The ant in the labyrinth: random walks and percolation

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Overview

(1) Percolation

(2) Random walks

(3) Divergence form PDEs

(4) PDE methods for random walks

(5) Critical percolation

(6) Video (?)

Percolation

This was introduced by Broadbent and Hammersley (1957).

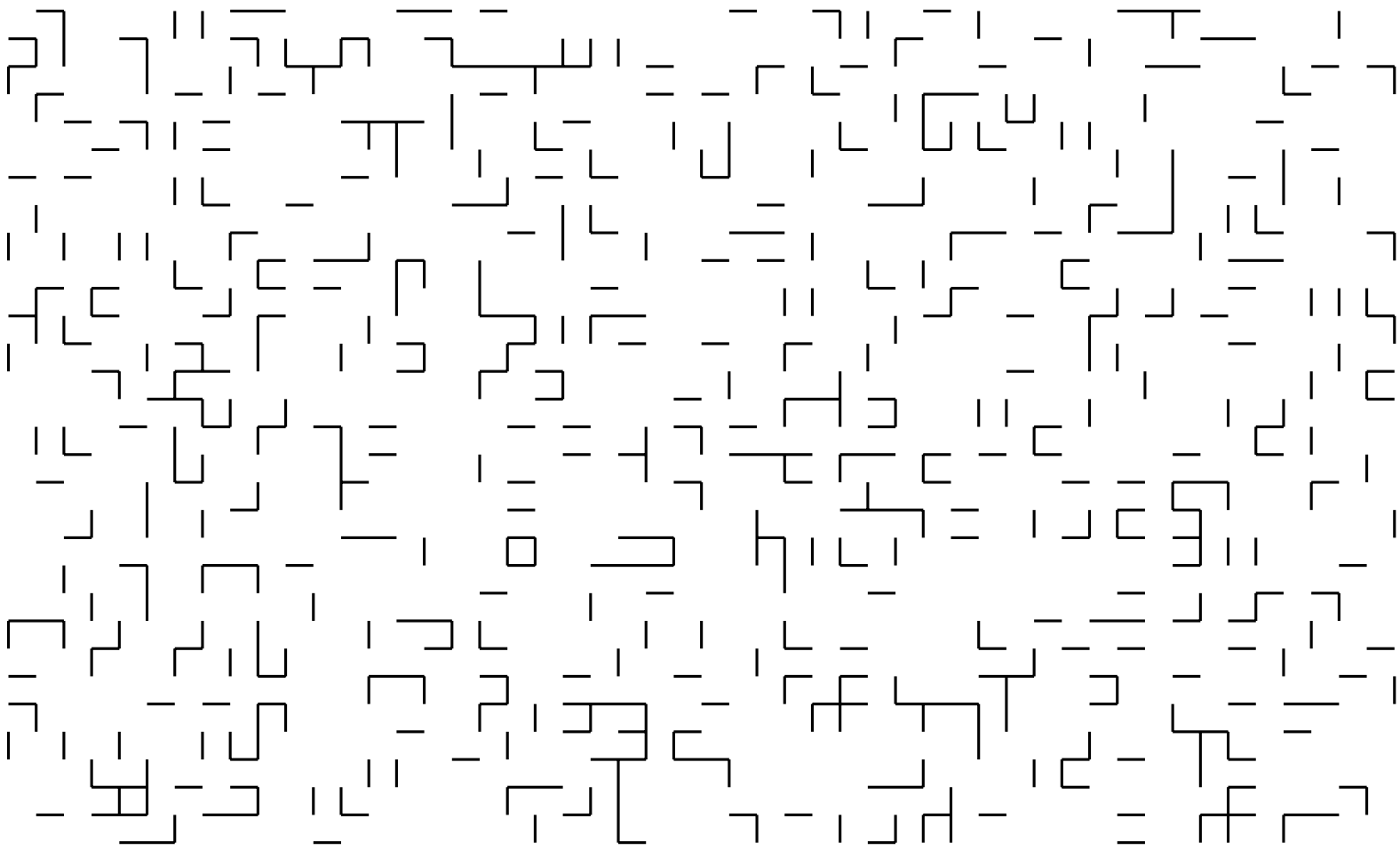
Consider the Euclidean lattice $\mathbb{Z}^d$, with edges (bonds) E_d.

