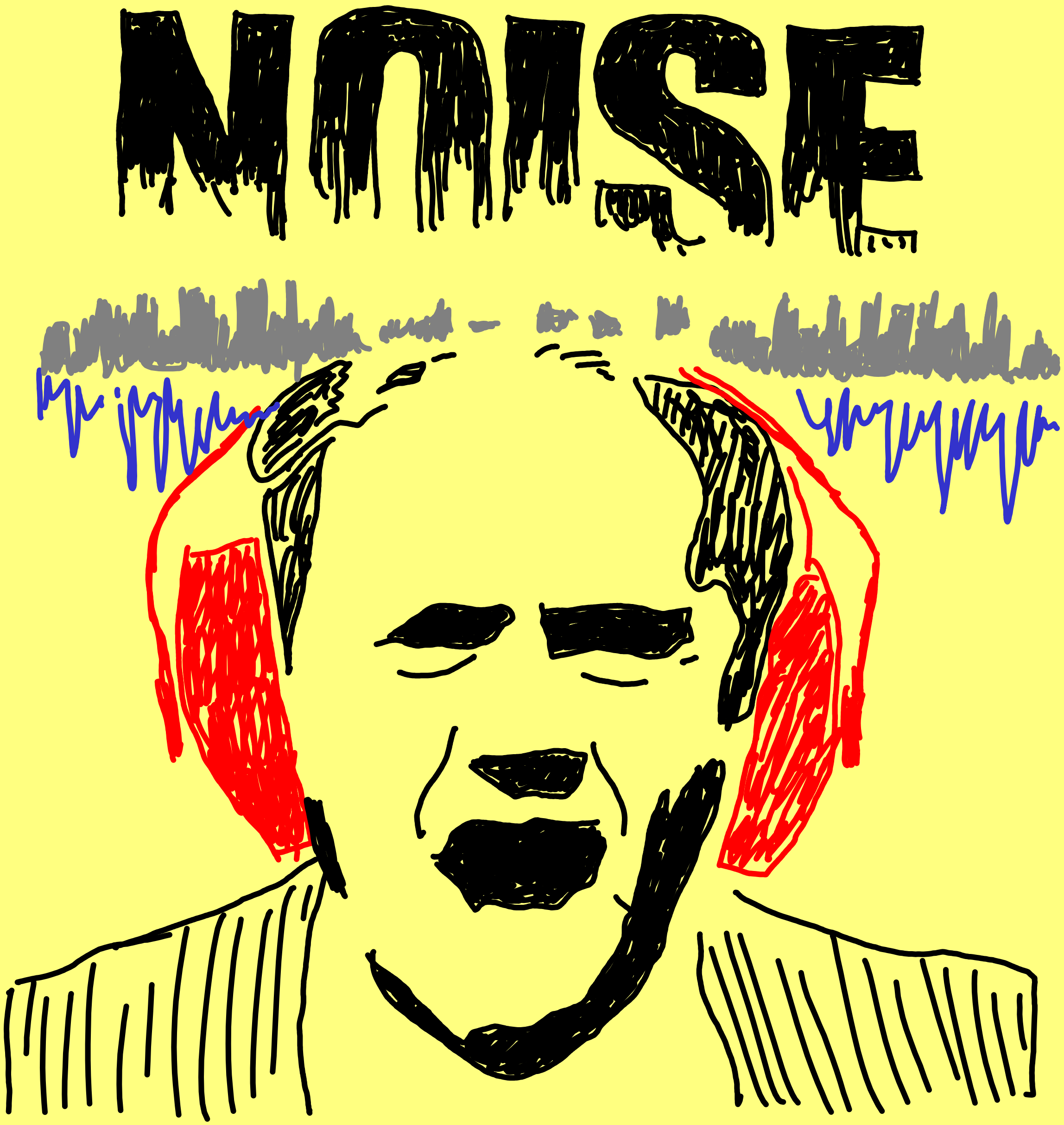
Measuring with Algebra

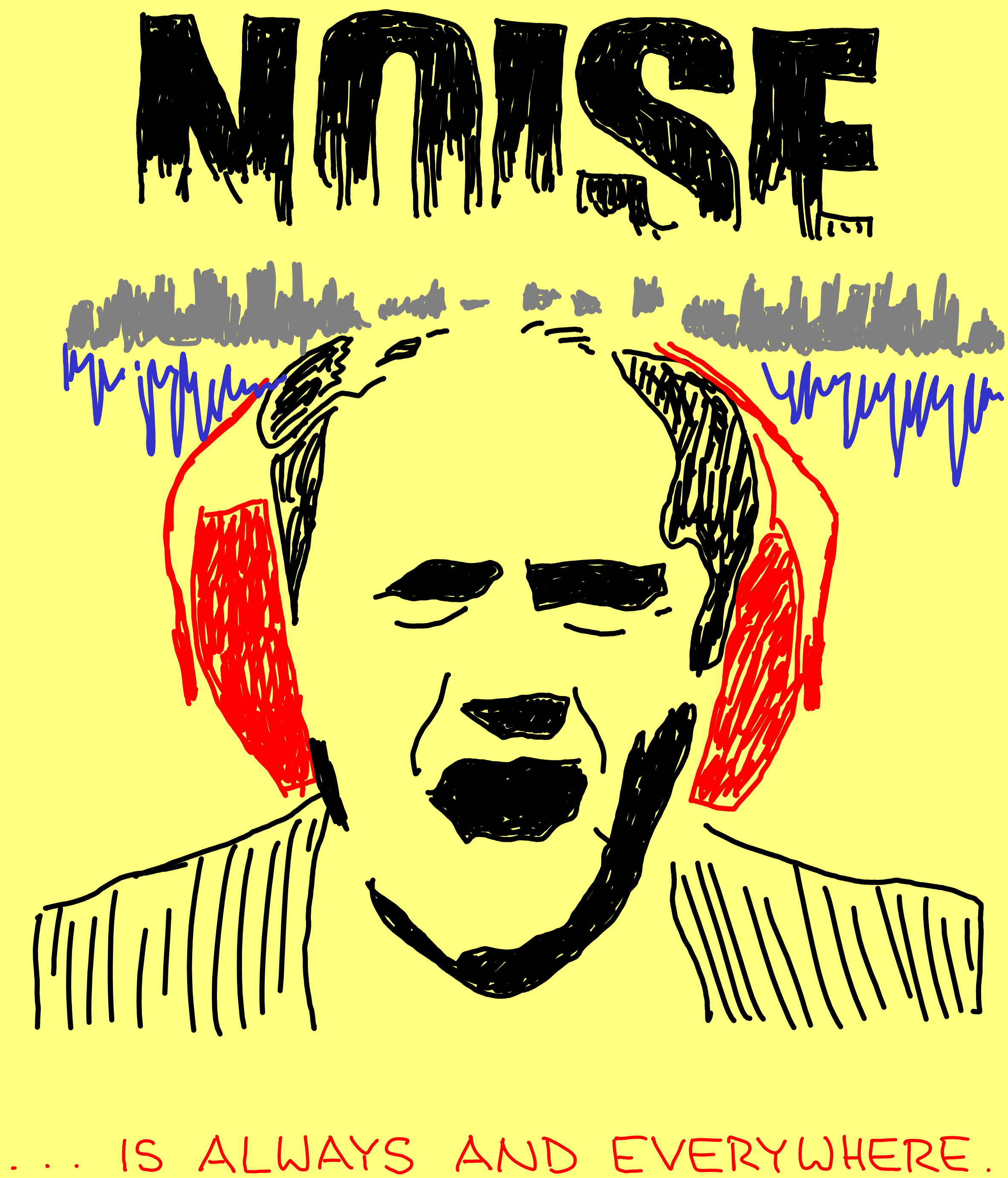
Univ. Vienna

Measuring with Algebra

Herbert Edelsbrunner

IST Austria,
Duke University & Geomagic


NOISE


NOISE
. . . IS ALWAYS AND EVERYWHERE.

I Persistent Homology

II L_p-Stability and Somites

III Contour Stability

I.1 One-D Functions

function

$f: \mathbb{S}^1 \to \mathbb{R}$

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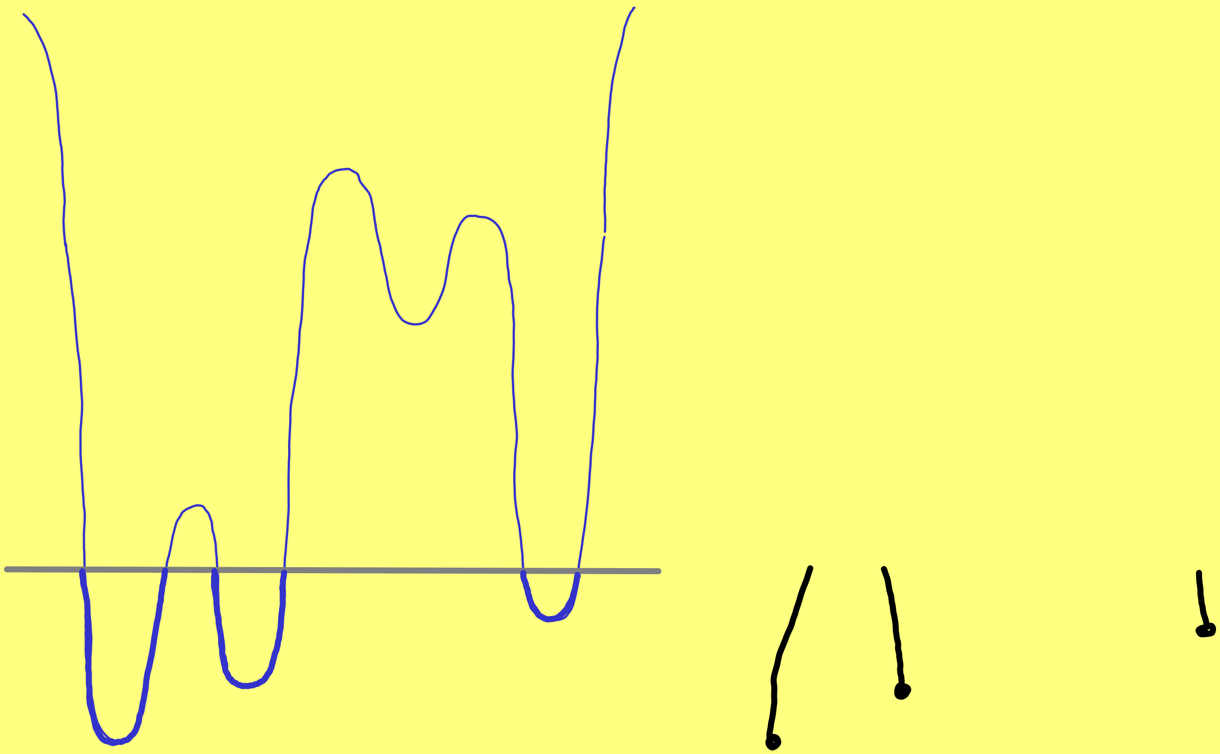

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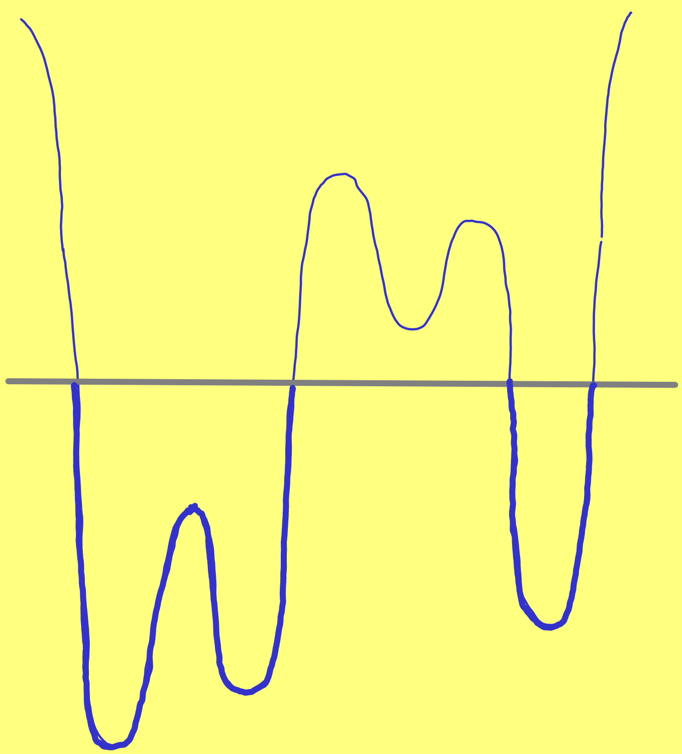

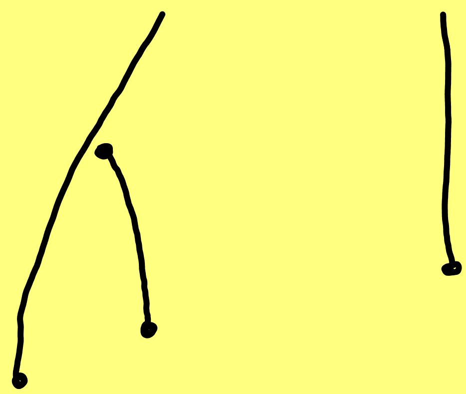

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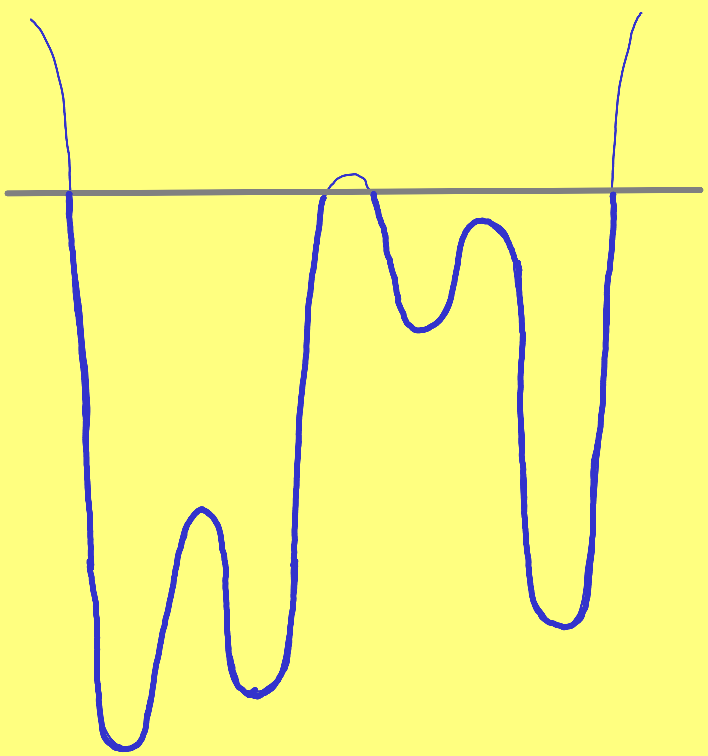

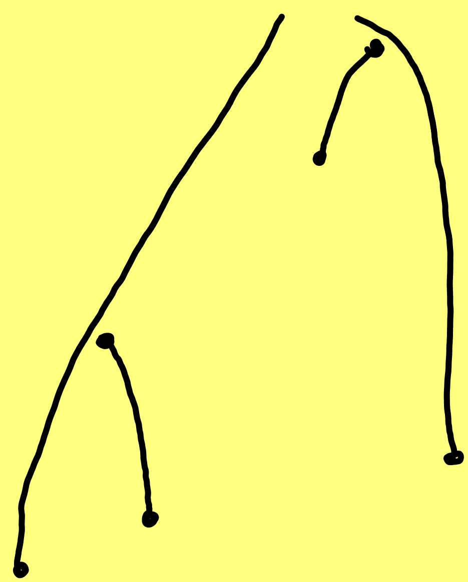

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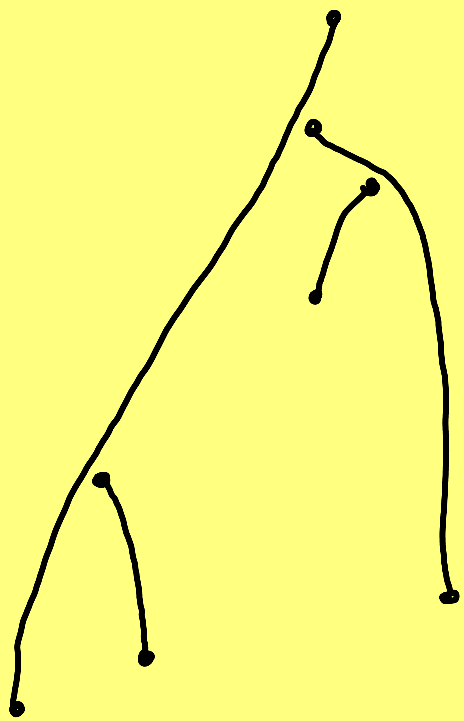

merge tree

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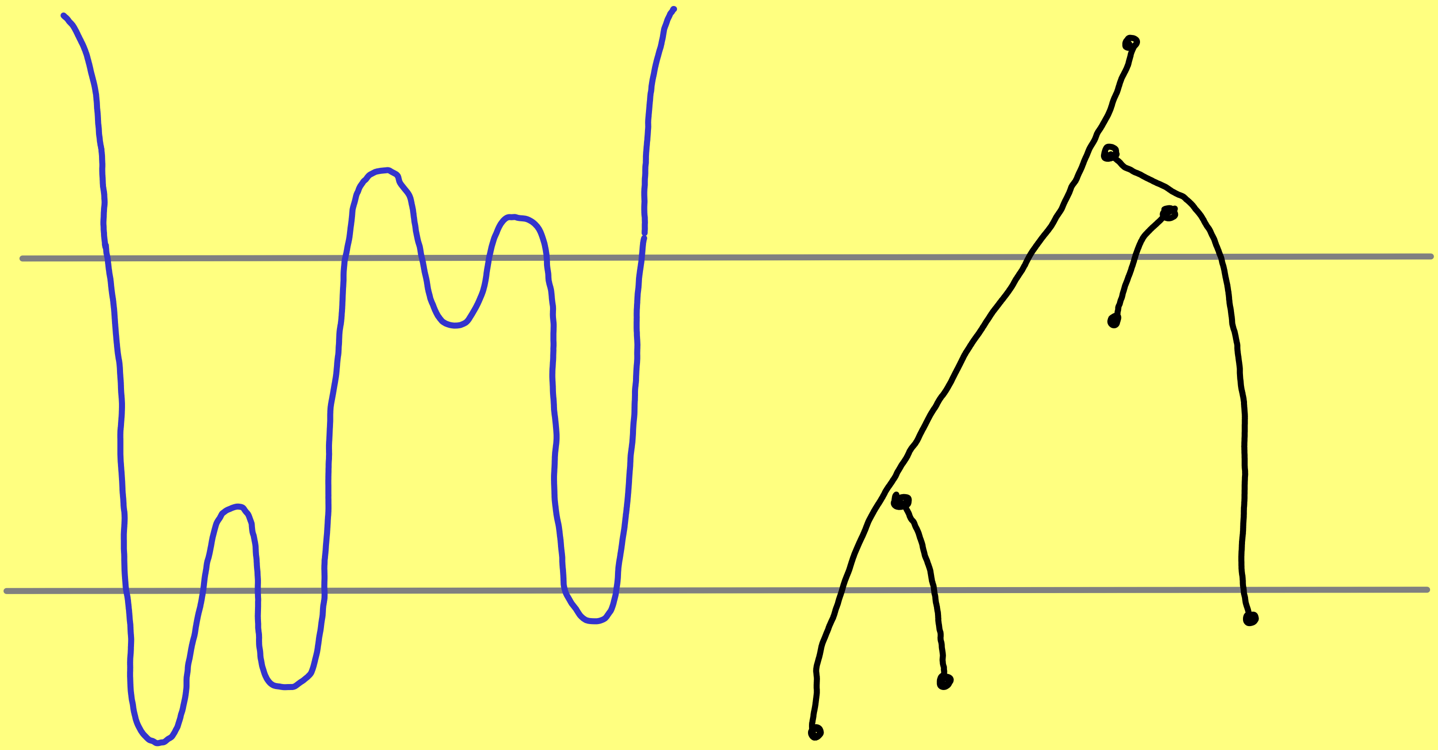

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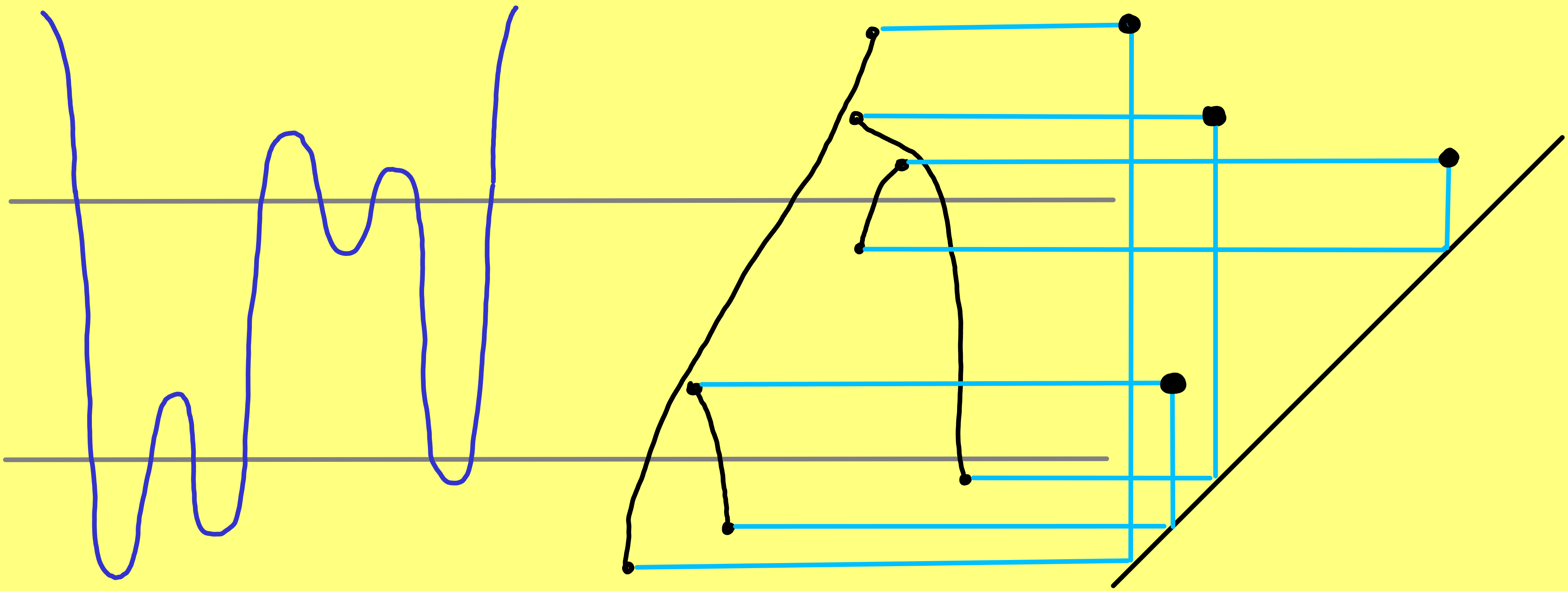

