picard-Vessiot extension (but there are still issues of uniqueness to be settled).

<u>Q</u>. Can every H-field be extended to one that satisfies (2) ?

If so, it would immediately tell us a lot about existentially closed H-fields.

The Newton Polynomial

Let $P \in K\{Y\}$, $P \neq 0$. Then

$$P = f \cdot D_P + R$$

with $D_P \in C\{Y\}$, $f \in K^\times$, $v(f) = v(P) < v(R)$

This determines D_P uniquely up to a factor $u \in K^\times$ with $v(u) = 0$.

D_P = dominant part of P

Suppose now that $K \models (3)$.
Then something interesting happens:
rewrite P in terms of the derivation $\phi^{-1}\partial$
where $v(\phi) < \partial(\Gamma^{>0})$; this leads to
$D_{P\phi} \in C\{Y\}$, varying with ϕ, and as $v\phi$
increases, $D_{P\phi}$ will eventually settle down
to a differential polynomial $N_P \in C\{Y\}$,
the <u>Newton polynomial of</u> P (over K)

JvdH: <u>If</u> $K = \mathbb{T}$, <u>then</u> $N_P \in \mathbb{R}[Y]\cdot(Y')^{\mathbb{N}}$

<u>Proposition</u> If K is Liouville closed <u>and</u> an inductive union of H-subfields satisfying (2), then again
$$N_P \in C[Y](Y')^{\mathbb{N}}$$

Unfortunately, we have an example of a Liouville closed K and a $P \in K\{Y\}$, $P \neq 0$, such that
N_P does <u>not</u> have this special form.

JvdH: If $K = \mathbb{T}$, then

$$N_P(0) = 0 \implies N_{AP+BP'}(0) = 0$$

<u>Q</u>: Is this true for every H-field $K \models (3)$?

If not, we are in big trouble

If true, then every $K \models (3)$ has an immediate spherically complete H-field extension (probably unique up to K-isomorphism). This would be very nice.

For example, it would follow that every H-field extends to one satisfying (2).