Fix $p \in [0,1]$. For each edge $e = \{x, y\}$ keep the edge with probability p, delete it with probability $1 - p$, independently of all the others.
Let $\mathcal{O}$ be the set of edges which are kept, which are called *open edges*.
The connected components of the graph $(\mathbb{Z}^d, \mathcal{O})$ are called *(open) clusters*.

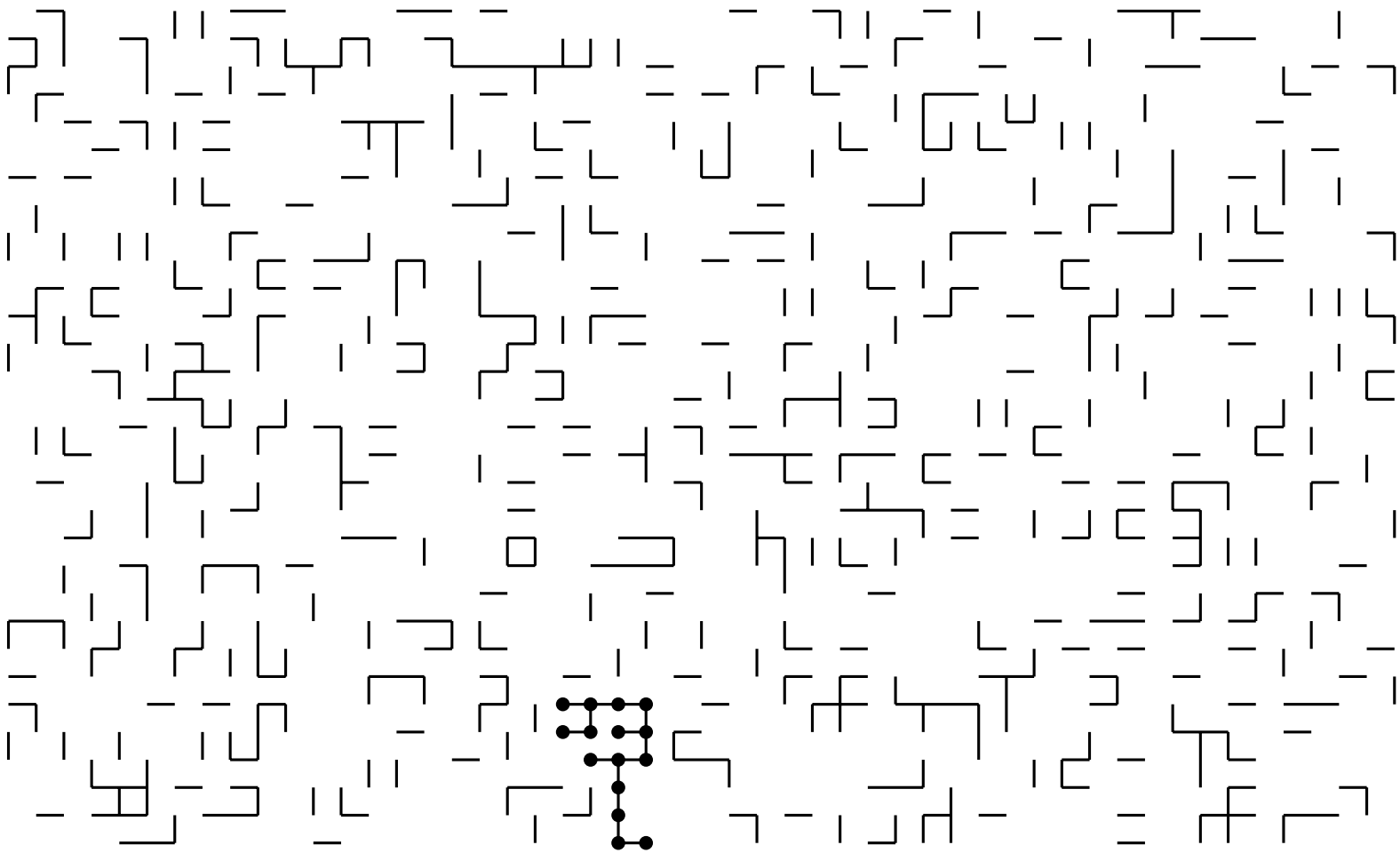
We are particularly interested in the infinite clusters.
There exists $p_c = p_c(d) \in (0,1)$ such that, with probability 1 (wp1):