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persistence diagram

$\mathrm{Dgm}_0(f)$

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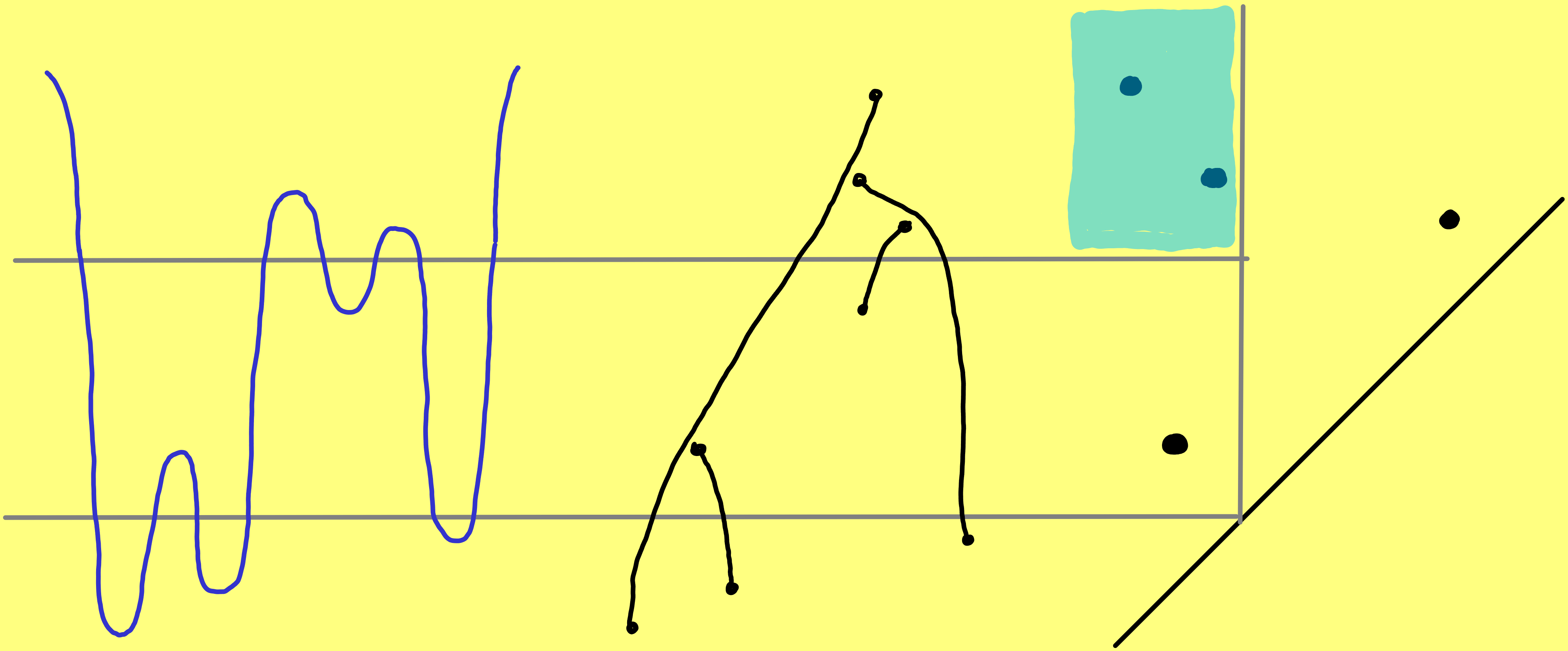

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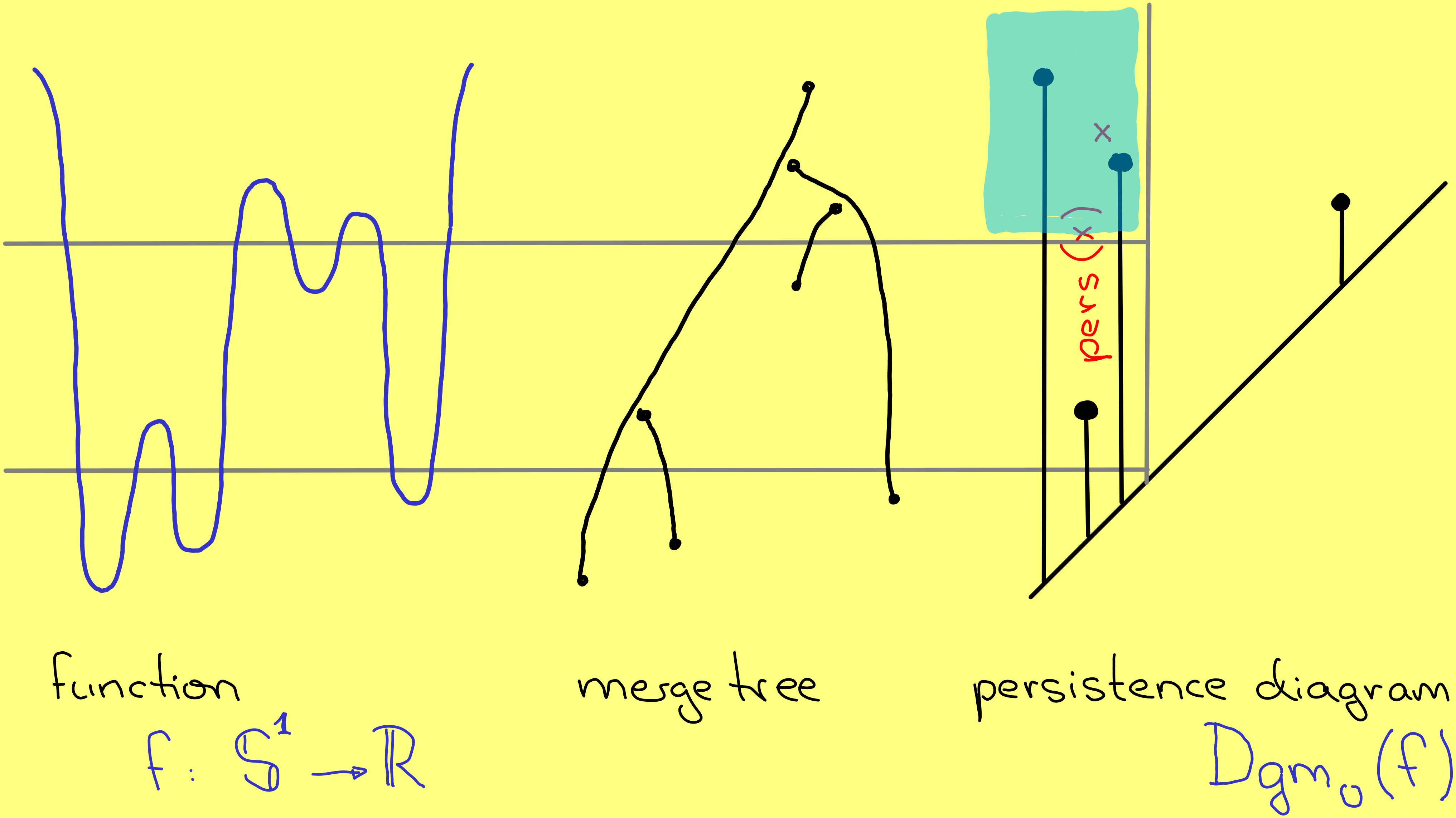

I.2 Filtrations

$$\dots \subseteq K_{i-1} \subseteq K_i \subseteq \dots \subseteq K_{j-1} \subseteq K_j \subseteq \dots$$

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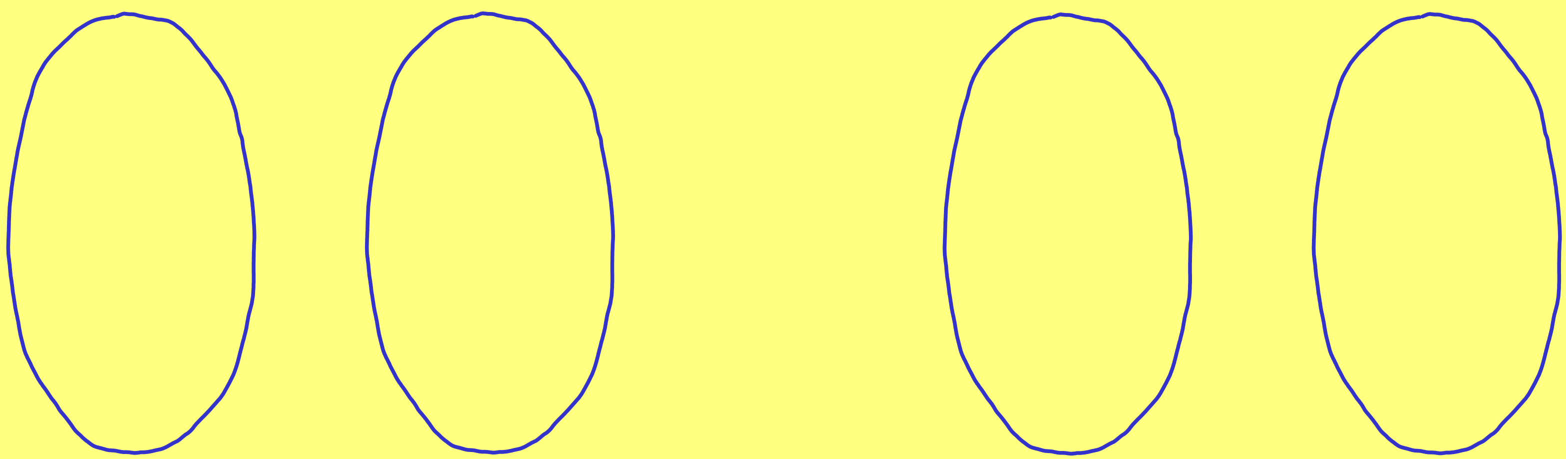

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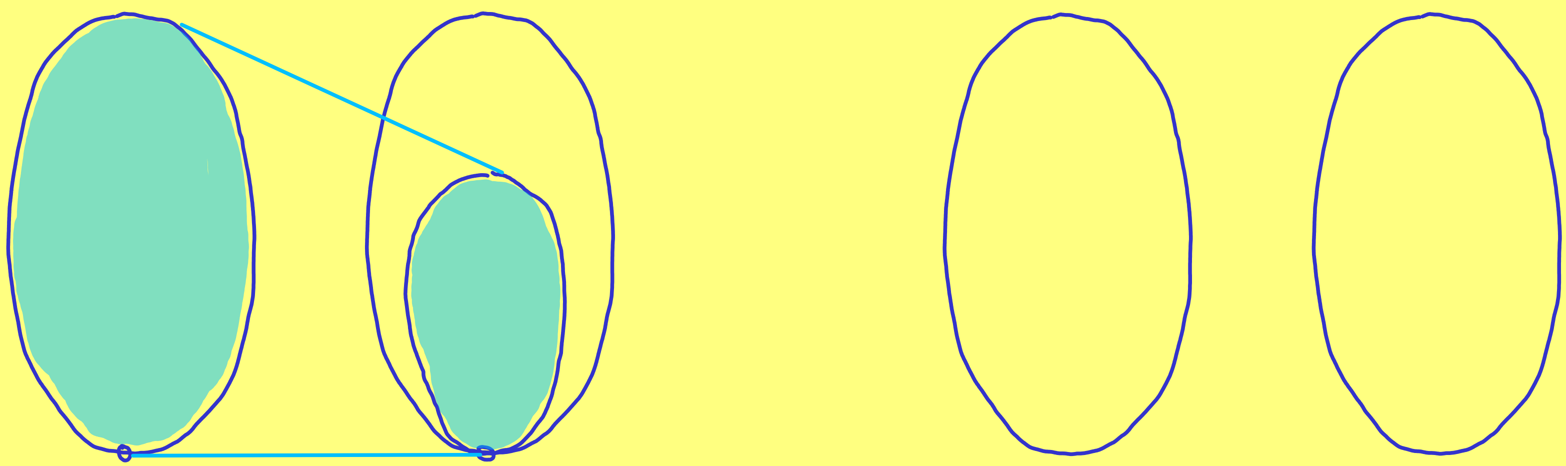

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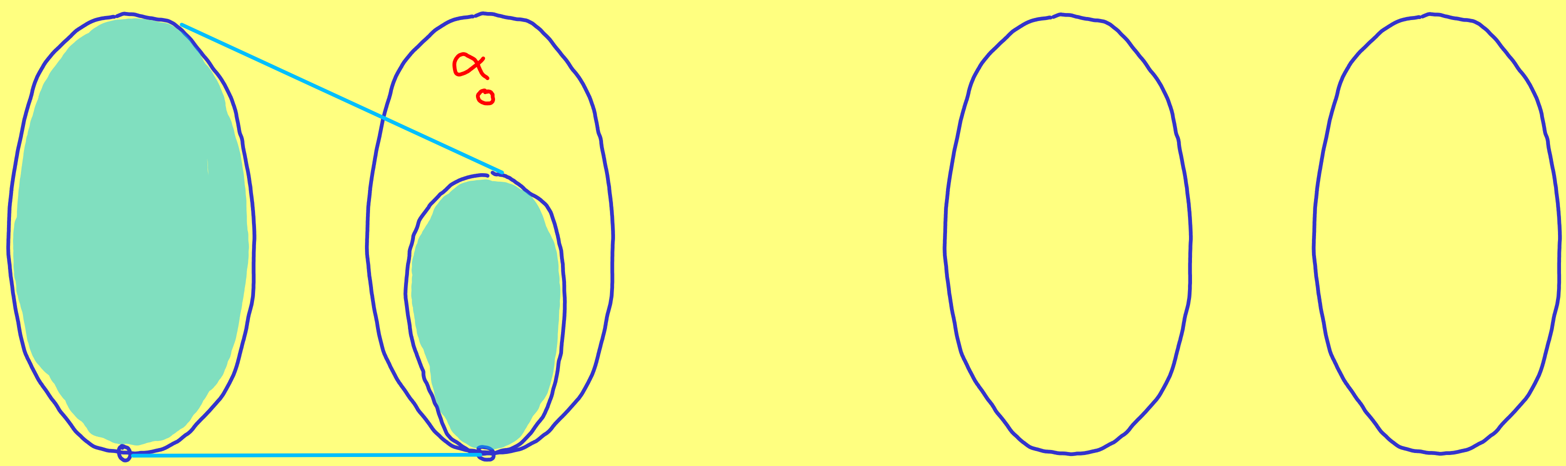

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α is born at K_i

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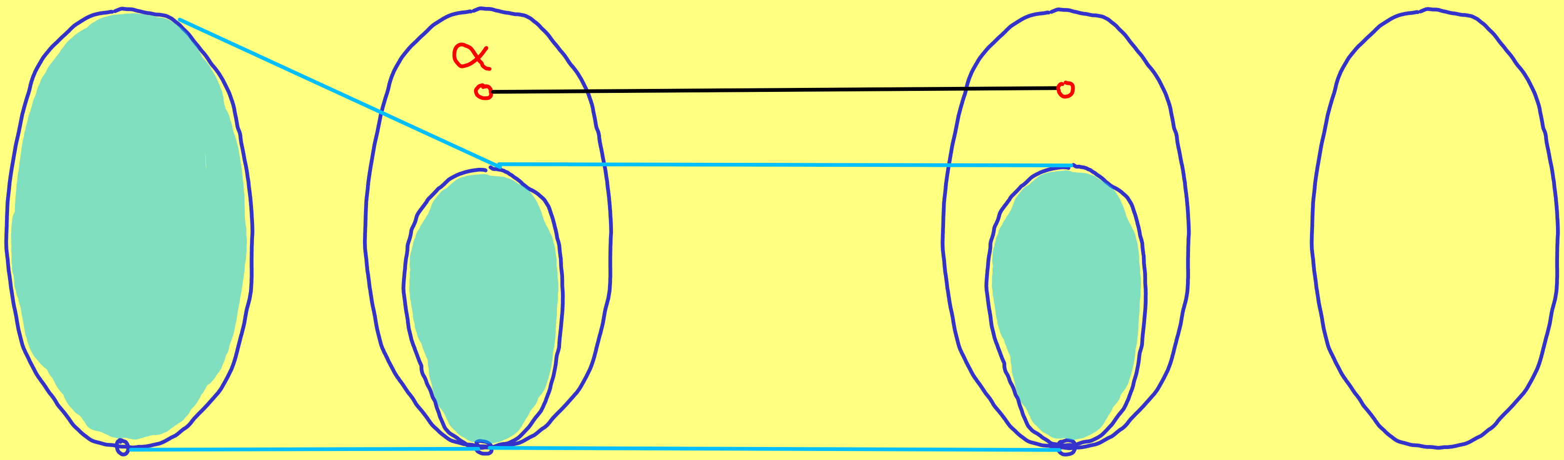

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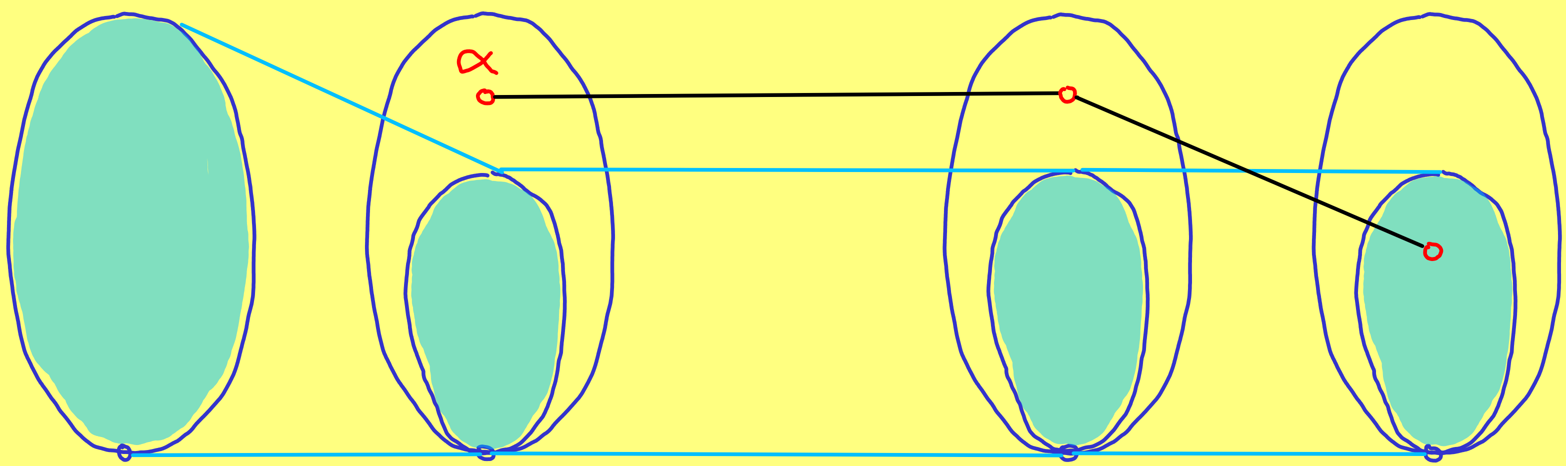

$$\cdots \to H_p(K_{i-1}) \to H_p(K_i) \to \cdots \to H_p(K_{j-1}) \to H_p(K_j) \to \cdots$$

α is born at K_i and dies entering K_j

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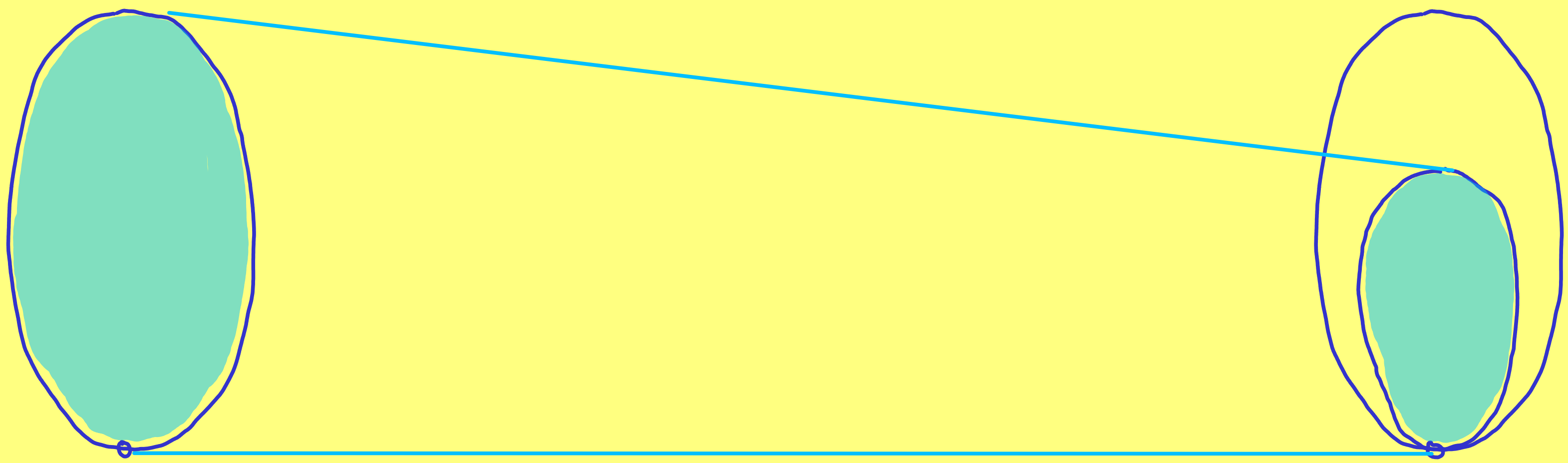

$$H_p(K_{i-1}) \to \quad \dots \quad \to H_p(K_j)$$

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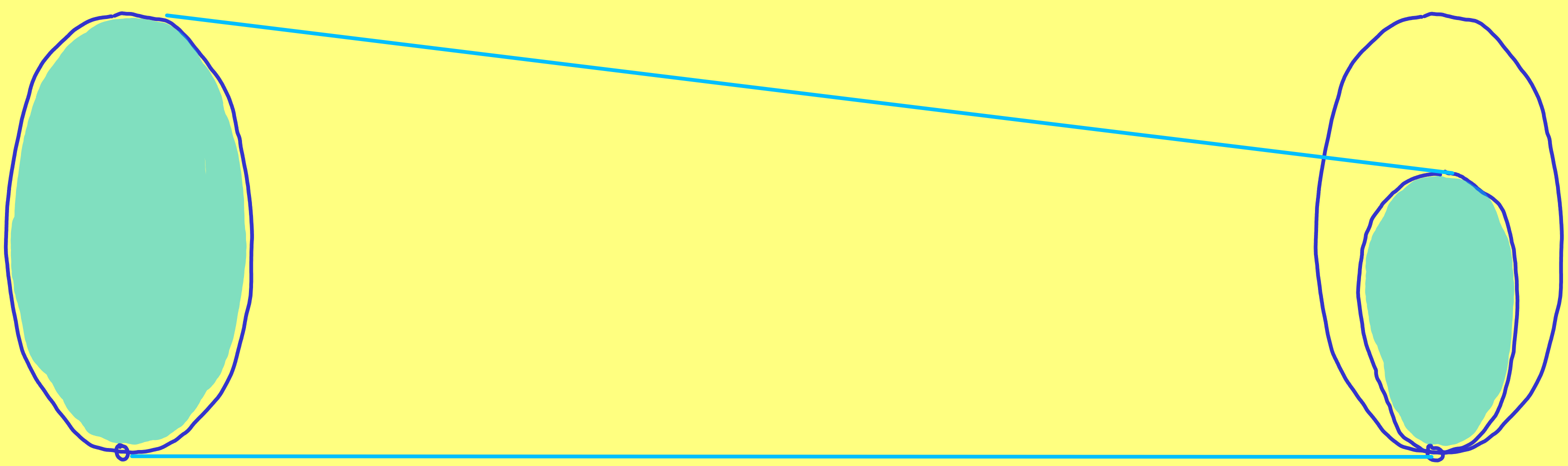

$$h_p^{i-1,j} : H_p(K_{i-1}) \to \ldots \to H_p(K_j)$$

induced homomorphism

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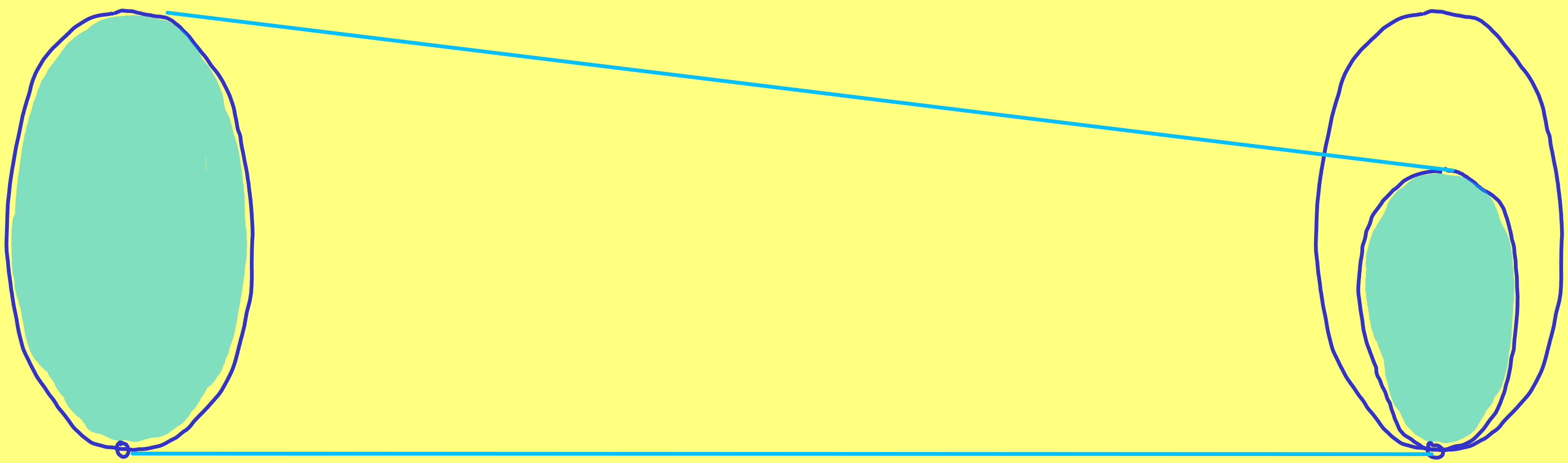


