Homotopy and twisted homology for hyperplane arrangements

Richard Randell

University of Iowa

1 Introduction

Let M be the complement of a complex hyperplane arrangement in $\mathbf{C}^\ell$. For reflection arrangements, M is aspherical, that is, the higher homotopy groups $\pi_k(M)$ vanish, $k > 1$. For generic arrangements, the higher homotopy groups don't vanish. So generally one would like to understand these groups. Thanks to Brieskorn and Orlik-Solomon, we think we understand the homology $H_k(M; Z)$ pretty well. So a natural question is: What is the image of the higher Hurewicz homomorphism $h_k : \pi_k(M) \to H_k(M)$ for arrangement complements M?

Theorem 1 *(Vanishing Theorem, R., [8], 1997) For* $k > 1$,

$$h_k : \pi_k(M) \to H_k(M)$$

is the zero homomorphism.

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This concludes my talk. Thank you

But..., ten years later Yoshinaga proved

Theorem 2 *(Non-Vanishing Theorem, M. Yoshinaga [11]) Let $\mathcal{L}$ be a non-resonant complex rank r local system on M, and let $M' = M \cap H$ be the intersection of M with a generic hyperplane H. Then*

$$h'_{\ell-1} : \pi_{\ell-1}(M) \otimes_{\mathbb{Z}} \mathcal{L}_{x_0} \to H_{\ell-1}(M'; \mathcal{L}')$$

is onto (where $\mathcal{L}' = i^*\mathcal{L}$ is the restriction, and $H_{n-1}(M'; \mathcal{L}')$ has positive rank equal to the absolute value of the euler characteristic of M'.)

So, what gives? Here we will give general results which put the above two theorems in a more unified setting and which clarifies what topological properties of a complex arrangement complement are important in their proofs.

2 Some background on topology of arrangements

There are three fundamental topological properties of arrangement complements:

- (Locality) The Brieskorn mapping

  $$i_* : \oplus_Z H^r(M_Z) \to H^r(M)$$

  is an isomorphism, the direct sum taken over all lattice elements Z of rank r. M_Z being the complement associated to the arrangement of all hyperplanes containing Z.

- (Toroidality) $H^*(M)$ is generated by degree one classes $\frac{d\alpha_i}{\alpha_i}$, where α_i are the defining linear forms for the arrangement.(Arnol'd, Brieskorn)

- (Minimality) M admits the structure of a minimal n-dimensional CW complex. That is, the number of cells in each dimension equals the corresponding betti number. (Dimca-Papadima[3], R.[9])

The proof of the vanishing theorem above uses toroidality in a crucial way. The proof of Yoshinaga's non-vanishing theorem uses CW structures in what seems a crucial way. Part of the point of this work is to understand whether these properties are essential to these proofs.

3 Some background on higher homotopy groups of arrangements.

Here are some known results for higher homotopy groups of complements of complex hyperplane arrangements.

- Sometimes M is aspherical, such as for the braid arrangement (Fadell-Neuwirth), many real reflection arrangements (Brieskorn), all real simplicial arrangements, hence all real reflection arrangements (Deligne), fiber-type (=supersolvable) arrangements (Falk-R.), complex reflection arrangements (most by Orlik-Solomon, proof for all by Bessis).

- Sometimes M isn't aspherical (Hattori for generic arrangements, Falk-R. for certain fundamental groups or for non-formal arrangements, R. for sections of aspherical, Falk for some line arrangements)

- More precise calculations in some specific cases are known. See Papadima-Suciu for the hypersolvable case, Dimca-Papadima for iterated generic sections.

- Unless there is a good special reason, you shouldn't expect much. e.g. Free arrangements need not be aspherical (Edelman-Reiner).

- For real arrangements, recent advances (Yoshinaga, Salvetti-Settepanella, Delucchi-Settepanella, Hager) give one tools to understand the attaching maps in a minimal CW-structure.

4 Some background on local (twisted) coefficient systems

Let X be a topological space which has a universal cover $\widetilde{X}$, and let $\pi = \pi_1(X)$. Let N be a left $\mathbf{Z}[\pi]$-module, characterized by the left action

$$\varphi : \pi_1(X) \times N \to N$$

or equivalently by the homomorphism

$$\varphi : \pi_1(X) \to Aut(N).$$

Then as usual there is an associated local system N on X. (See Hatcher [5] for details about local systems). Then one has homology with coefficients in N, defined by the chain complex

$$C_j(X; N) = C_j(\widetilde{X}) \otimes_{Z[\pi]} N$$

Recall that since we are in general working here over a non-commutative ring we need to make the left $Z[\pi]$ module $C_j(\widetilde{X})$. into a right module by defining the right action as left multiplication by the inverse element.