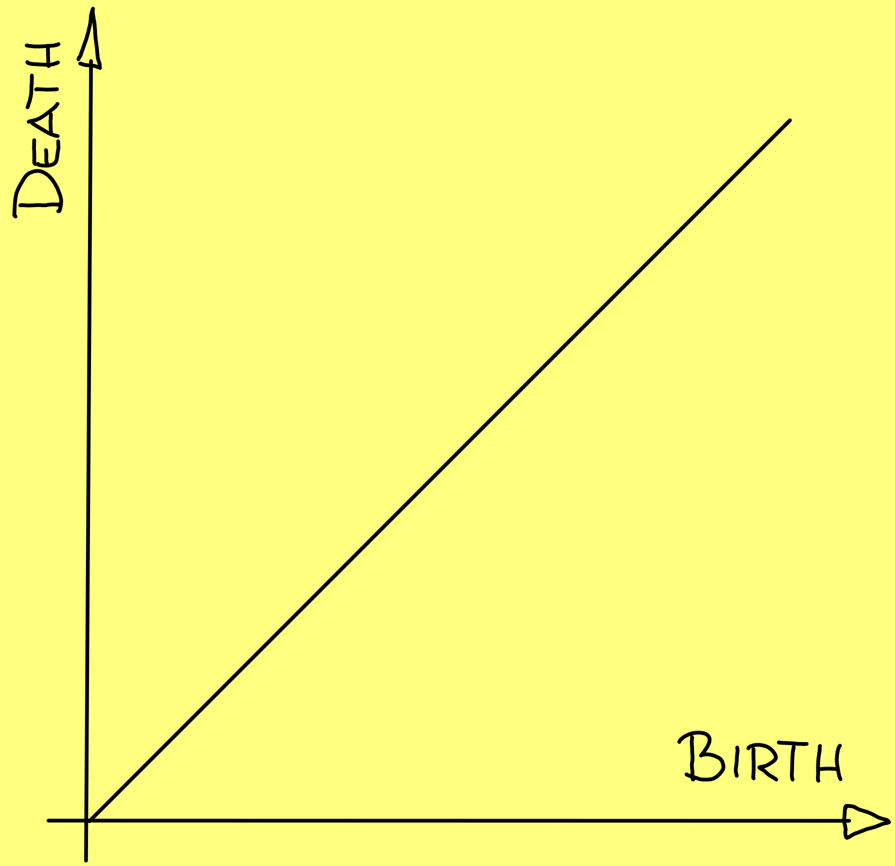
$$h_p^{i-1,j} : H_p(K_{i-1}) \to \ldots \to H_p(K_j)$$

induced homomorphism

$\mathrm{im}\, h_p^{i-1,j}$ is persistent homology group

I.3 Diagrams

$\mathrm{Dgm}_p(f)$:



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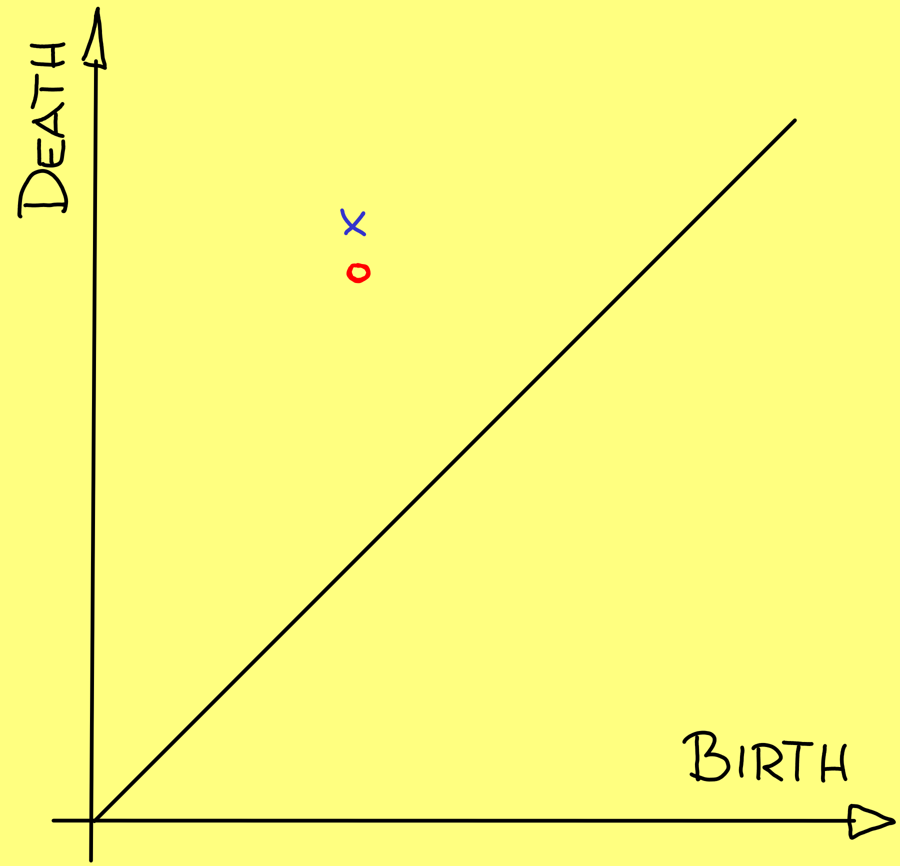

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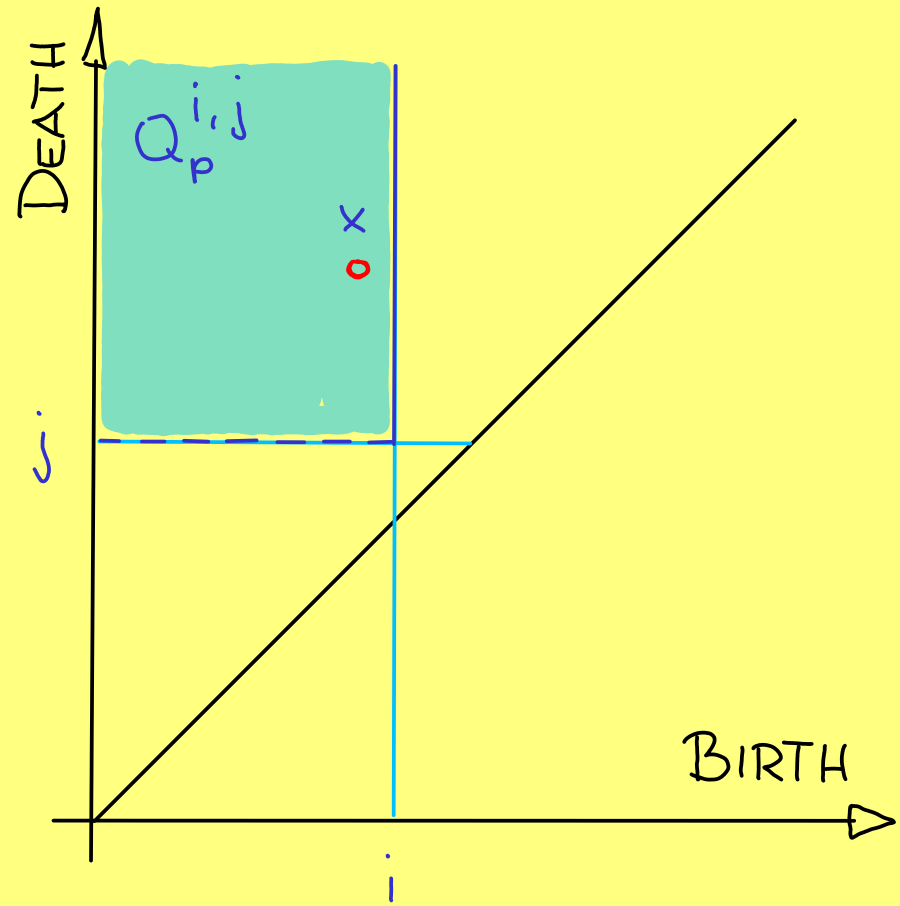

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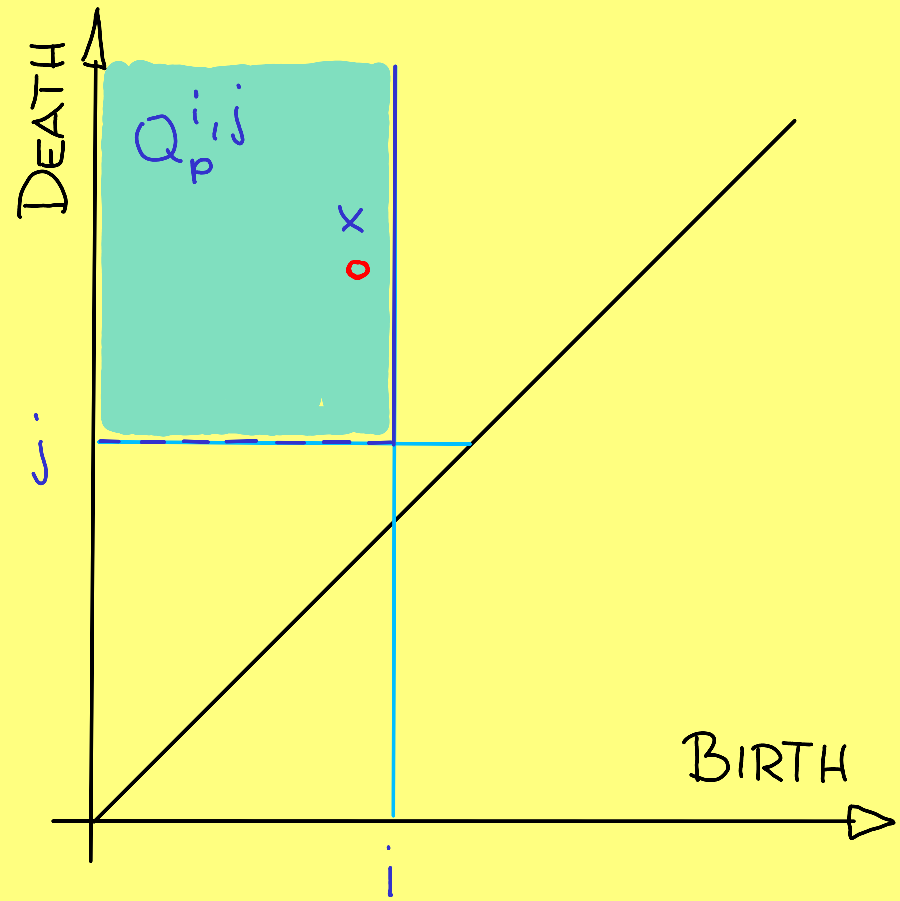
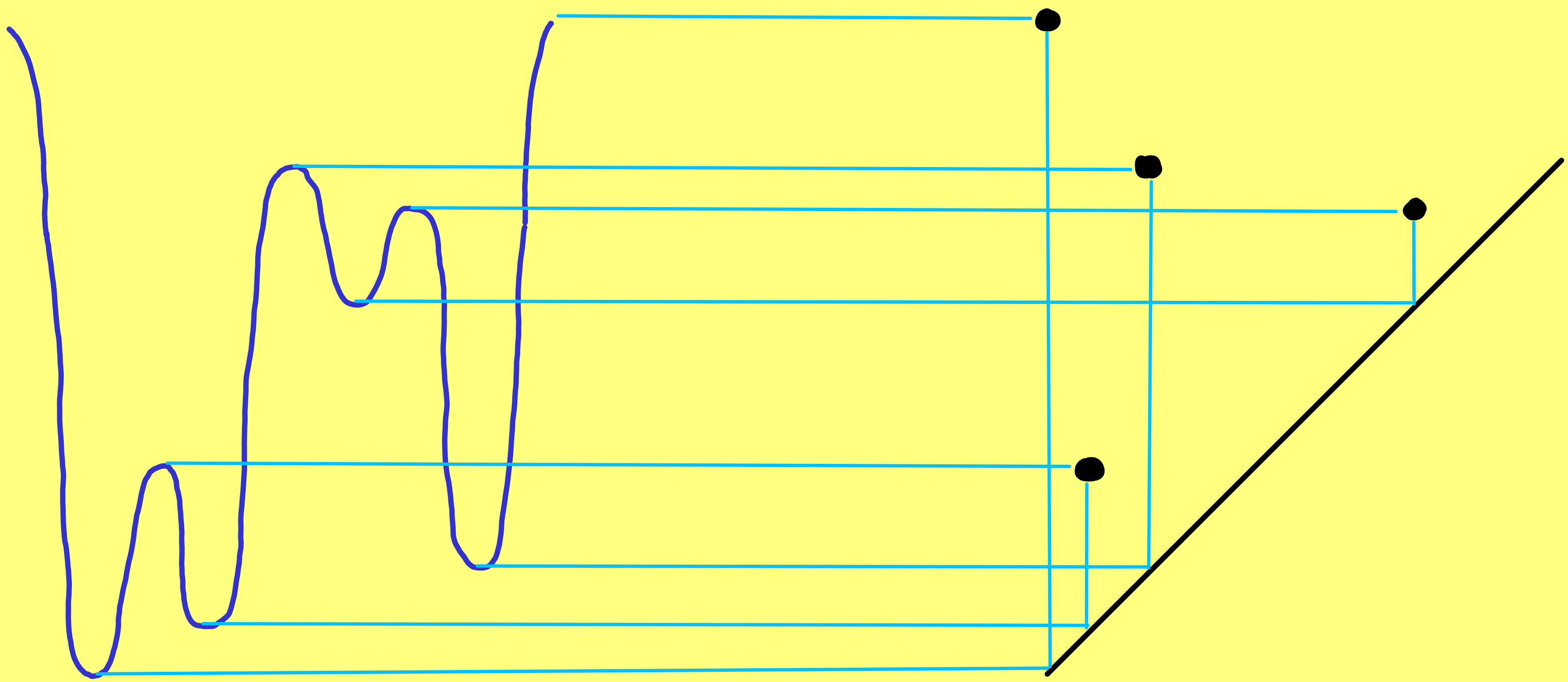


FUNDAMENTAL LEMMA OF PERSISTENT HOMOLOGY.

#points in $Q_p^{i,j}$ $=$ rank im $h_p^{i,j}$.

I.4 Bottleneck Distance



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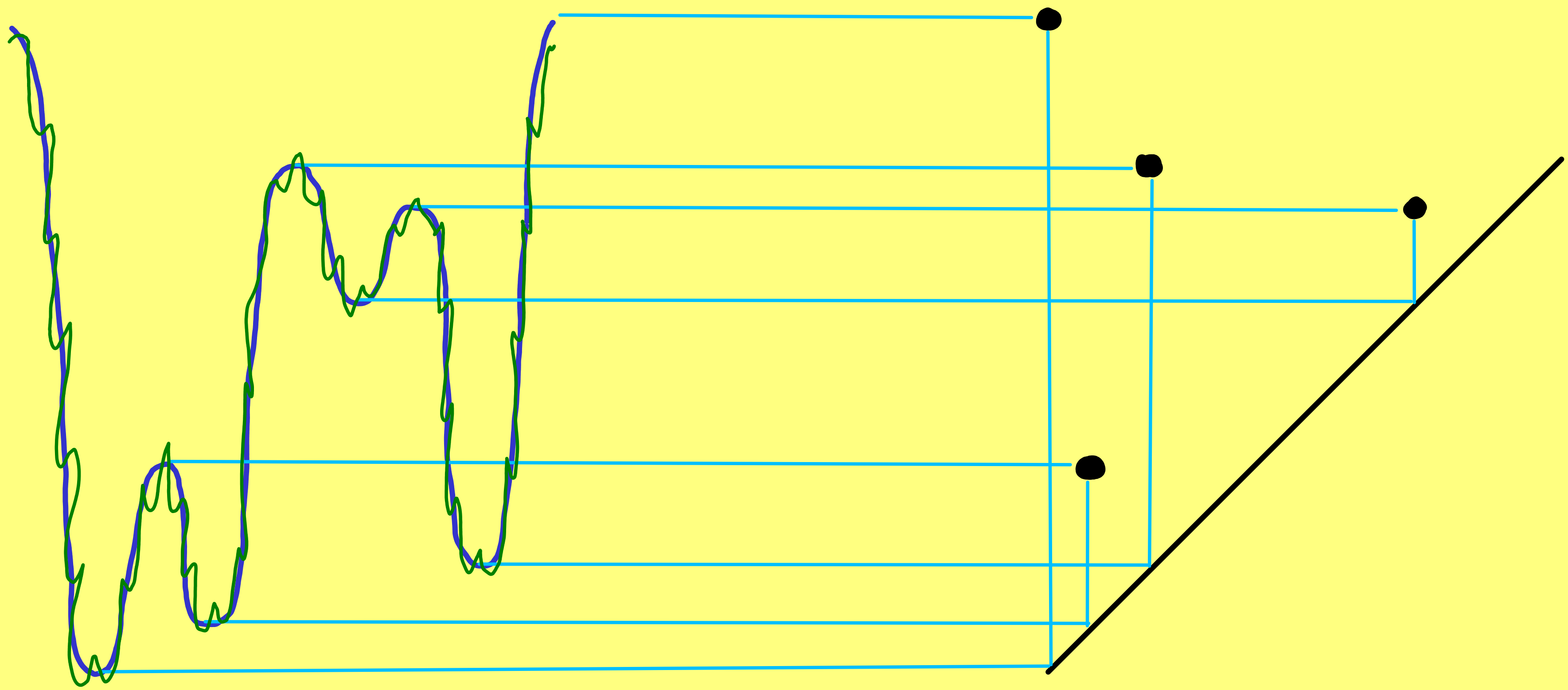

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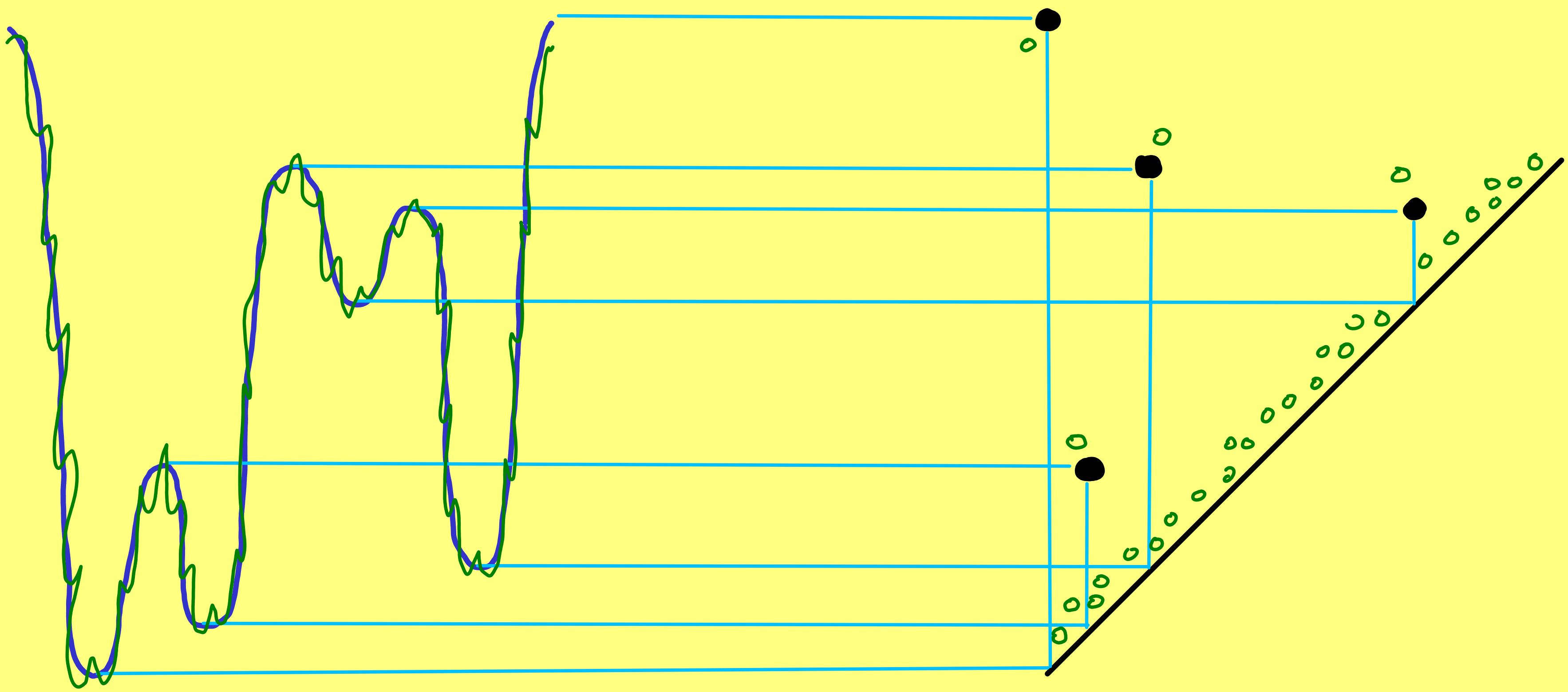

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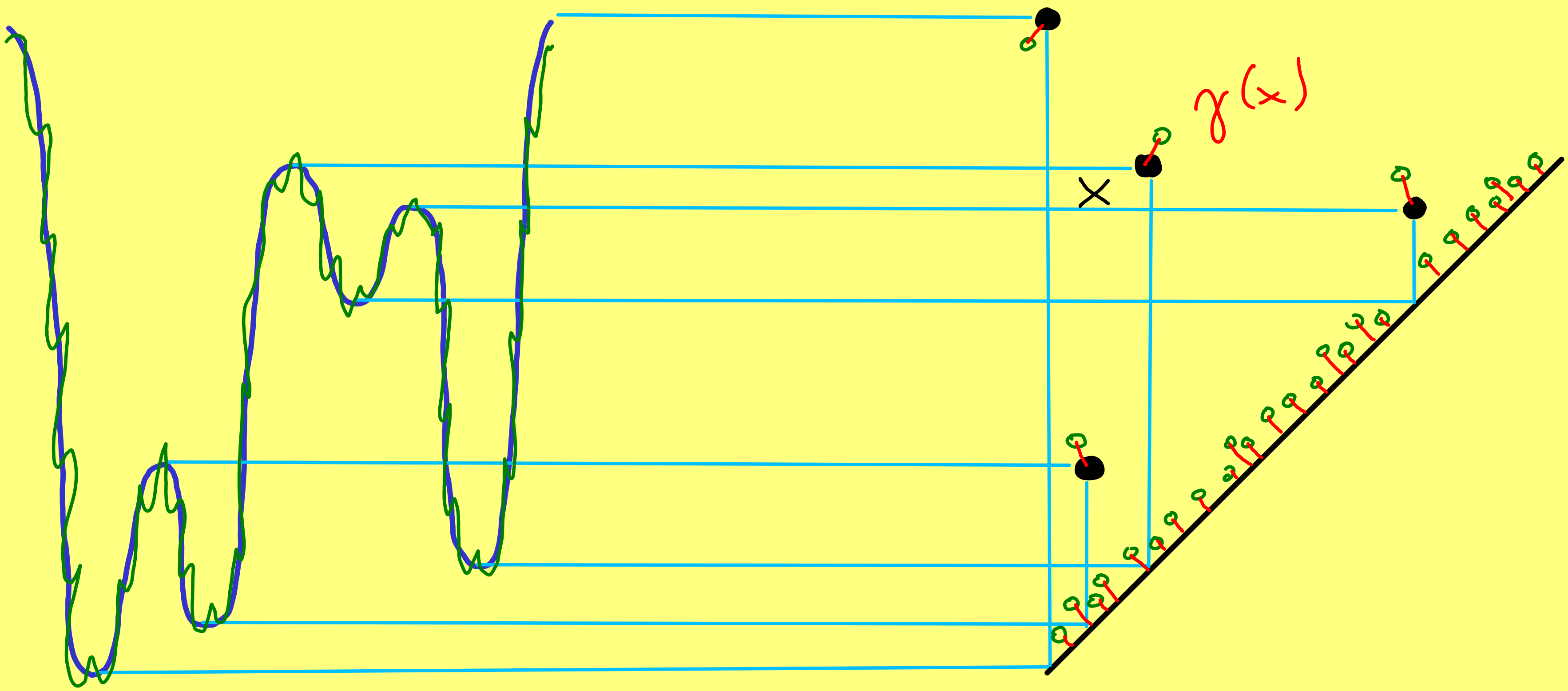

The bottleneck distance between the diagrams is

$$W_\infty\left(Dgm_p(f), Dgm_p(g)\right) = \inf_\gamma \sup_x \|x - \gamma(x)\|_\infty .$$

I.4 Bottleneck Distance

L_∞-STAB. THM. For tame functions $f, g : \mathbb{X} \to \mathbb{R}$ the bottleneck distance between their diagrams is bounded by the max-difference between the functions,

$$W_\infty(\mathrm{Dgm}_p(f), \mathrm{Dgm}_p(g)) \leq \|f - g\|_\infty .$$

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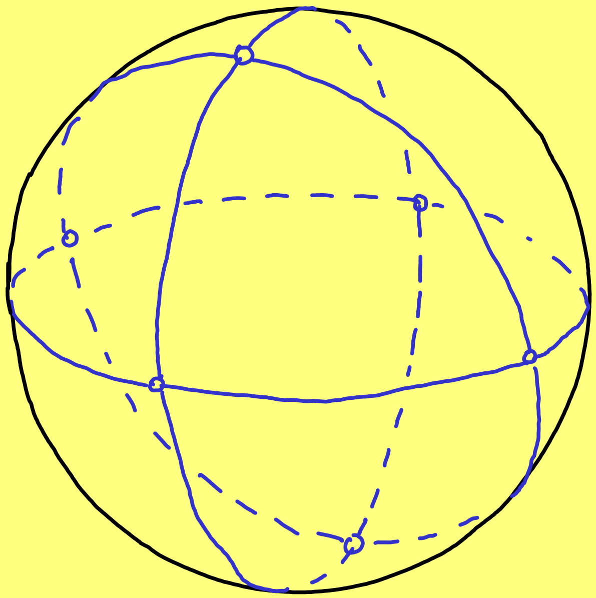
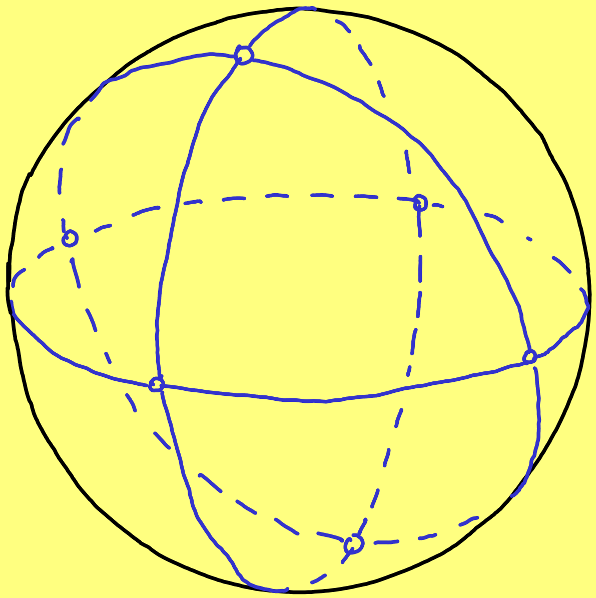
II.1 Size and Growth



II.1 Size and Growth



mesh (K) = max. diameter of a simplex

size (K) = number of simplices

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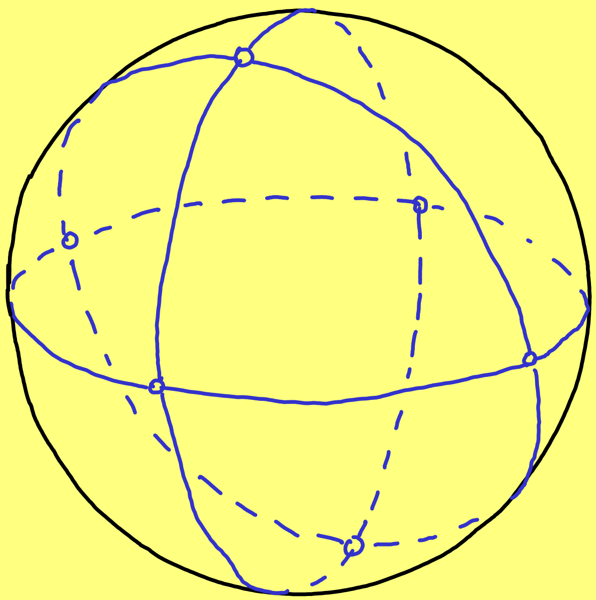

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$$N(r) = \min_{\mathrm{mesh}(K) \le r} \mathrm{size}(K)$$

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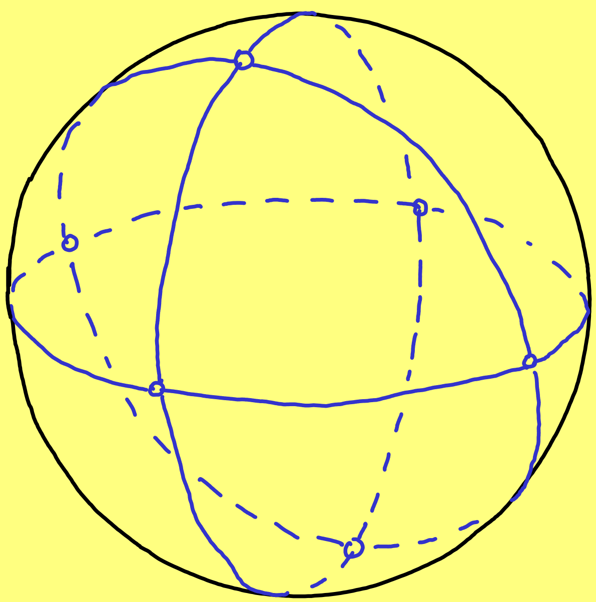

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$$N(r) = \min_{\mathrm{mesh}(K) \le r} \mathrm{size}(K)$$

The triangulation grows polynomially if $\exists$ constants C, M s.t.

$$N(r) \le C/r^M.$$

II.2 Total Persistence

The q-th total persistence of f is

$$\mathrm{Pers}_q(f) = \sum_p \sum_x \mathrm{pers}(x)^q$$

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f Lipschitz and $N(r) \le C / r^M$

$\Longrightarrow \mathrm{Pers}_q(f) \le \mathrm{const}$ for all $q > M$

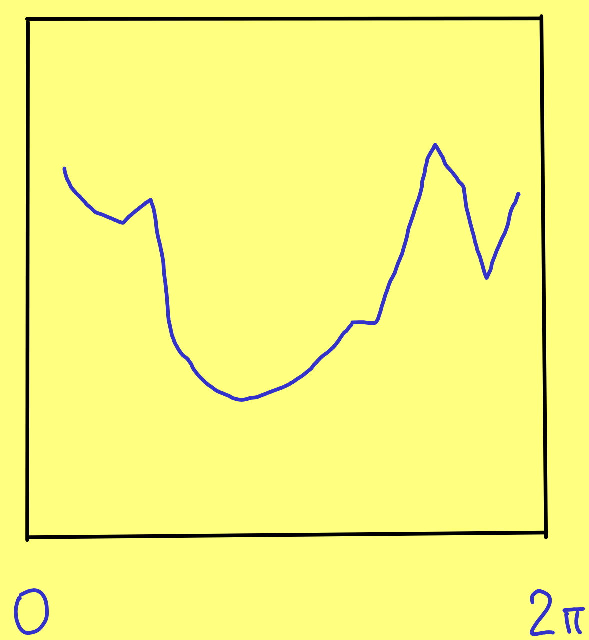
II.2 Total Persistence

TP-STAB. THM. Let $\mathbb{X}$ be a compact metric space with polynomially growing triangulation and $f, g : \mathbb{X} \to \mathbb{R}$ Lipschitz functions. Then

$$\mathrm{Pers}_q(f) - \mathrm{Pers}_q(g) \leq \mathrm{const} \cdot \|f - g\|_\infty$$

for every $q > M + 1$.

II.2 SIMPLIFICATION



normalized to $\mathrm{amp}(f) = \max_x f(x) - \min_x f(x) = 1$

II.2 SIMPLIFICATION

An ε-simplification of f is a function $f_\varepsilon : \mathbb{X} \to \mathbb{R}$ with $\| f - f_\varepsilon \|_\infty \leq \varepsilon$ whose diagrams $\mathrm{Dgm}_p(f_\varepsilon)$ are same as $\mathrm{Dgm}_p(f)$ without points of persistence at most ε.

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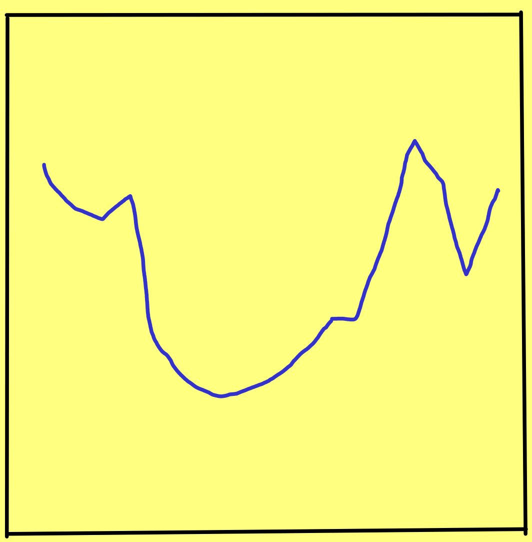

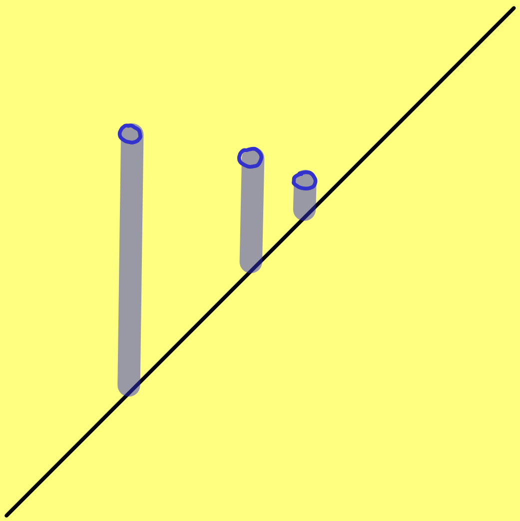

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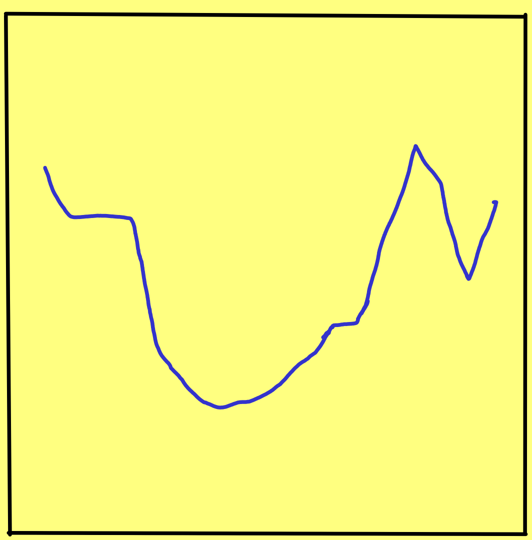

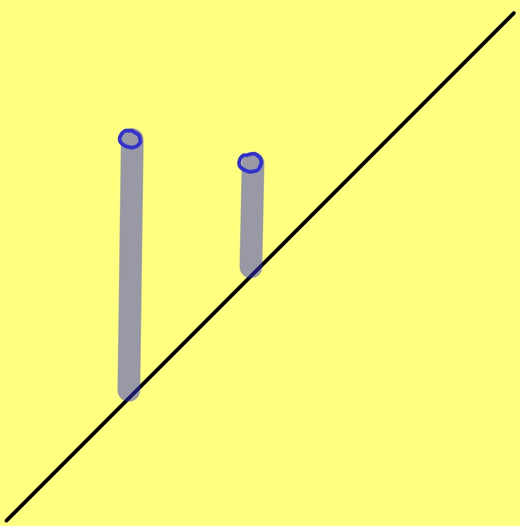

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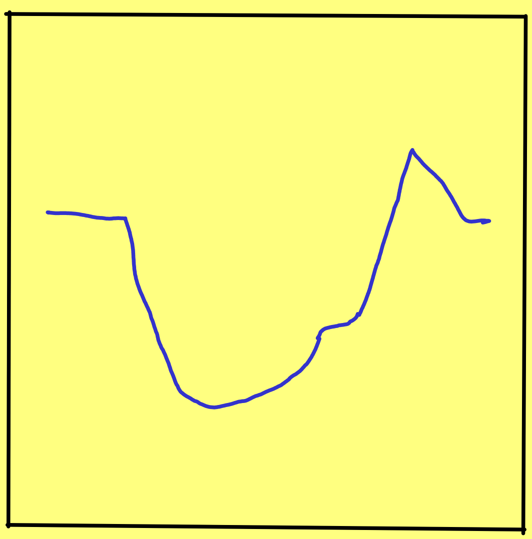

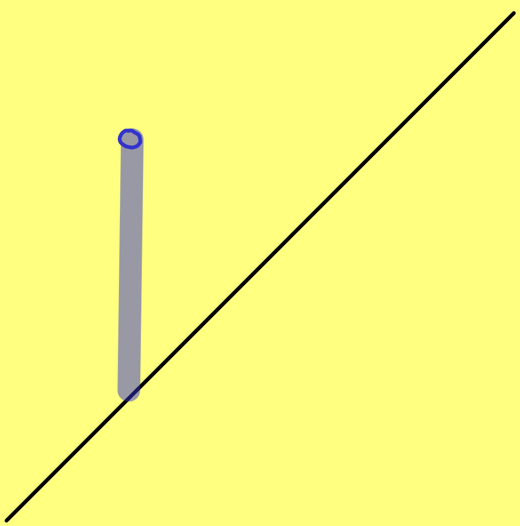

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etc.

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$f : \mathbb{S}^1 \to \mathbb{R}$

$$M_0(f) = \frac{1}{2} \# \text{crit. pts.} = \mathrm{Pers}_0(f)$$

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$$M_2(f) = \int_{\varepsilon=0}^{1} M_1(f_\varepsilon)\, d\varepsilon = \mathrm{Pers}_2(f)$$

etc.

II.2 Simplification

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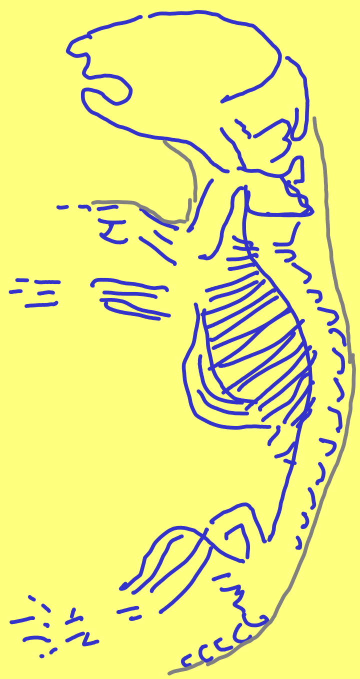
$$\text{not stable}\left\{\begin{array}{l} M_0(f) = \frac{1}{2}\ \#\text{crit. pts.} = \mathrm{Pers}_0(f) \\[2ex] M_1(f) = \displaystyle\int_{\varepsilon=0}^{1} M_0(f_\varepsilon)\,d\varepsilon = \mathrm{Pers}_1(f) \end{array}\right.$$

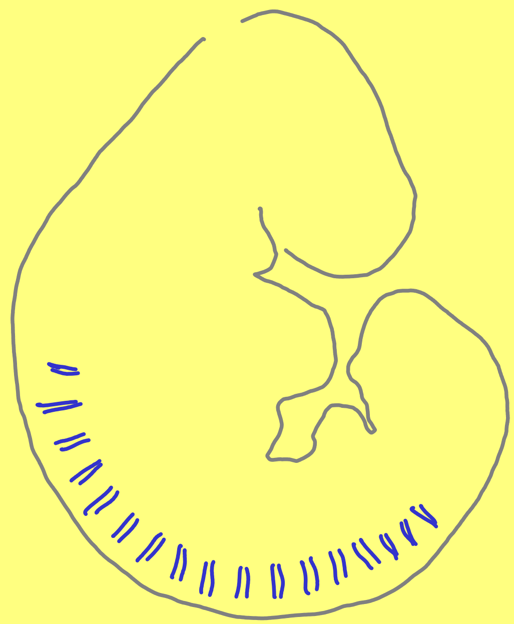
$$\text{stable}\left\{\begin{array}{l} M_2(f) = \displaystyle\int_{\varepsilon=0}^{1} M_1(f_\varepsilon)\,d\varepsilon = \mathrm{Pers}_2(f) \\[2ex] \quad \text{etc.} \end{array}\right.$$