Furthermore, by the usual left action of π on $\pi_n(X, Y)$ we may form the tensor product

$$\pi_n(X, Y) \otimes_\pi N$$

where we use the subscript"π" to indicate the tensor product over $\mathbf{Z}[\pi]$. In what follows we will switch as appropriate among the various viewpoints and notations for local systems, but we will always have the above set-up in mind. We will generally suppress φ from the notation.

In the context of local systems locality is not usually relevant, since the Brieskorn homomorphisms are usually not defined: to do so one needs a local system on $M(\mathcal{A})$ which is the restriction of one on $M(\mathcal{A}_Z)$, and so is trivial around hyperplanes not containing Z. Deletion-restriction for this situation (when $rk(Z) = 1$ is examined by D. Cohen in [1].

Minimality in the local systems setting has been examined by Dimca and Papadima in [4]. In particular they show there that the π-equivariant chain complex associated to a Morse-theoretic minimal CW structure on an arrangement complement is independent of the CW structure.

Now for a non-resonant (or "generic") local system it is known that the first homology and cohomology groups vanish ([2]). Thus for generic local systems, toroidality fails badly. Toroidality is a key ingredient in the proof that the usual Hurewicz homomorphism is zero. In place of that result we have a result below.

5 A more general Hurewicz theorem

There is the following general version of the Hurewicz theorem. See for example [10]

Theorem 3 *(Twisted Hurewicz Theorem) Let* (X, Y) *be an* $(n-1)-$*connected topological pair with* $n \geq 3$ *and let* N *be a local system on* X. *Then there is a natural isomorphism*

$$h : \pi_n(X, Y) \otimes_\pi N \to H_n(X, Y; N) \qquad (1)$$

Notice that the right hand side involves local system homology, while the left hand side is an algebraic tensor product. Here "Naturality" means that with these assumptions there exist homomorphisms

$$h : \pi_n(X, Y) \otimes_\pi N \to H_n(X; N)$$

and

$$h : \pi_n(X,Y) \otimes_\pi N \to H_n(Y;N)$$

so that the following diagram commutes.

$$\begin{array}{ccccccccc}
\to & H_n(X;N) & \to & H_n(X,Y;N) & \xrightarrow{i_*} & H_{n-1}(Y;N) & \to \\
 & \uparrow h & & \uparrow h \cong & & \uparrow h & \\
\to & \pi_n(X) \otimes_\pi N & \to & \pi_n(X,Y) \otimes_\pi N & \to & \pi_n(Y) \otimes_\pi N & \to
\end{array} \quad (2)$$

$$\begin{array}{ccccc}
\to H_{n-1}(X;N) & \to & H_{n-1}(X,Y;N) & \to \\
\uparrow h & & \uparrow h & \\
\to \pi_{n-1}(X) \otimes_\pi N & \to & \pi_{n-1}(X,Y) \otimes_\pi N & \to
\end{array}$$

Note that since tensor product is not exact, we have no guarantee that the lower row of this diagram is exact. It will be exact, of course, if the module N is flat.

6 Consequences of the twisted Hurewicz theorem

We first have the following general consequence of the twisted Hurewicz result.

Theorem 4 *Let* (X, Y) *be an* $(n-1)-$*connected topological pair with* $n \geq 3$*. Then* $\ker(i_* : H_{n-1}(Y; N) \to H_{n-1}(X; N)) \subset im(h_Y : \pi_{n-1}(Y) \otimes_\pi N \to H_{n-1}(Y; N))$.

Proof. Since $\pi_{n-1}(X, Y) \cong 0$, and tensor product is right exact, the sequence

$$\pi_n(X, Y) \otimes_\pi N \to \pi_{n-1}(Y) \otimes_\pi N \to \pi_{n-1}(X) \otimes_\pi N \to 0$$

is exact, so that the commuting diagram above yields a commuting diagram with exact rows

$$\begin{array}{ccccc} H_n(X,Y;N) & \to & H_{n-1}(Y;N) & \xrightarrow{i_*} & H_{n-1}(X;N) \\ \uparrow h \cong & & \uparrow h_Y & & \uparrow h \\ \pi_n(X,Y)\otimes_\pi N & \to & \pi_{n-1}(Y)\otimes_\pi N & \to & \pi_{n-1}(X)\otimes_\pi N \end{array} \tag{3}$$

A simple diagram chase then gives the result. ■

The following result follows immediately.