II.3 Rhythmic Gene Expression

adult mouse

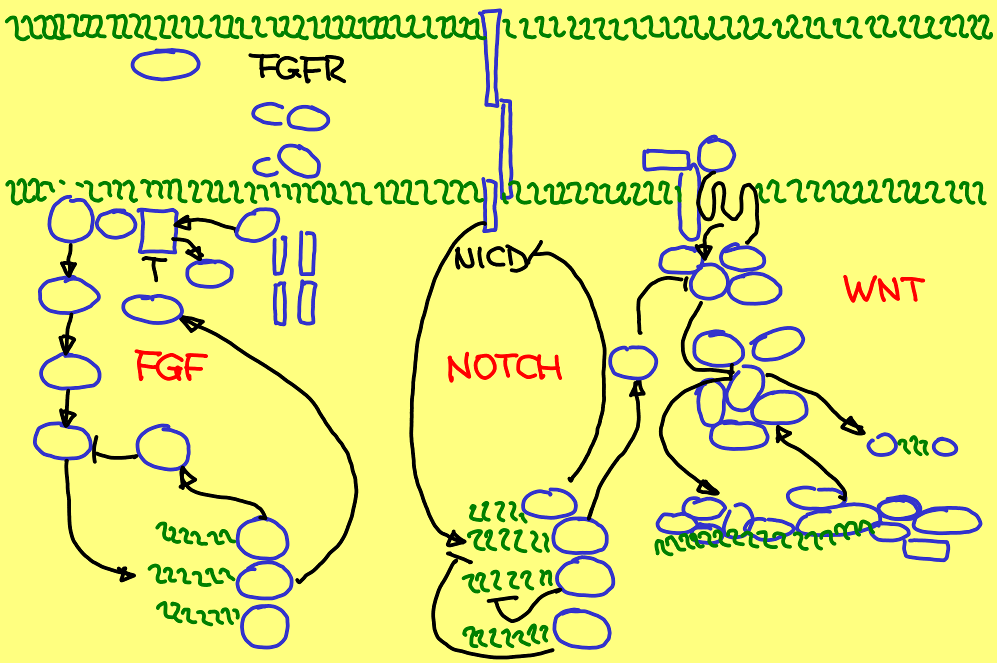


mouse embryo



somite development is a rythmic process

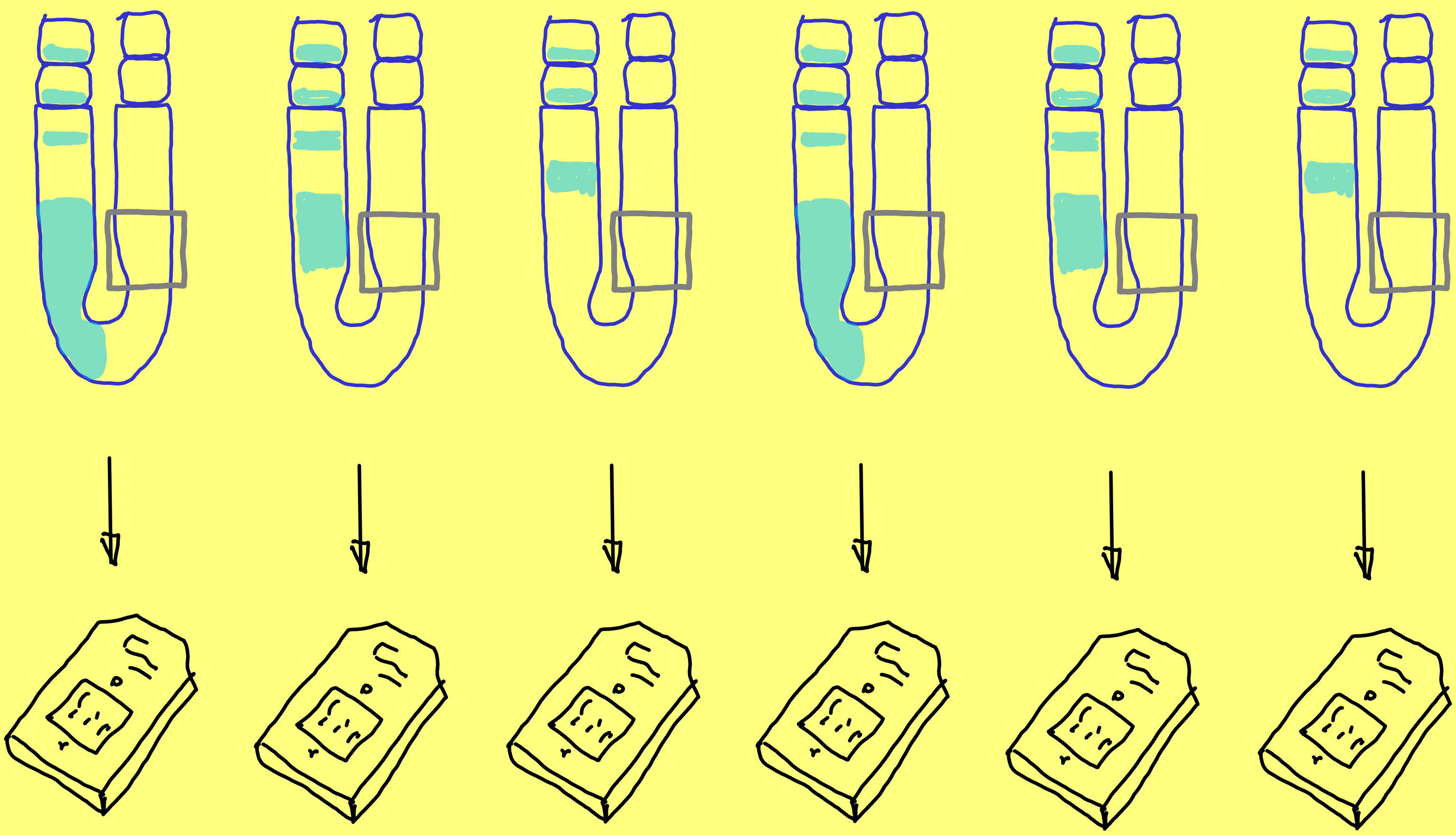
II.3 Rhythmic Gene Expression



the proposed clock

II.3 Rhythmic Gene Expression

... time series microarray analysis



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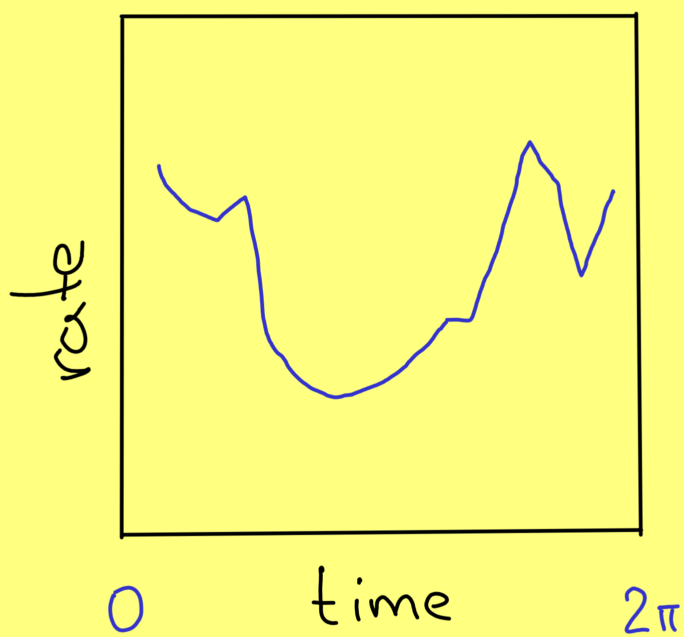
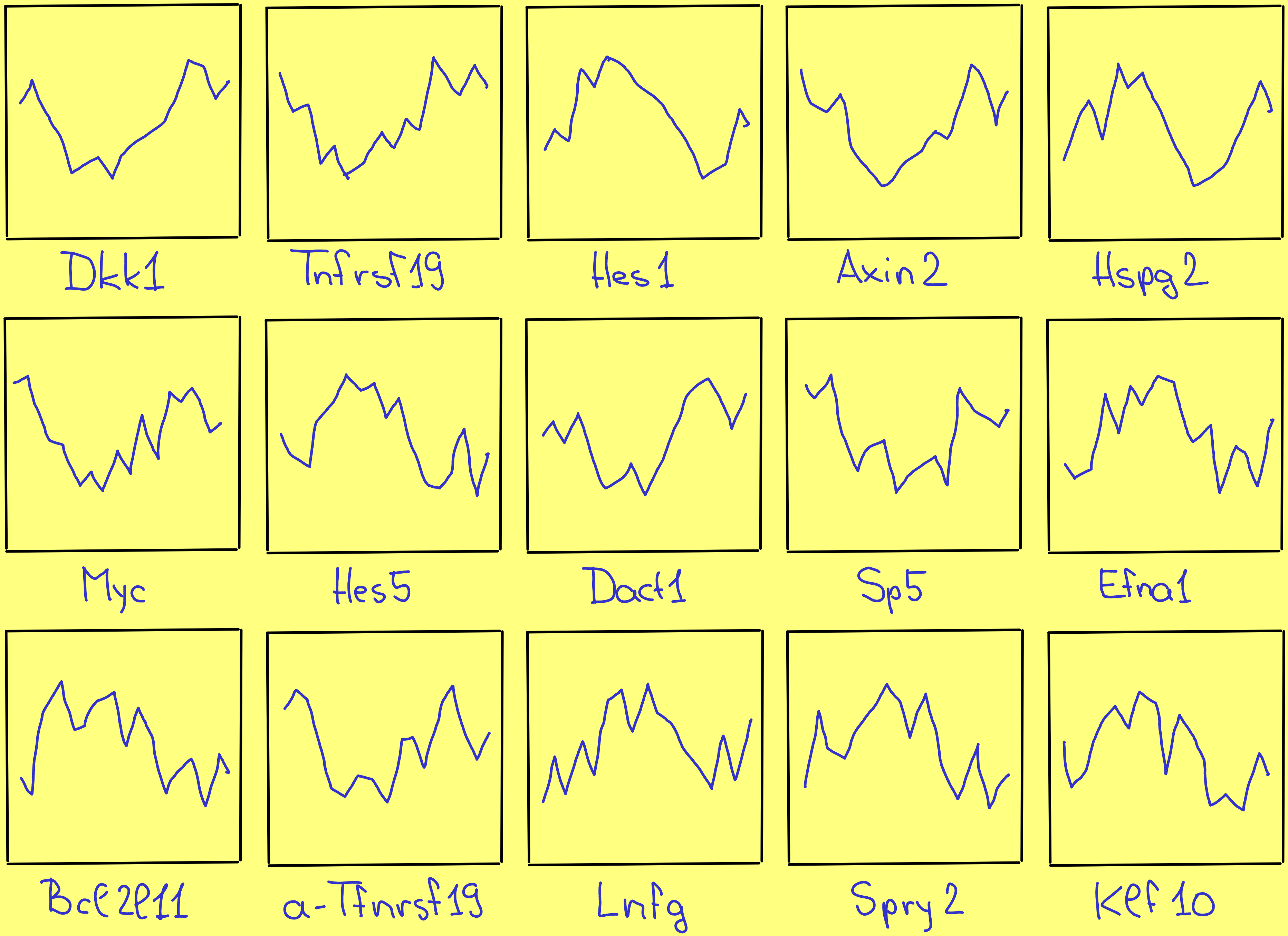
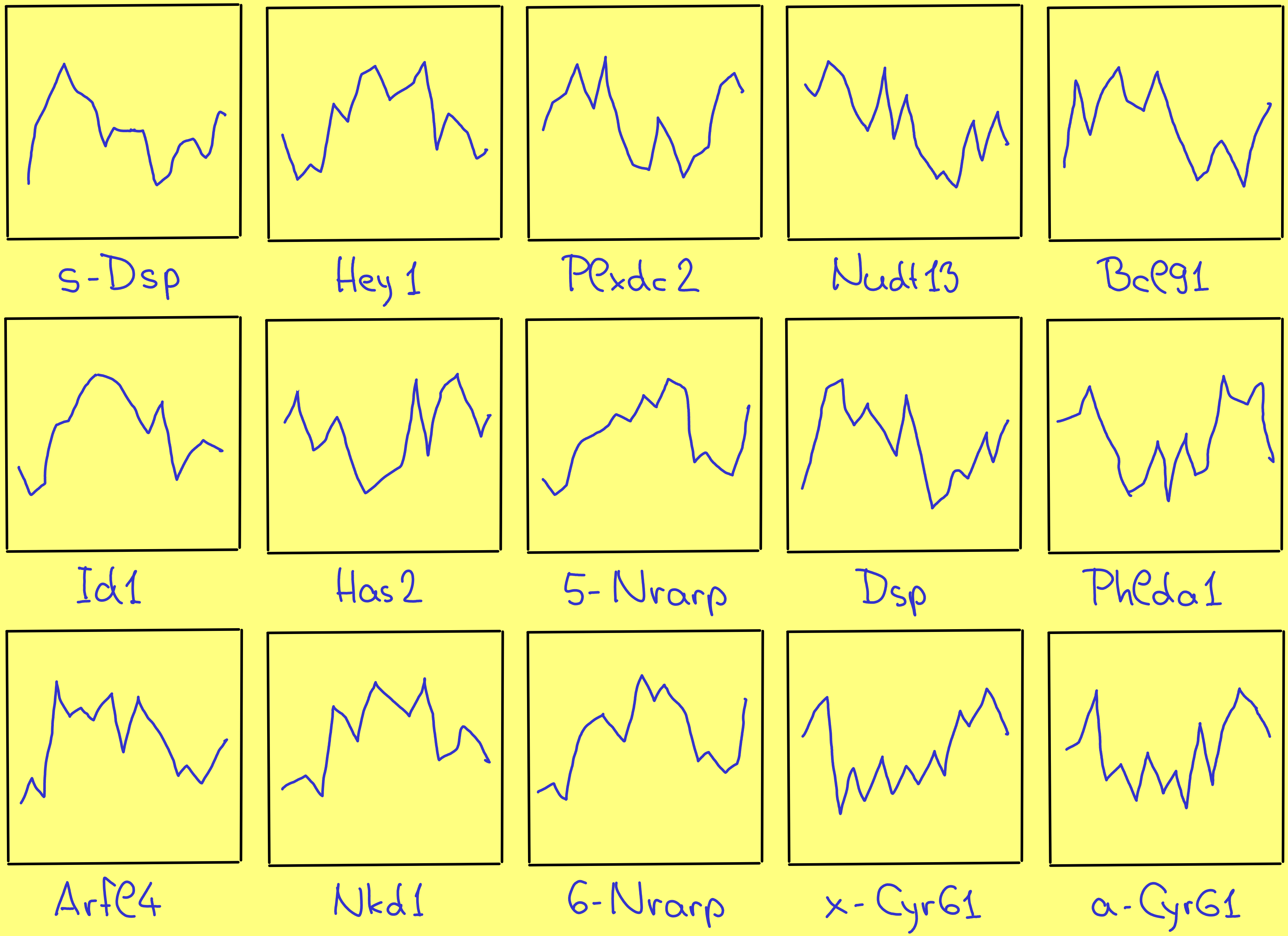


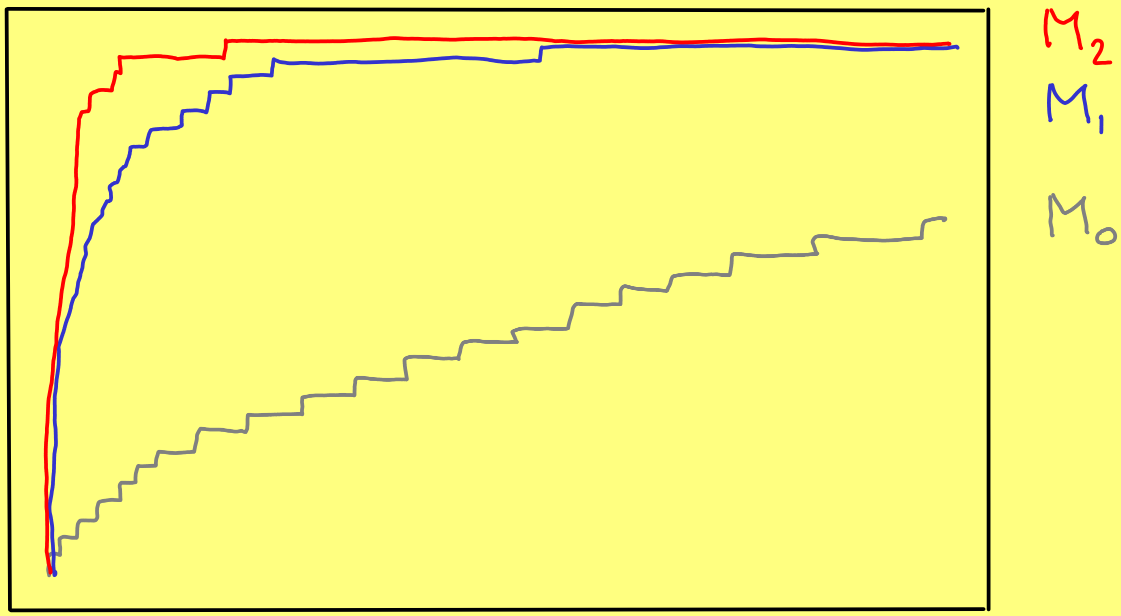

Dkk1
Tnfrsf19
Hes1
Axin2
Hspg2
Myc
Hes5
Dact1
Sp5
Efna1
Bcl2l11
a-Tfnrsf19
Lnfg
Spry2
Klf10


s-Dsp
Hey1
Plxdc2
Nudt13
Bcl9l
Id1
Has2
5-Nrarp
Dsp
Phlda1
Arl4
Nkd1
6-Nrarp
x-Cyr61
a-Cyr61

II.3 Rhythmic Gene Expression

yield of 30 biologically confirmed genes

ROC Curves



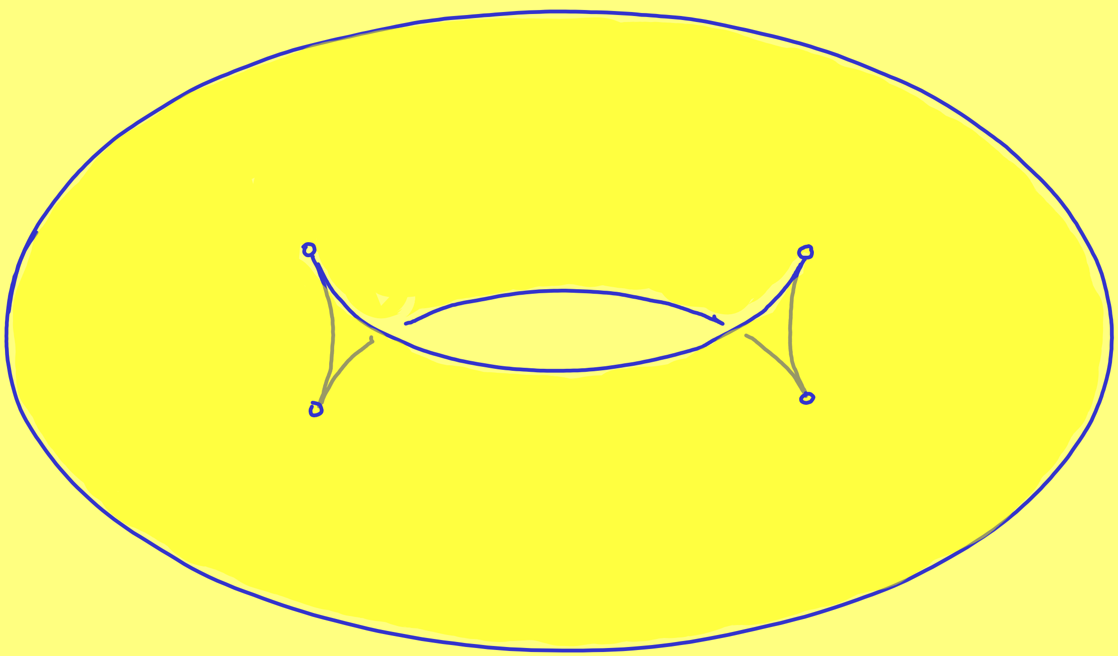
	M_0	M_1	M_2	M_3	M_4	max
Area	4.42	9.75	10.18	10.19	10.15	10.50

I Persistent Homology

II L_p-Stability and Somites

III Contour Stability

III.1 THE CONTOUR

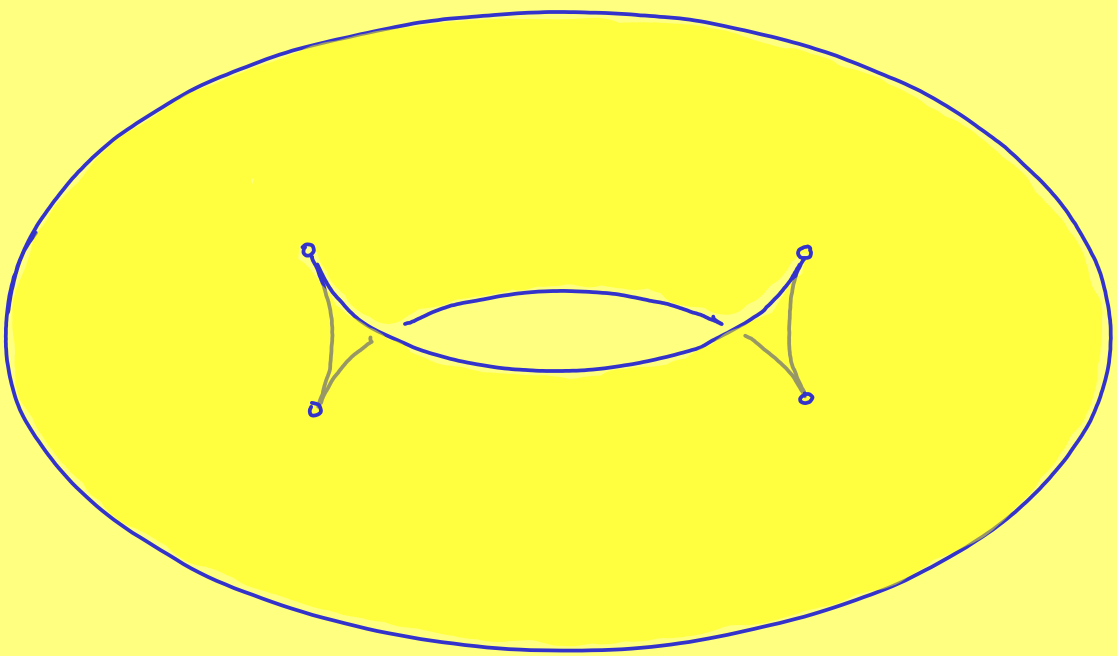


III.1 The Contour

$$f : \mathbb{M} \to \mathbb{R}^2$$

generic smooth

compact, orientable 2-mfld w/o boundary



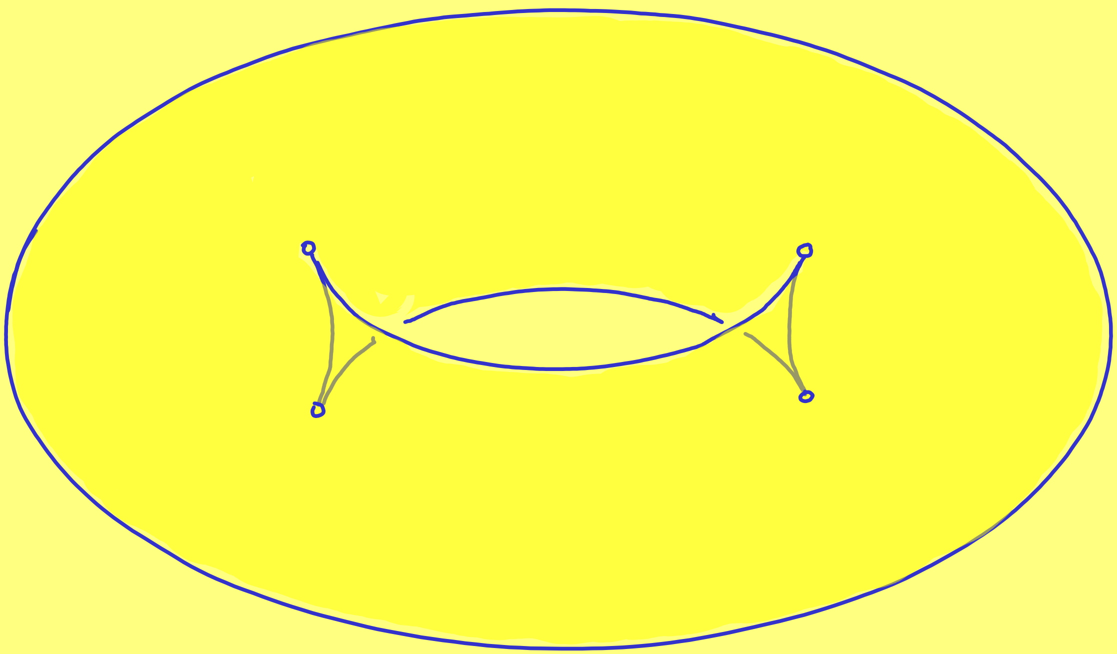
III.1 The Contour

$f : \mathbb{M} \to \mathbb{R}^2$

generic smooth

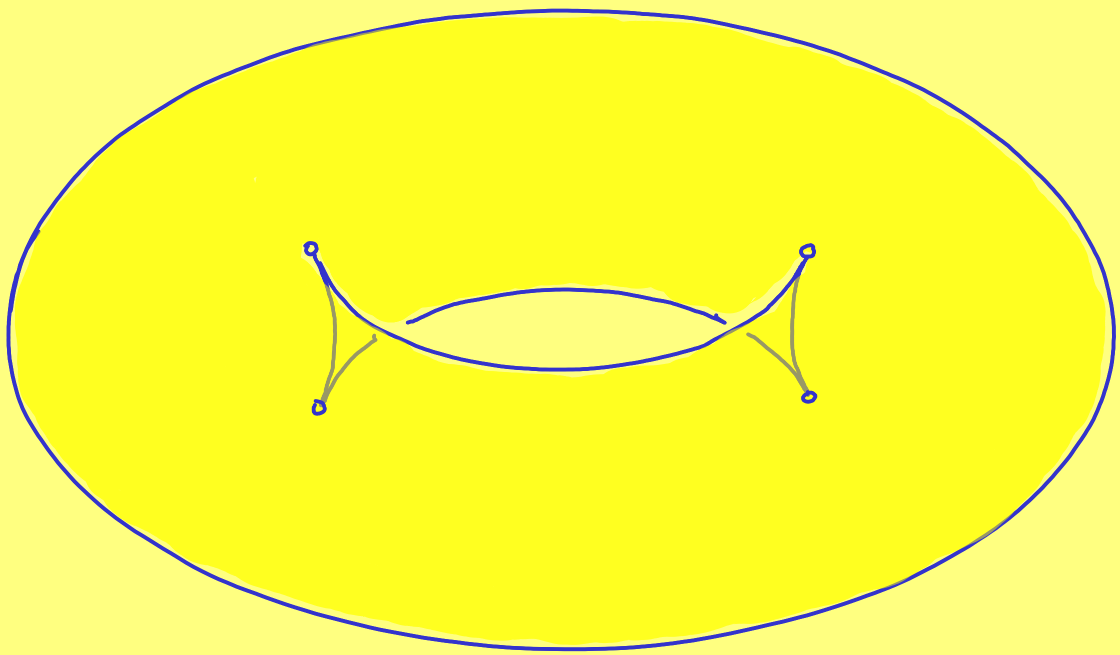
compact, orientable 2-mfld w/o boundary

Contour (f) = set of critical values



III.2 A Perturbation

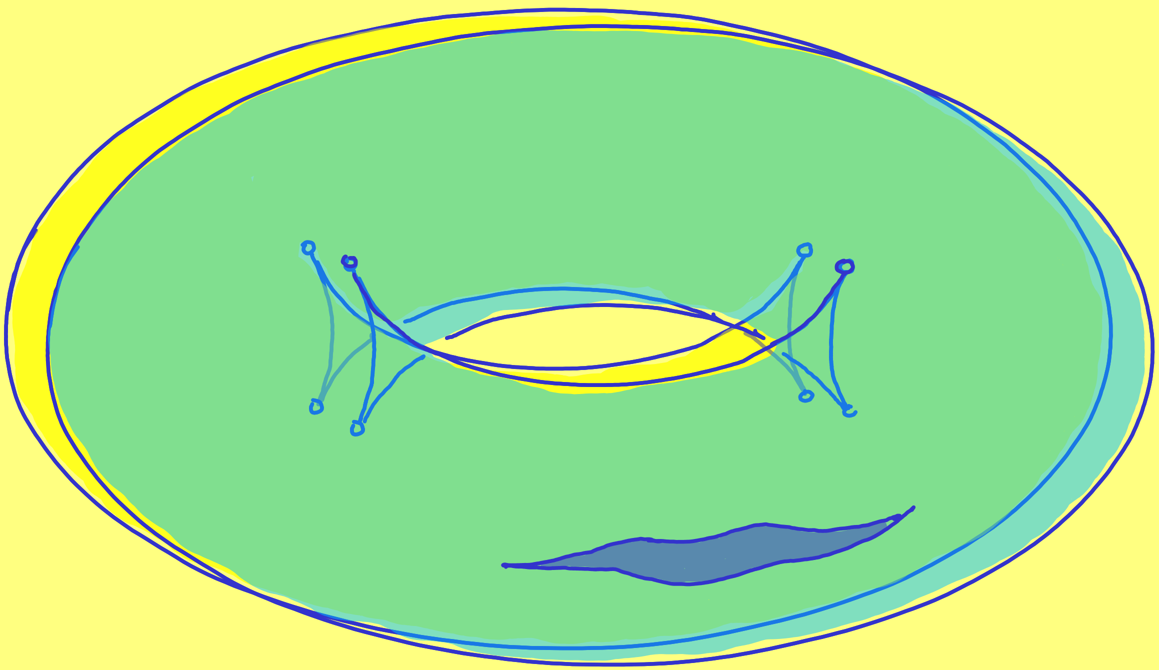
$f : M \to \mathbb{R}^2$



III.2 A Perturbation

$f, g : \mathbb{M} \to \mathbb{R}^2$

$\varepsilon = \max_{x \in \mathbb{M}} \| f(x) - g(x) \|_2$

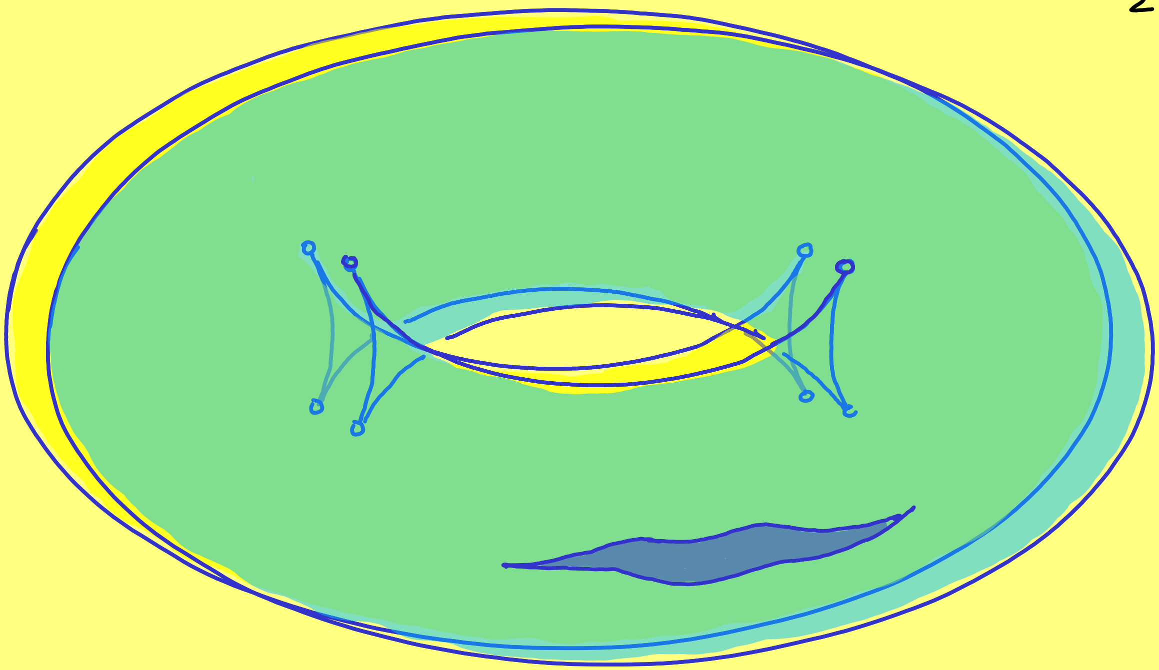


III.2 A Perturbation

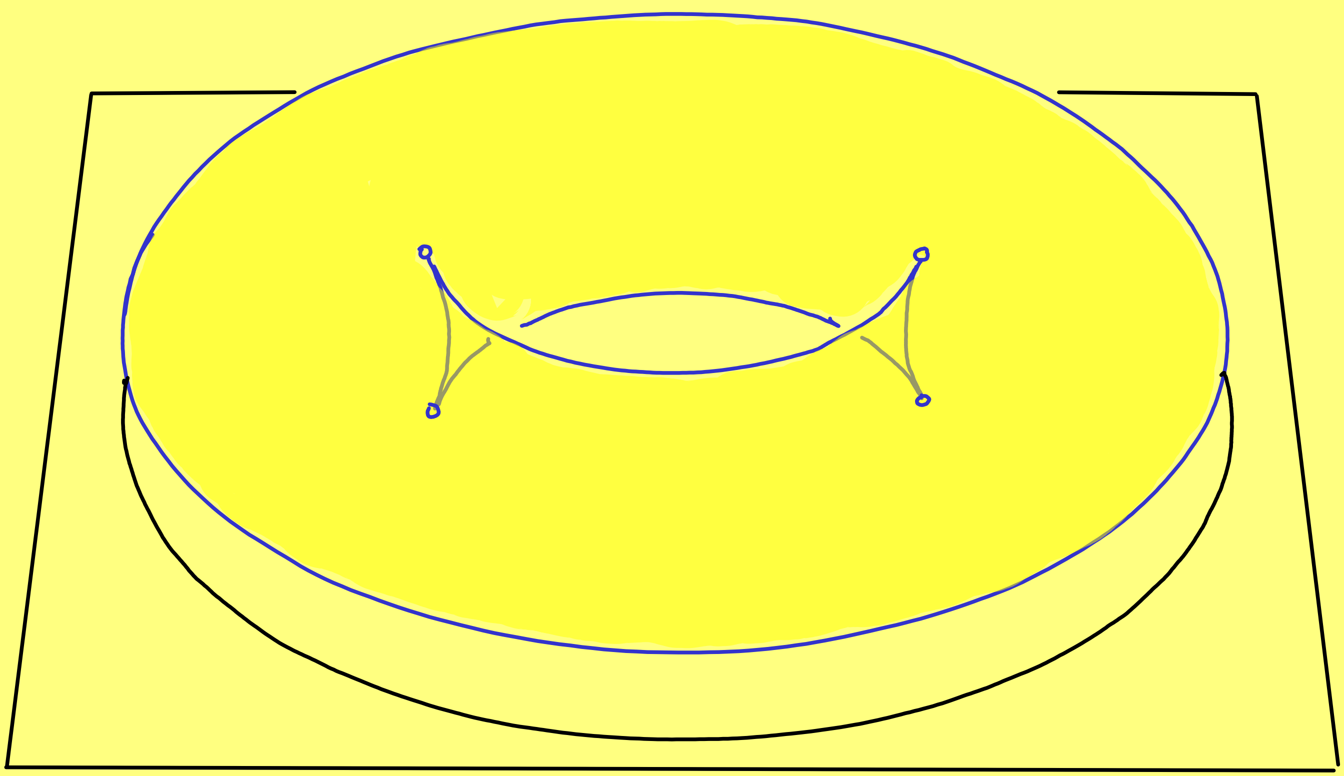
$$f, g : \mathbb{M} \to \mathbb{R}^2$$

$$\varepsilon = \max_{x \in \mathbb{M}} \| f(x) - g(x) \|_2$$

1. close contour lines;
2. thin creases.



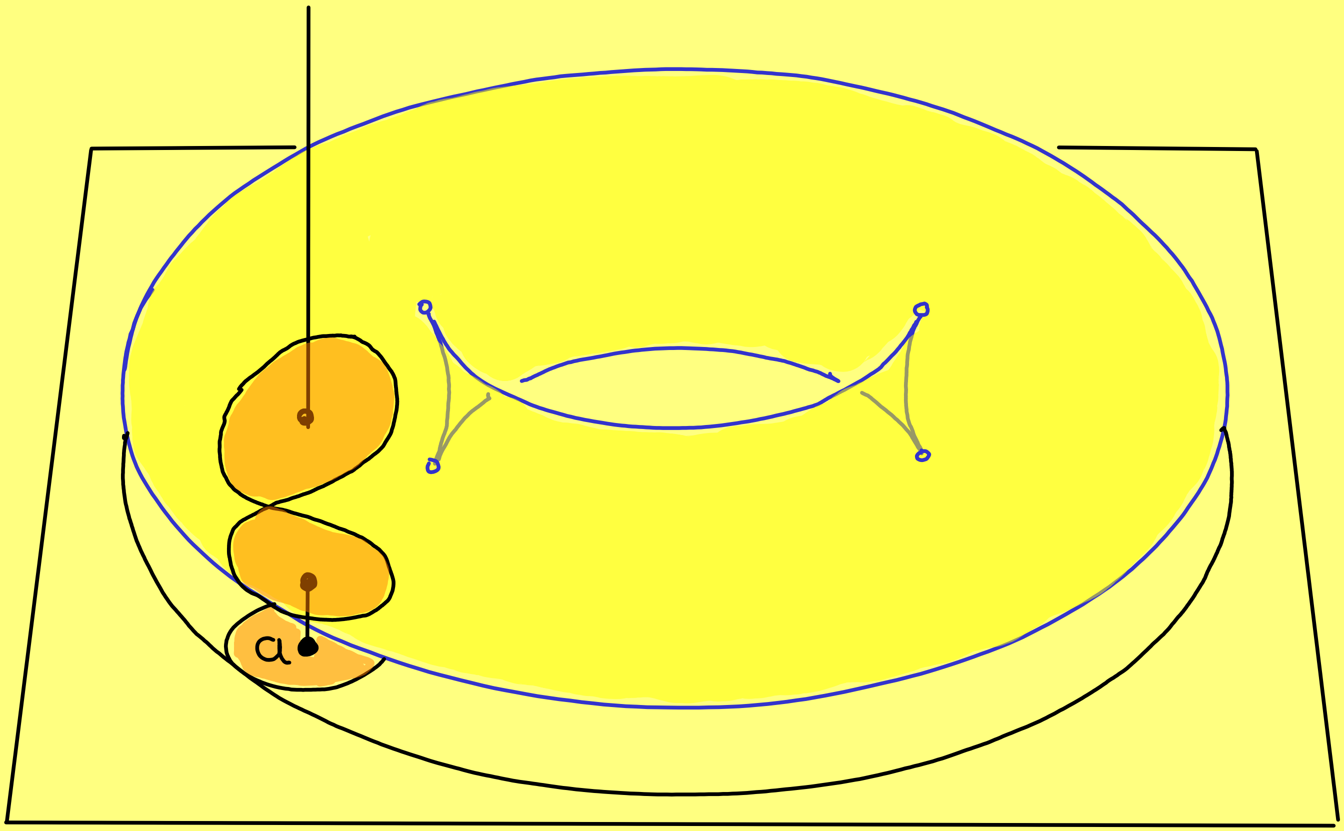
III.3 Distance Function



III.3 Distance Function

$f_a : \mathbb{M} \to \mathbb{R}$ defined by

$$f_a(x) = \| f(x) - a \|_2$$



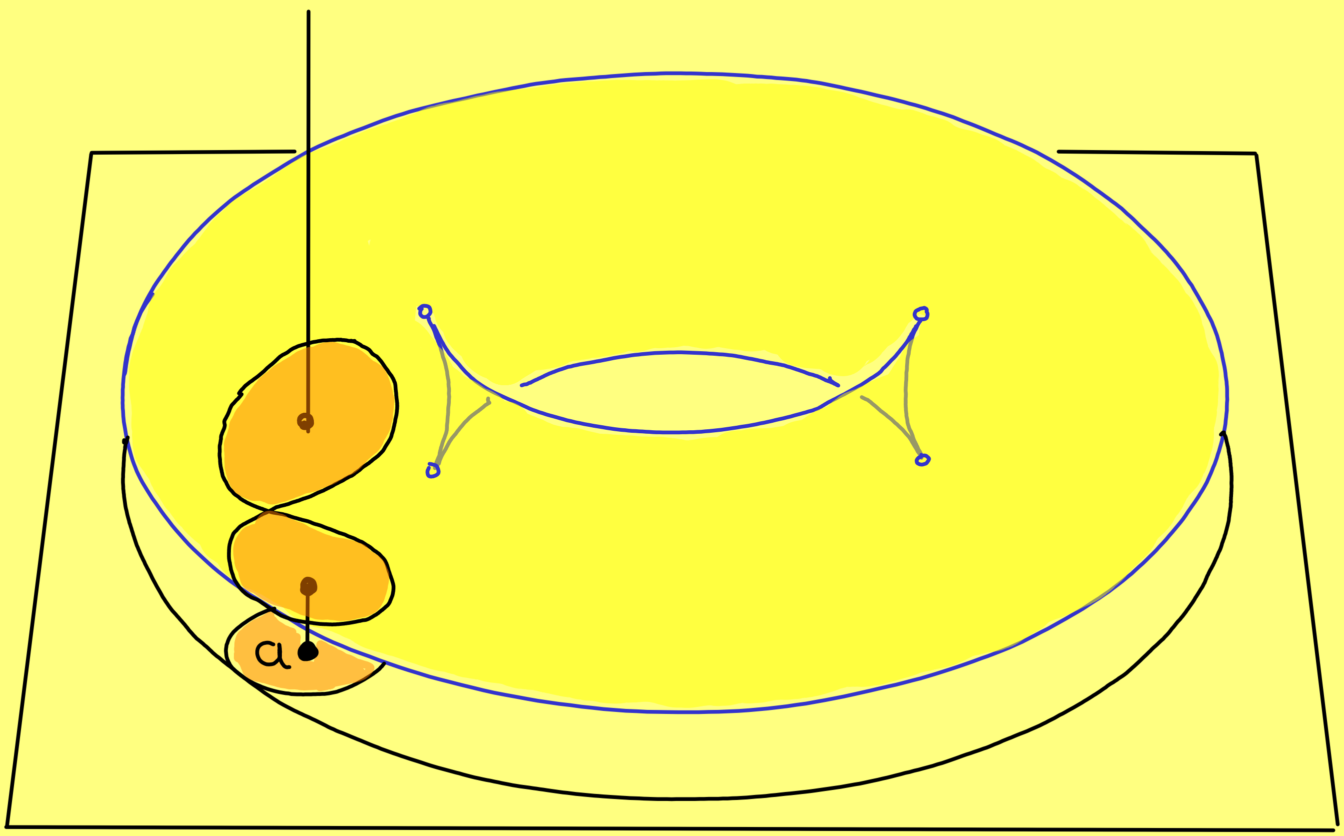
III.3 Distance Function

$f_a : \mathbb{M} \to \mathbb{R}$ defined by

$$f_a(x) = \| f(x) - a \|_2$$

sublevel set is

$$\mathbb{M}_r(a) = f_a^{-1}[0, r]$$



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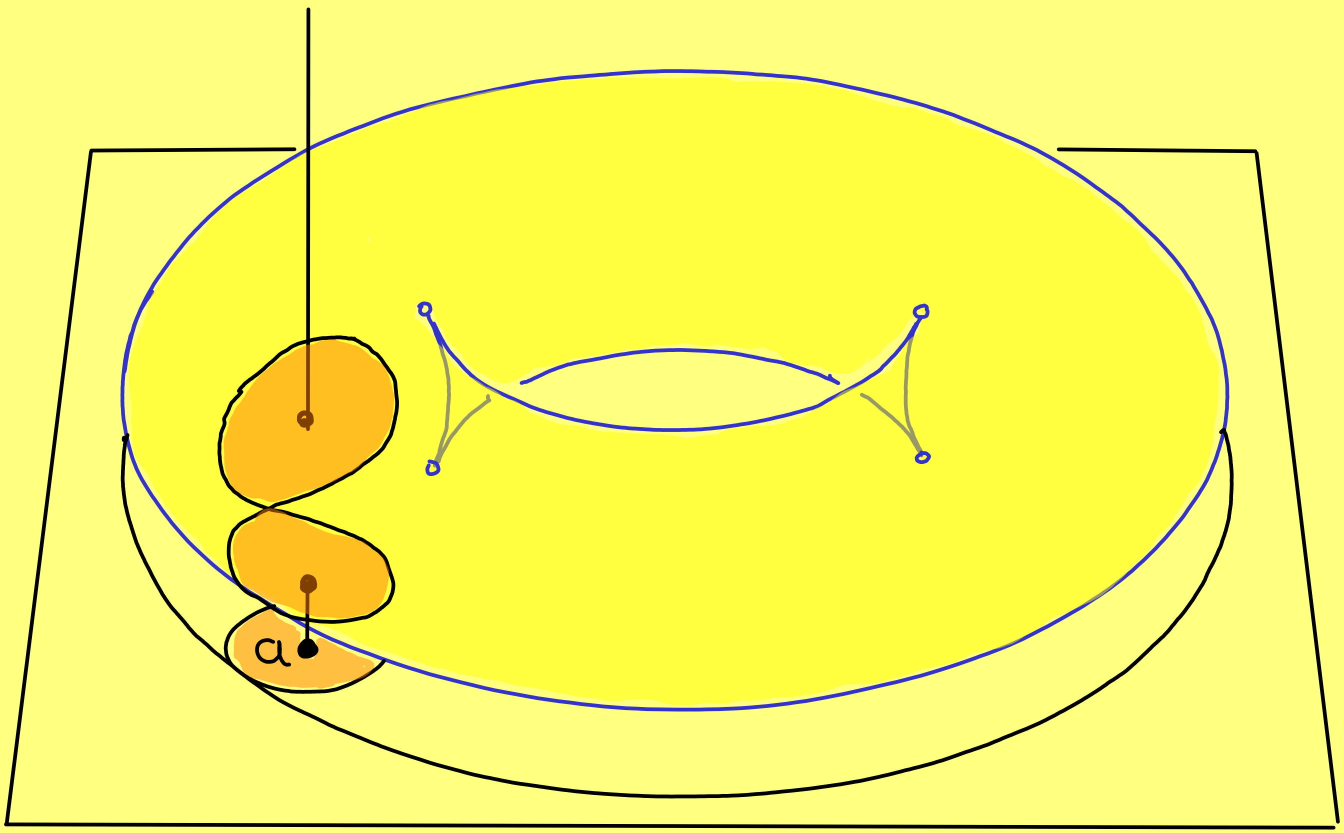