Proposition 5 *Let (X,Y) be an $(n-1)-$connected topological pair with $n \geq 3$. If h_Y is the zero homomorphism, then $i_* : H_{n-1}(Y;N) \to H_{n-1}(X;N)$ is injective.*

If the local system is trivial, $N = Z$, then h_Y is indeed the zero homomorphism for arrangement complements, and the injectivity of i_* with trivial integral coefficients is well-known. On the other hand, if i_* is not injective, say when $H_{n-1}(X;N) = 0$ and $H_{n-1}(Y;N) \neq 0$, then h_Y cannot be the zero homomorphism. We consider this in more detail.

We now want to consider arrangements–we switch notation and set $\ell = n$.

Definition 6 *An* arrangement pair (M, M') *is an* $(n-1)$*-connected pair of topological spaces with* M *an arrangement complement and* M' *a hyperplane section of* M*.*

It is well-known that a generic section M' of M yields an arrangement pair with the homotopy type of a CW pair, and that in fact the number of n-cells attached to M'' to yield M is exactly equal to the n-th betti number of M. (See ([3], [9]). Our definition of an arrangement pair assumes only $(n-1)$-connectedness, however, not minimality.

Proposition 7 *If* N *is the trivial system,* $N = \mathbb{Z}$*, and* (M, M') *is an arrangement pair, then* i_* *is an isomorphism, as is* $j_* : H_n(M;N) \to H_n(M, M';N)$*.*

Proof. The second statement holds because for the trivial system, it was shown in [8] that the Hurewicz map is always trivial on higher homotopy groups. Thus $h_{M''} = 0$. ■

In the case of hyperplane arrangments, the fact that i_* is an isomorphism follows from the Orlik-Solomon algebra (or, more basically from the Lefschetz theorem on hyperplane sections). The next result generalizes Yoshinaga's nonvanishing theorem. Our method of proof uses nothing about the CW structure of arrangement complements, however, only connectivity properties.

Corollary 8 *Suppose (X, Y) is an $(n-1)-$connected pair of topological spaces, $n \geq 3$. Suppose that N is a local system on X so that $H_{n-1}(X; N) = 0$. Then $h_Y : \pi_{n-1}(Y) \otimes_\pi N \to H_{n-1}(X; N)$ is onto.*

Stated for arrangement pairs we have

Theorem 9 *(Yoshinaga [11]) Suppose* (M, M') *is an arrangement pair and* N *is a non-resonant local system of rank* r *on* M*. Then* $h_{M''} : \pi_{n-1}(M') \otimes_{\pi} N \rightarrow H_{n-1}(M'; N)$ *is onto.*

Proof. For a non-resonant system $H_{n-1}(M; N) = 0$ ■

It should be noted that Yoshinaga obtains a more precise result: $H_{n-1}(X; N)$ is generated by the attaching maps for the n-cells, appropriately interpreted.

Actually the Hurewicz map here differs slightly from the one considered by Yoshinaga, but the the result above clearly implies that of Yoshinaga. Here is the relationship. Let X be a topological space with basepoint x. Then, as in [11] we have a twisted Hurewicz map

$$h_j : \pi_j(X, x) \otimes_{\mathbf{Z}} \mathcal{L}_x \rightarrow H_j(X, \mathcal{L})$$

defined by setting $h(f \otimes t)$ equal to the twisted cycle it determines. This Hurewicz homomorphism differs by a change of ring (from $\mathbf{Z}[\pi]$ to $\mathbf{Z}[1] \cong \mathbf{Z}$) from the one we considered earlier. The homomorphisms are related by the obvious commuting triangle.

7 The Hurewicz image for general arrangement covers

Now since M is an arrangement complement, it is path-connected, locally path-connected and semi-locally simply connected. Therefore M has a universal cover and there is a bijective correspondence between subgroups π' of $\pi = \pi_1(M)$ and connected covering spaces of M. Let M' be the cover corresponding to π'. Then the free abelian group $\mathbf{Z}[\pi/\pi']$ with basis the cosets $\gamma\pi'$ is a $\mathbf{Z}[\pi]$-module and the homology groups of M' with coefficients in $\mathbf{Z}$ are the same as the homology groups of M with coefficients in the local system $\mathbf{Z}[\pi/\pi']$. There are isomorphisms

$$H_j(M, \mathbf{Z}[\pi/\pi']) \cong H_j(M', \mathbf{Z}).$$

Let us now consider the case where we have a local system on M of the form $N = \mathbf{Z}[\pi/\pi'])$.