sublevel set is

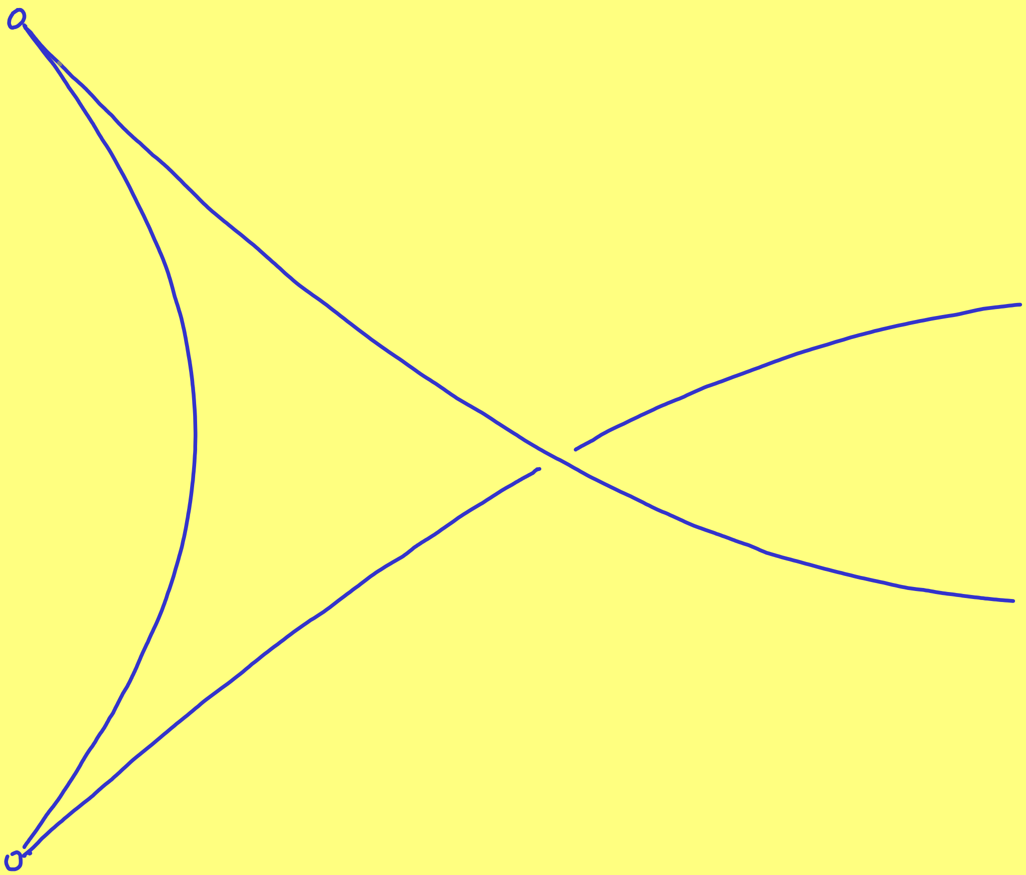
$$\mathbb{M}_r(a) = f_a^{-1}[0, r]$$

x crit. for f_a

$\Downarrow$

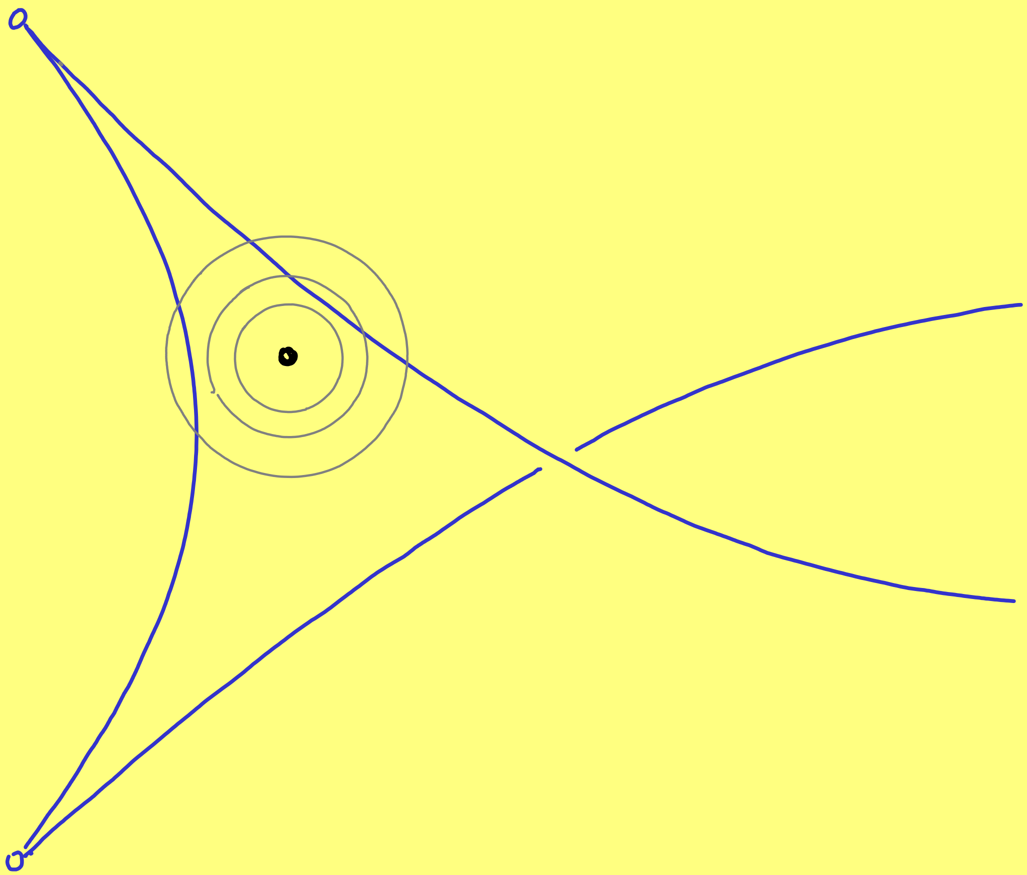
x crit. for f

III.4 Degree



III.4 Degree

$F_r = H_0(\mathbb{M}_r(a))$ indexed by radius



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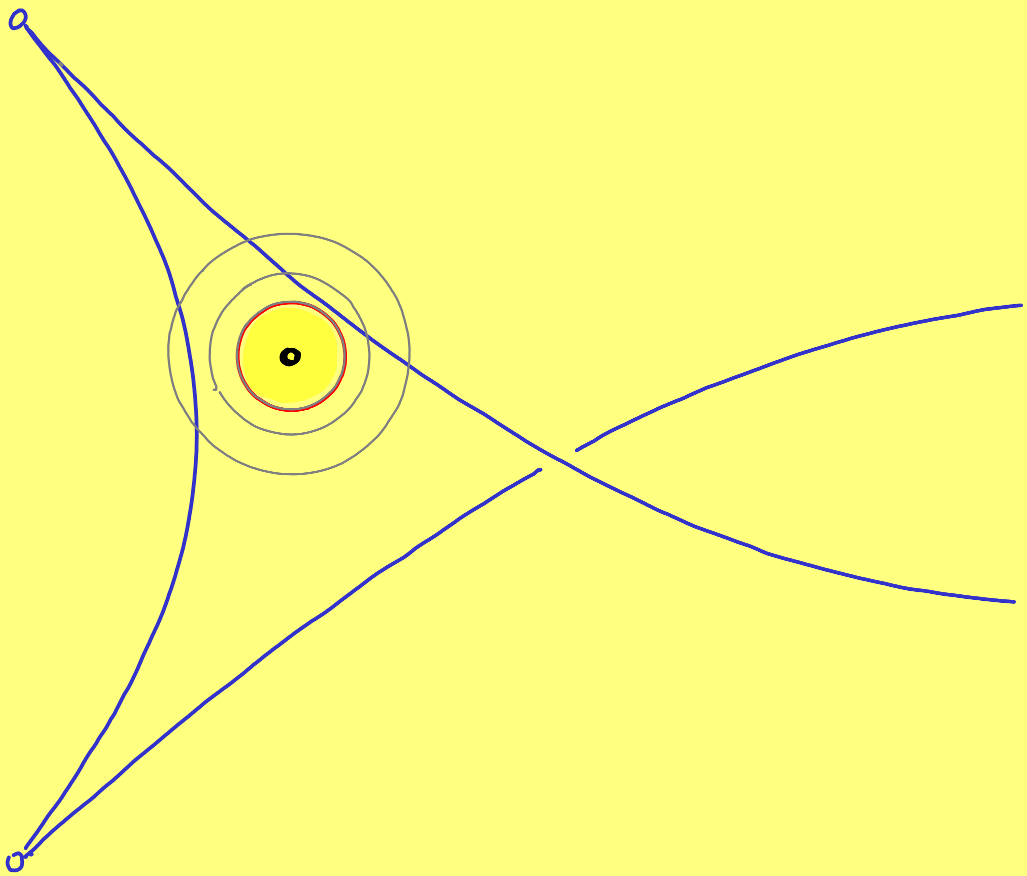

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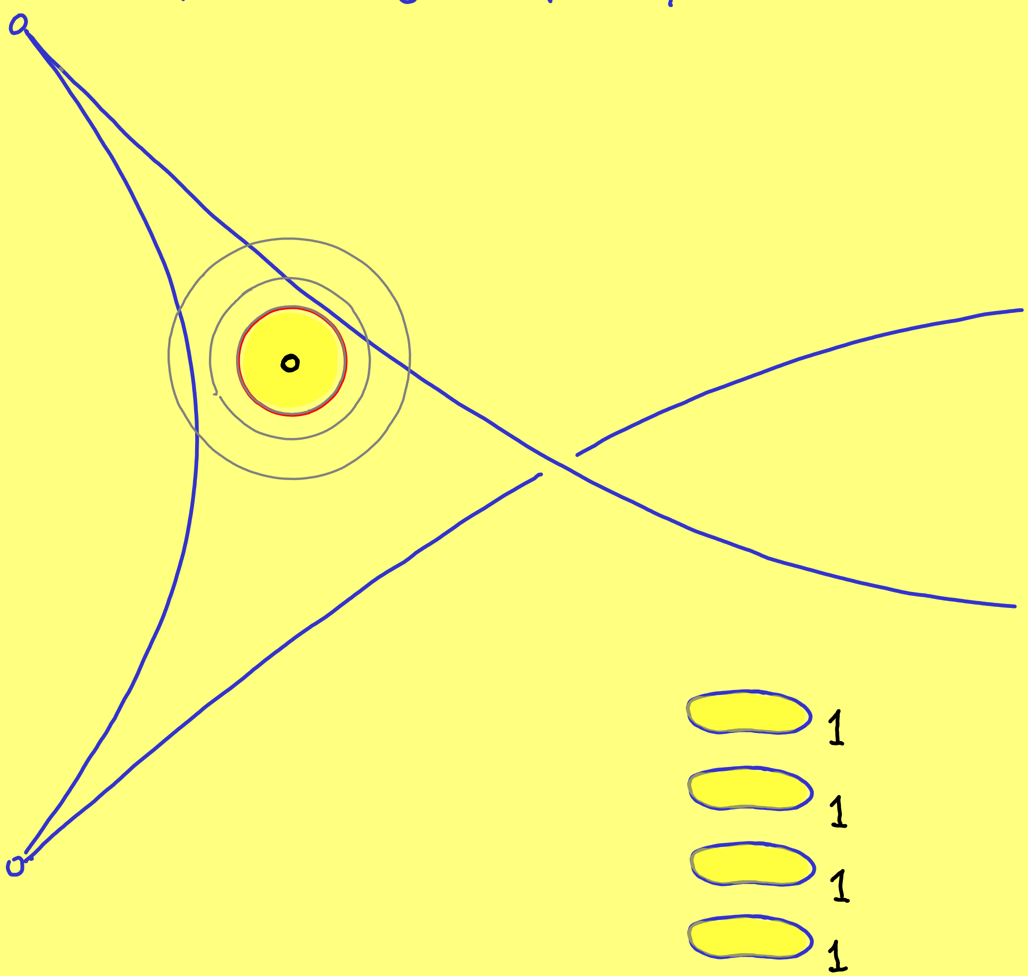
$\mathbb{F}_r = H_0(\mathbb{M}_r(a))$ indexed by radius



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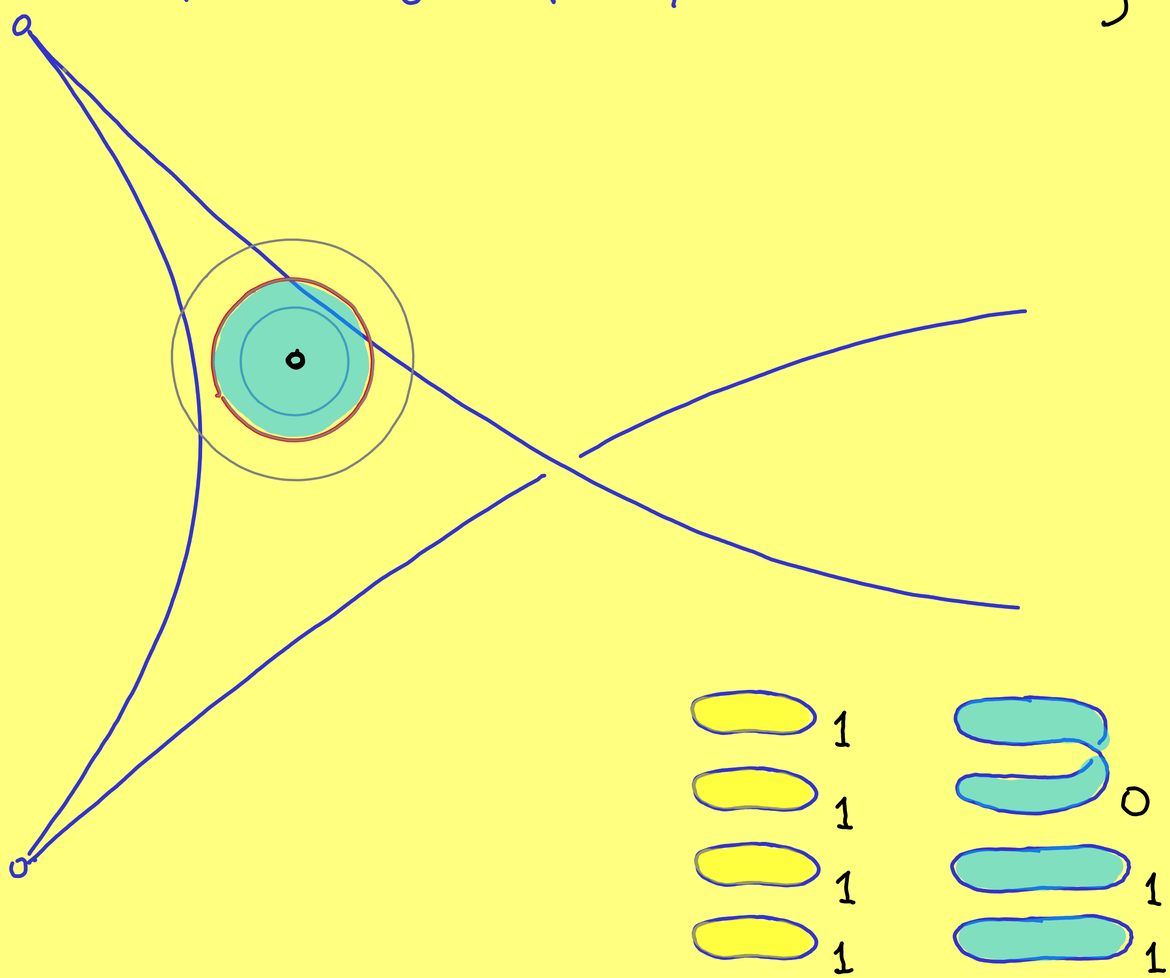

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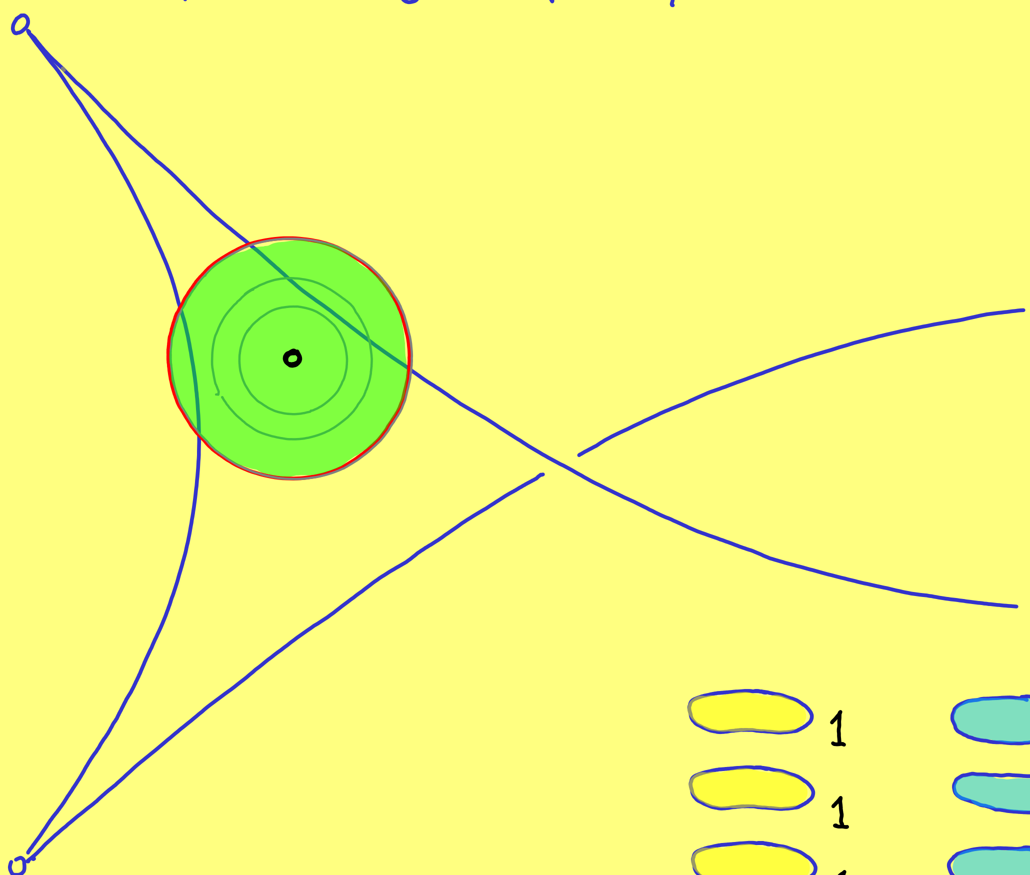

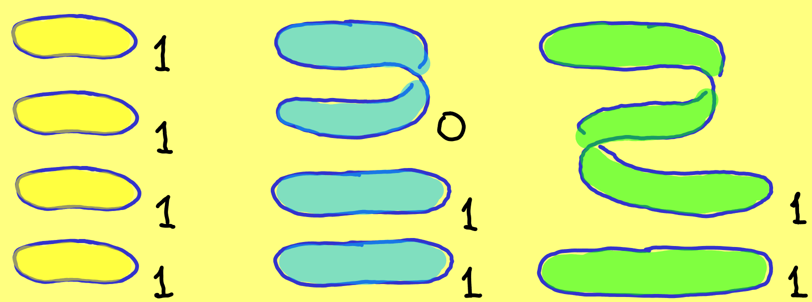

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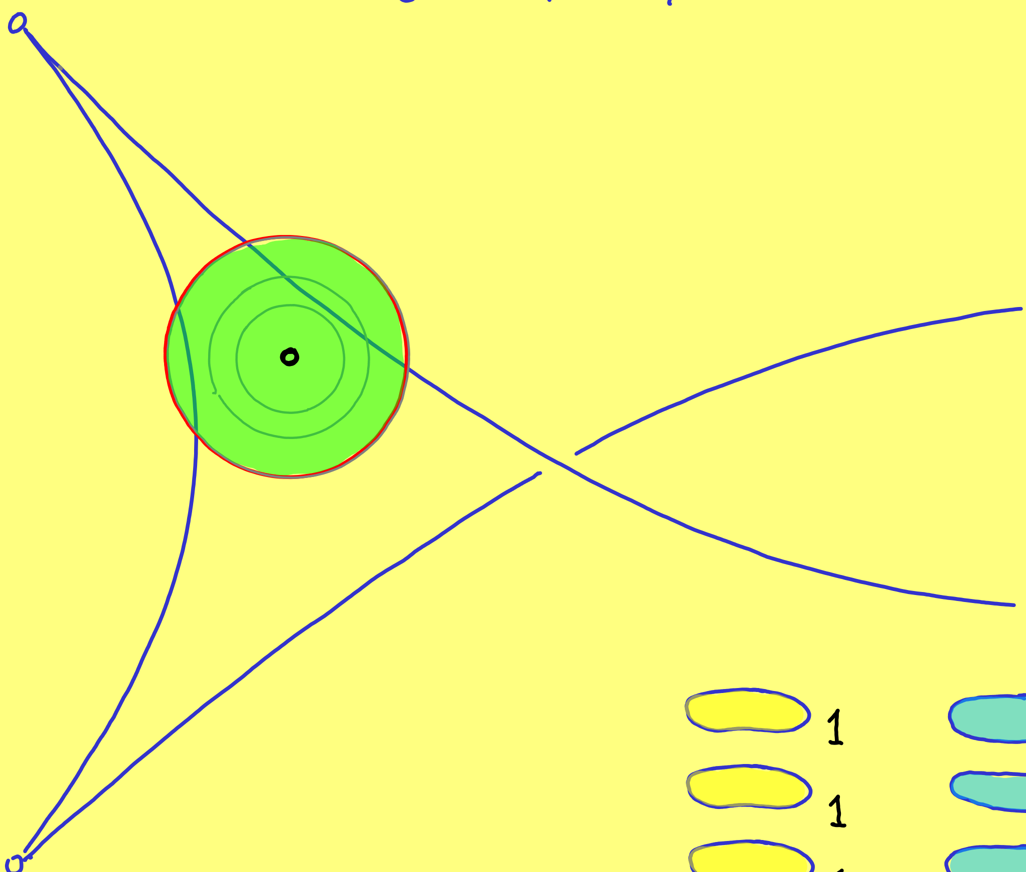
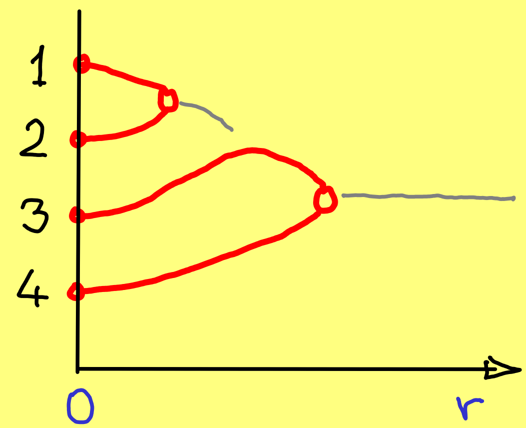
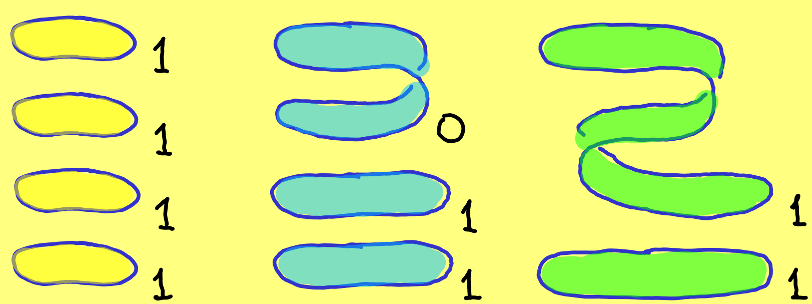
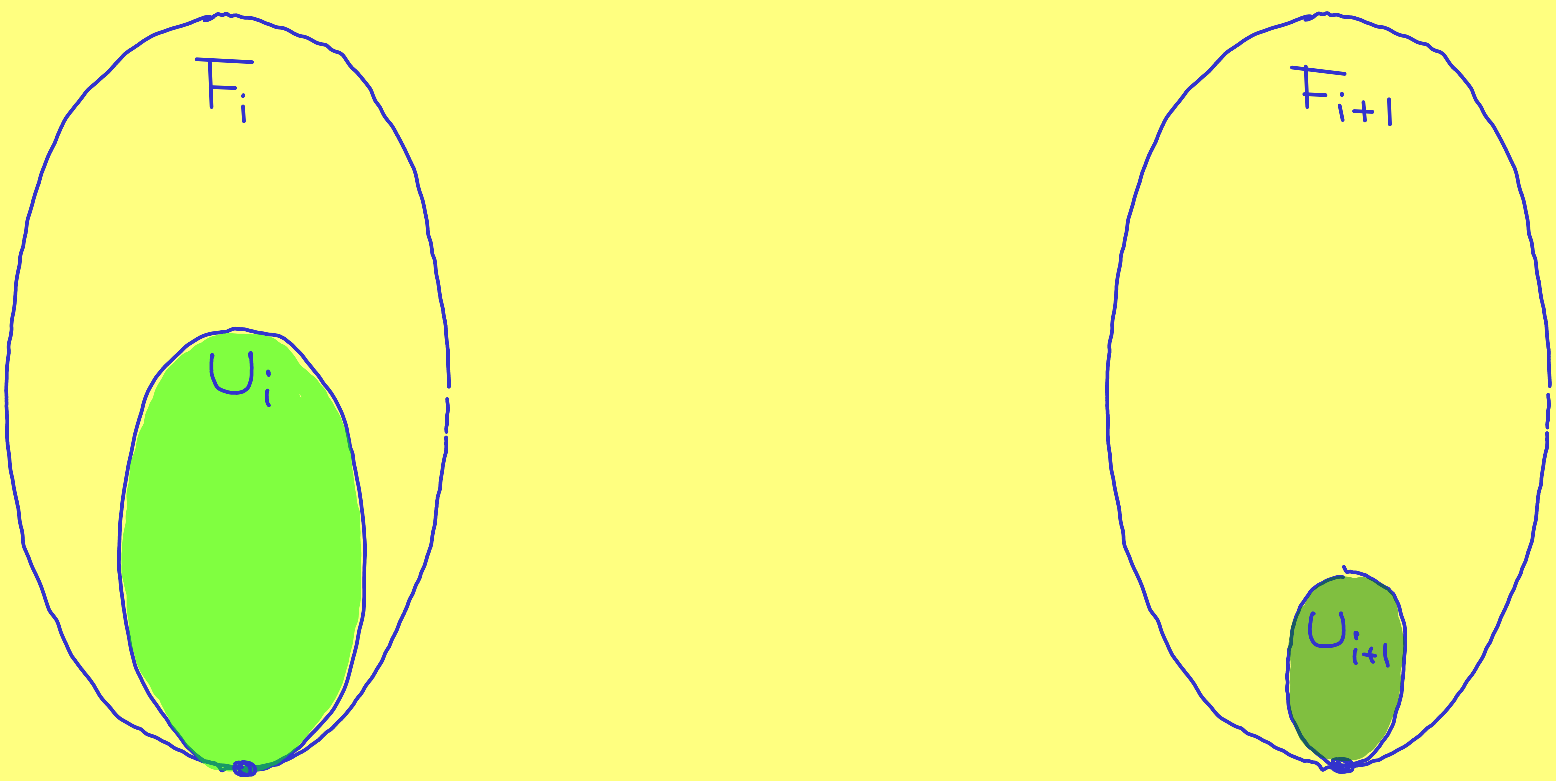






III.5 WELL GROUP

$U_i \subseteq F_i$ well subgroup of $H_0(\mathbb{M}_r(a))$



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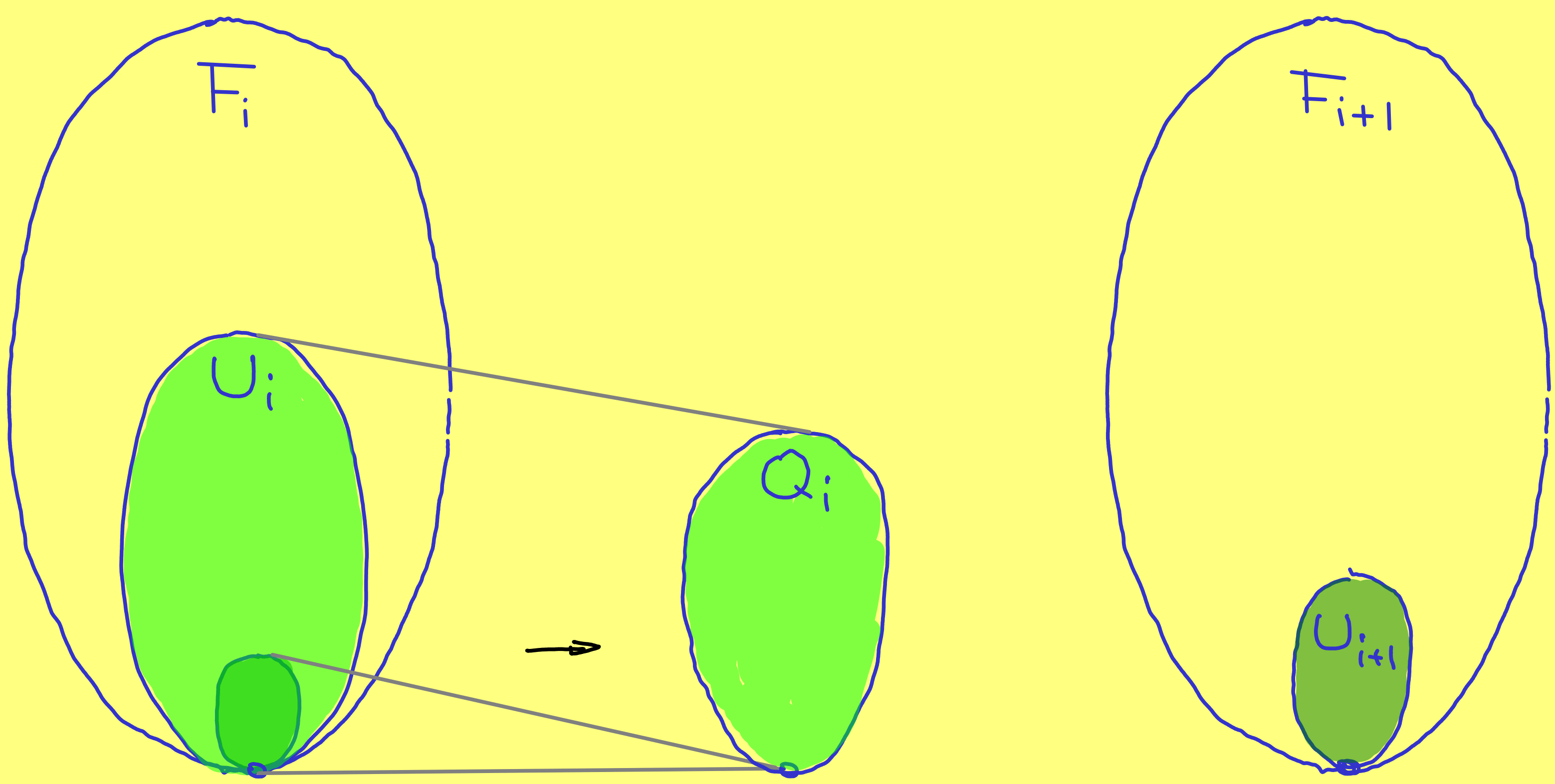

conventional death

III.5 Well Group

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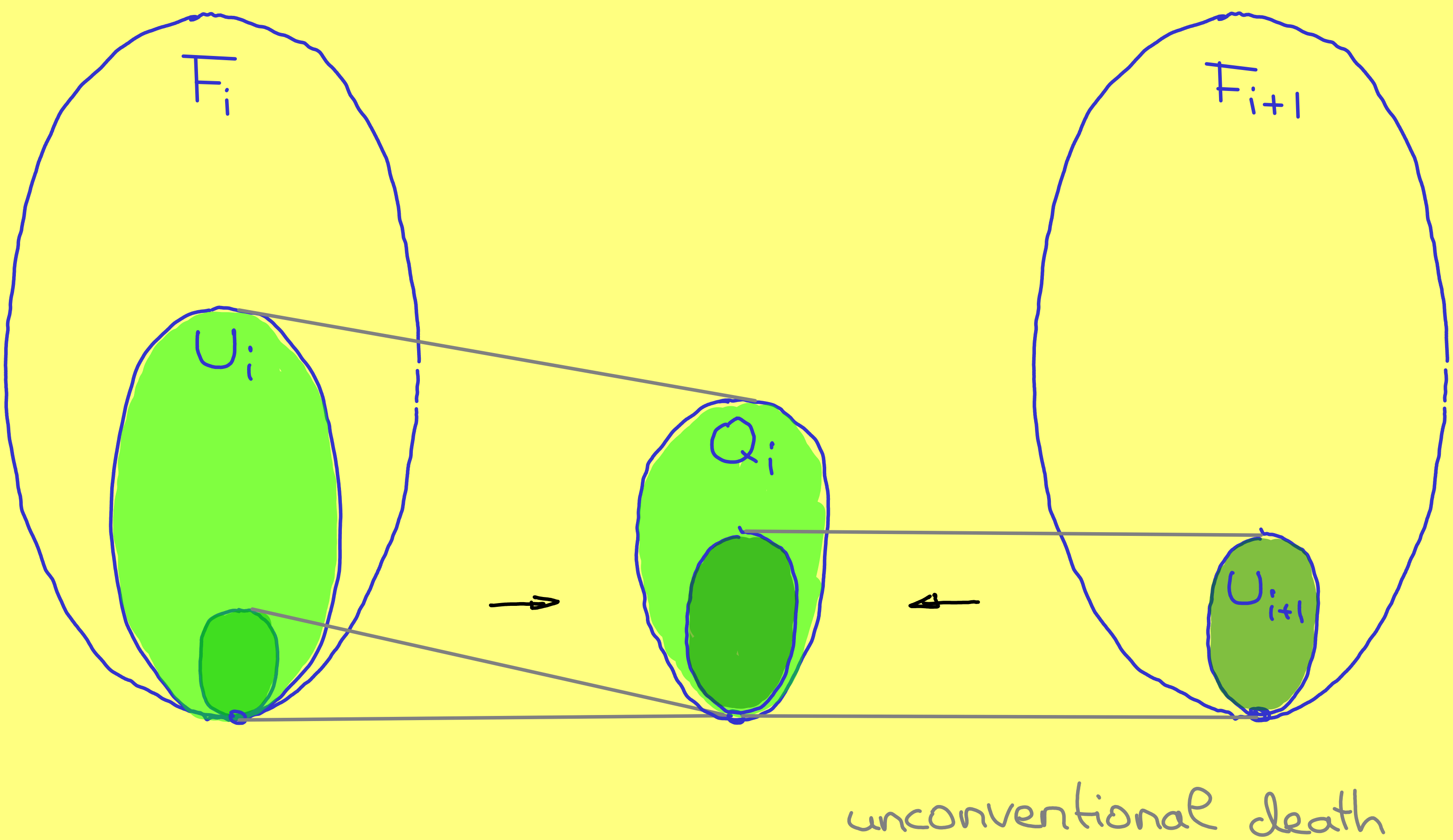
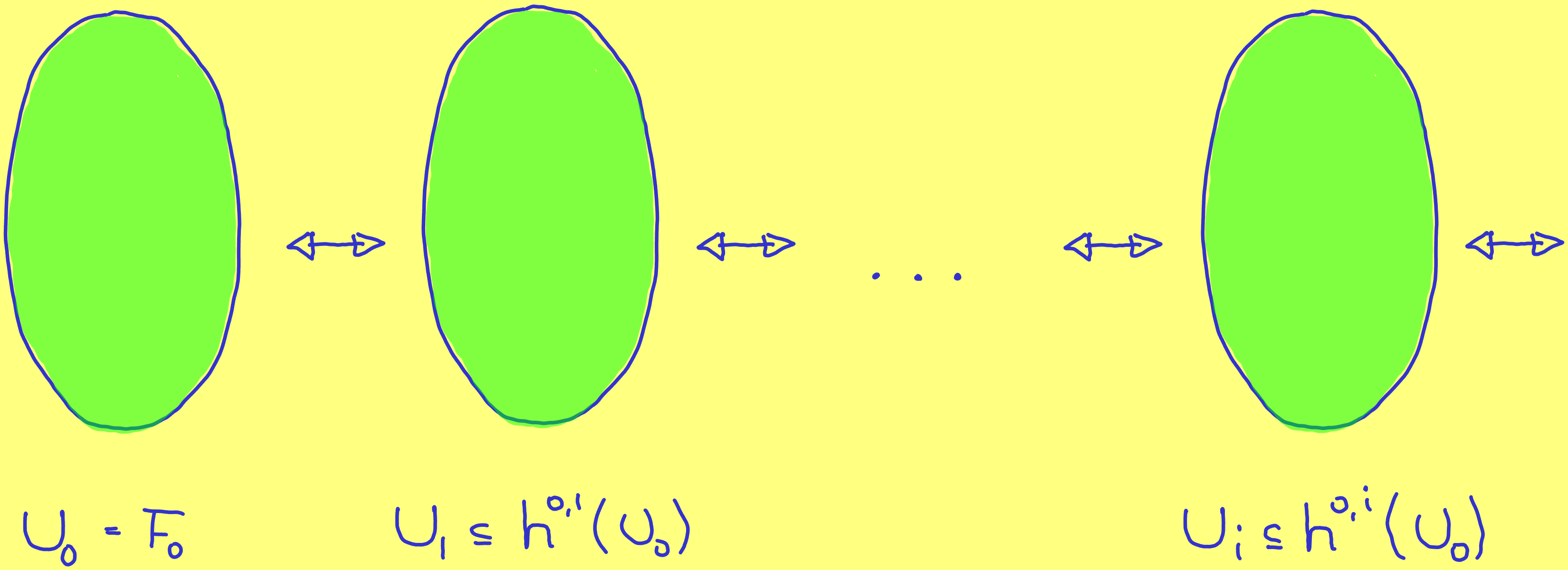
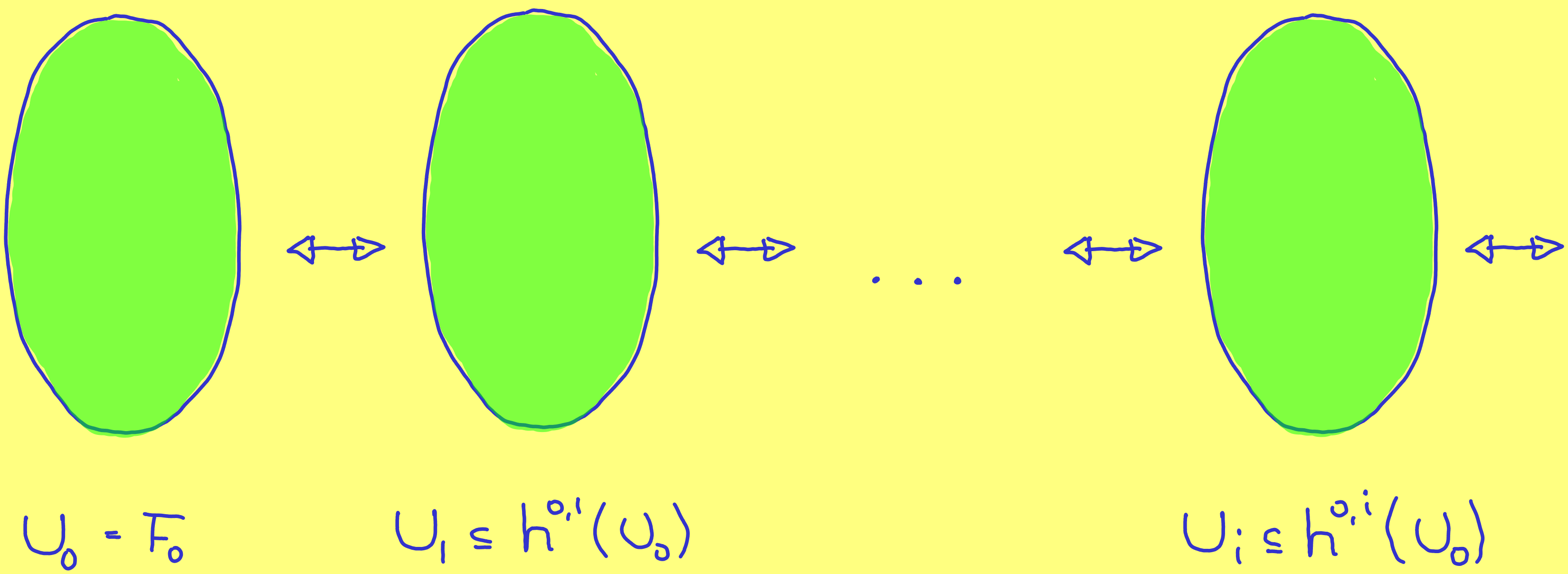


III.6 Zigzag Module

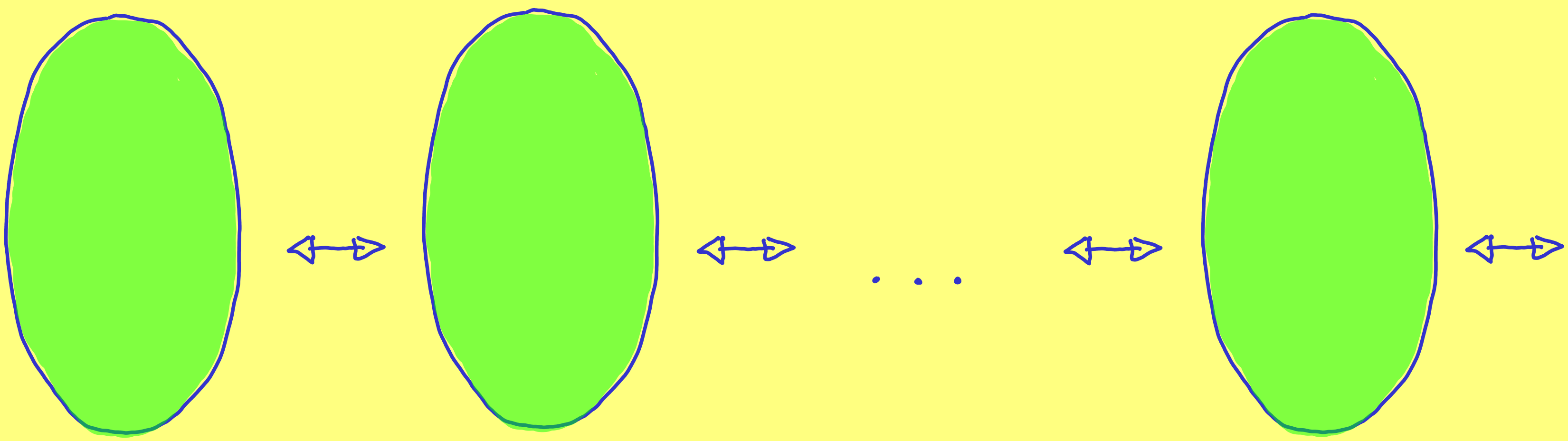


III.6 Zigzag Module



persistence diagram exists

III.6 Zigzag Module



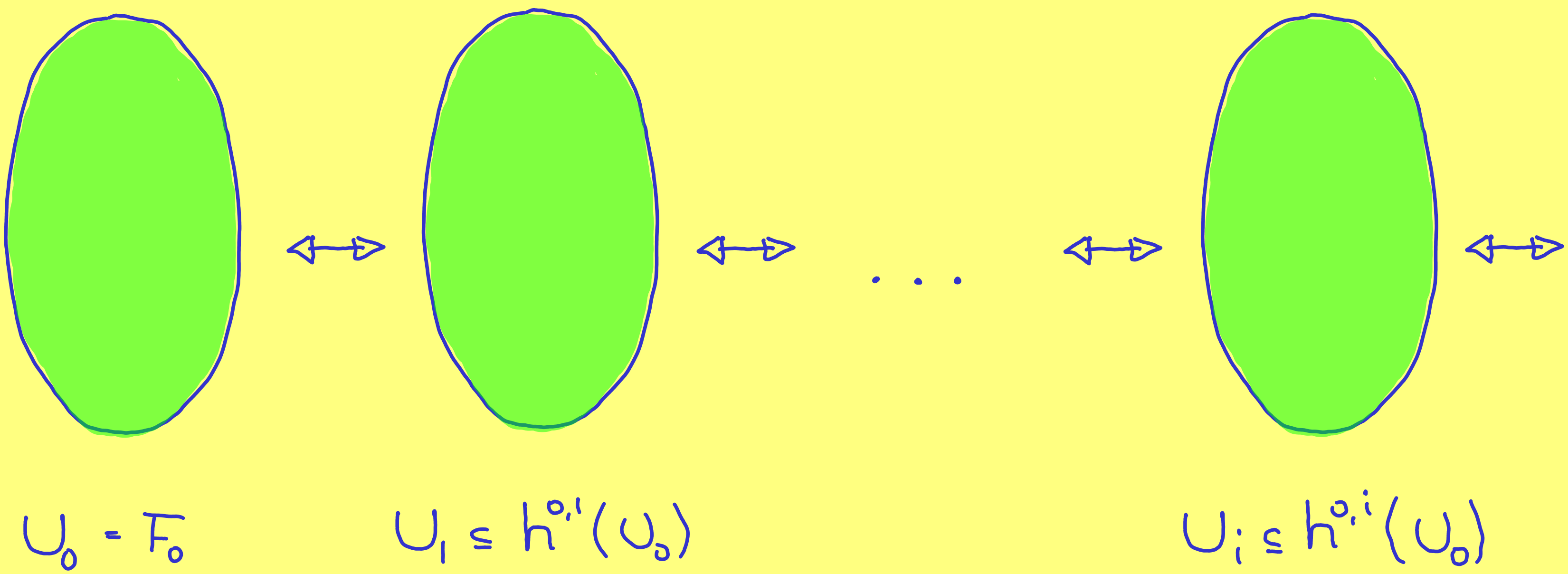
$U_0 = F_0$ $\quad$ $U_1 \subseteq h^{0,1}(U_0)$ $\quad$ $U_i \subseteq h^{0,i}(U_0)$

persistence diagram

exists

one-dimensional

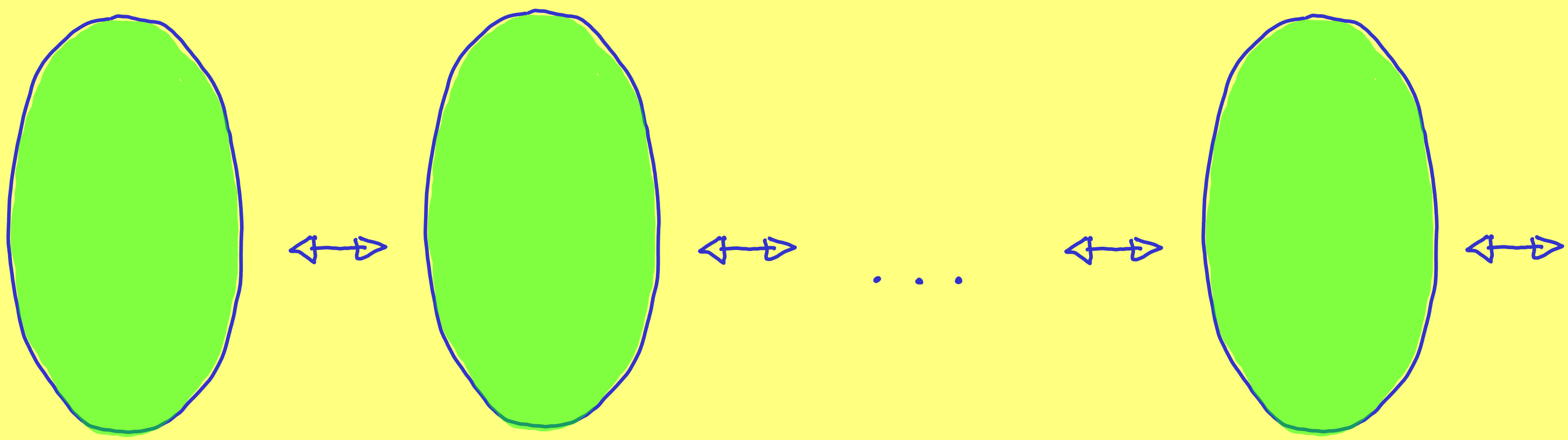
III.6 Zigzag Module



persistence diagram

exists
one-dimensional
stable

III.6 Zigzag Module



$U_0 = F_0$ $\qquad$ $U_1 \subseteq h^{0,1}(U_0)$ $\qquad$ $U_i \subseteq h^{0,i}(U_0)$

persistence diagram

exists
one-dimensional
stable $\implies$ contour stable

THANK YOU