Proposition 10 *$Im(h' : \pi_j(M') \to H_j(M')) \subseteq ker(p_* : H_j(M') \to H_j(M))$.*

Note that $H_j(M') = H_j(M, \mathbf{Z}[\pi/\pi'])$. In case $M = M'$ this is the result of [8].

Proof. Fix $j \geq 2$. Let $p : \tilde{M} \to M$ be the universal cover of M. Then there is a commuting diagram

$$\begin{array}{ccc} \pi_j(\tilde{M}) & \stackrel{\tilde{h}}{\to} & H_j(\tilde{M}) \\ p_* \downarrow\cong & & \downarrow p_* \\ \pi_j(M') & \stackrel{h'}{\to} & H_j(M') \\ p_* \downarrow\cong & & \downarrow p_* \\ \pi_j(M) & \stackrel{h}{\to} & H_j(M) \end{array} \tag{4}$$

Now by the result of [8] the bottom horizontal arrow is the zero homomorphism, and the result follows ■

Among interesting covers of arrangement complements are the universal cover, lots of abelian covers such as those considered in LKB representations, and finite cyclic covers, most notably the Milnor fiber associated to a central arrangement (which is a finite cyclic cover of the projectivization). For instance, in the case of the Milnor fiber one may see easily that one does not have equality in the above proposition.

Example 11 *Consider the reflection arrangement A_3, in projective space. In that case $H_1(M)$ is free abelian of rank five, while the six-fold cyclic cover M' which is the Milnor fiber F has $H_1(F)$ free of rank seven. One may use standard calculations and euler characteristic arguments to conclude that the second homology groups have ranks six and eighteen respectively. Also, M and F are aspherical, so that $Im(h' : \pi_j(M') \to H_j(M'))$ is zero, while it may be seen easily that $p_* : H_2(M') \to H_2(M)$ has non-trivial kernel (in fact, p_* is surjective, so that the kernel is free abelian of rank twelve.*

Example 12 *In general the image of the Hurewicz map is non-trivial. Consider for example any arrangement which is not aspherical. Then by the classical Hurewicz theorem the Hurewicz map on the first non-trivial homotopy group of the universal cover is an isomorphism.*

8 The Hurewicz image for complex local systems

Complex rank one local systems $N = \mathcal{L}$ on M are another case of particular interest. Here the module N is just the additive complex numbers as an abelian group, and the local system is given by a homomorphism of $\pi_1(M)$ to $Aut(\mathbf{C}) = \mathbf{C}^*$. Since $Aut(\mathbf{C})$ is abelian, such a homomorphism factors through the first homology group of M, and can thus be thought of as an assignment of a non-zero complex number to each hyperplane in the arrangement (the associated homology class being a small loop transverse to the hyperplane.)

Let $\mathcal{L}$ be a complex local system on the topological space X, of rank 1. Thus as a bundle of groups, the local system has fiber $\mathbf{C}$. Then by the universal coefficient theorem one has

$$H^p(M(\mathcal{A}),\mathcal{L}) \cong Hom_{\mathbf{C}}(H_p(M(\mathcal{A}),\mathcal{L}),\mathbf{C}) \qquad (5)$$

Via this isomorphism we identify homology and cohomology.

Definition 13 *Let $T^p(M,\mathcal{L})$ denote the submodule of $H^p(M,\mathcal{L})$ generated by products of degree one classes. We will refer to these submodules as the toroidal portion of cohomology. We have the dual notion in homology, giving submodules $T_p(M,\mathcal{L})$.*

We also need a twisted Hurewicz homomorphism for this setting. Let M have basepoint m. Then, as in [11] we have a twisted Hurewicz map

$$h_j : \pi_j(M,m) \otimes_{\mathbf{Z}} \mathcal{L}_m \to H_j(M,\mathcal{L})$$

defined as before by setting $h(f \otimes t)$ equal to the twisted cycle it determines.

The general result is

Theorem 14 *For a rank* 1 *complex local system,* $im(h_j) \cap T_j(\mathcal{L}) = 0$

If the rank one system is trivial, than $T_j(\mathcal{L}) = H_j(M, \mathcal{L})$, giving the vanishing result which began this talk.

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