> if $p < p_c$ all clusters are finite (subcritical regime),
>
> if $p > p_c$ then there exists a unique infinite cluster, C_∞ (supercritical regime),

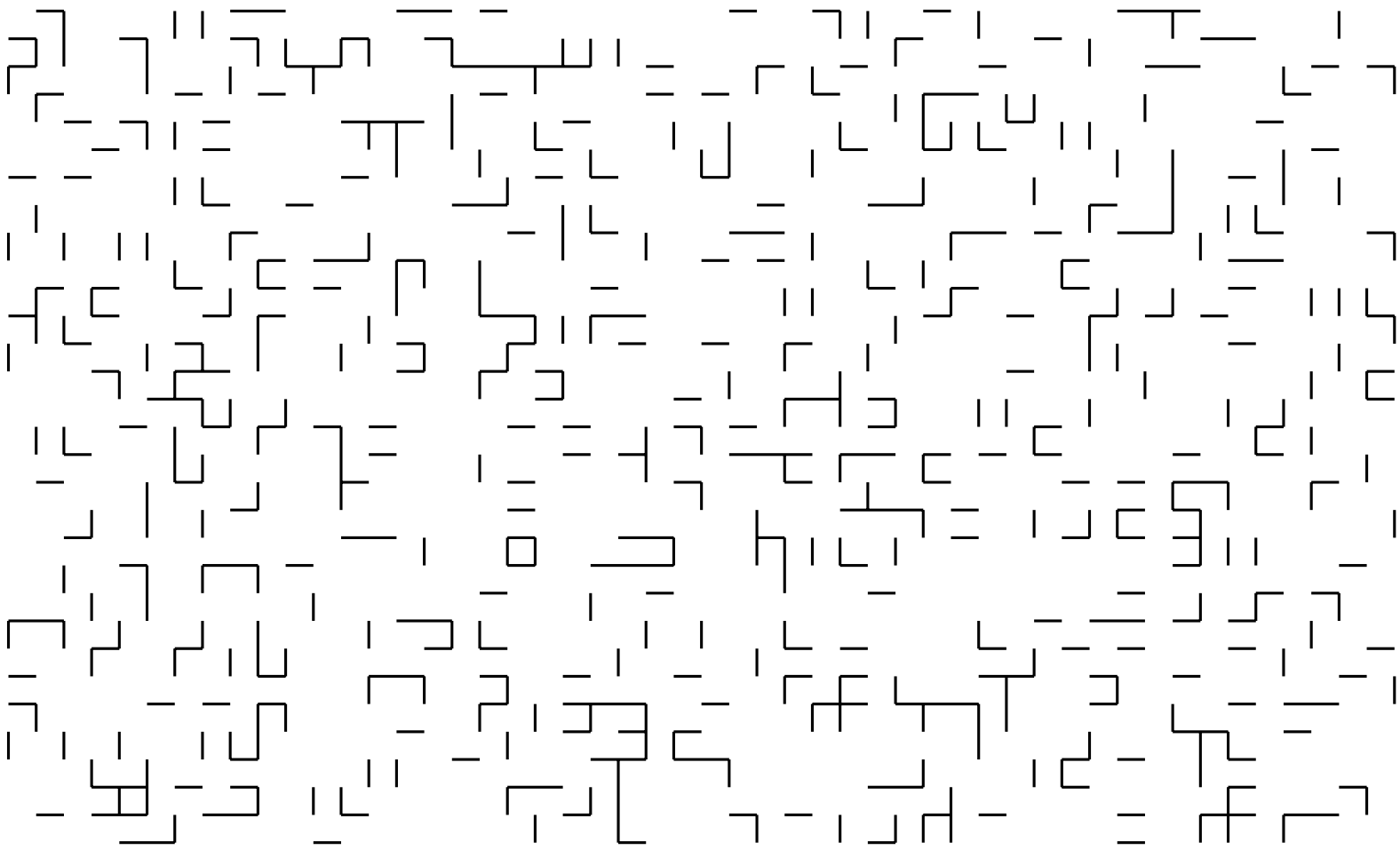
If $p = p_c$ (critical regime) it is conjectured that all clusters are finite, but only proved in some cases ($d = 2$, $d \geq 19$).



$p = 0.2$



$p = 0.2$, largest cluster marked



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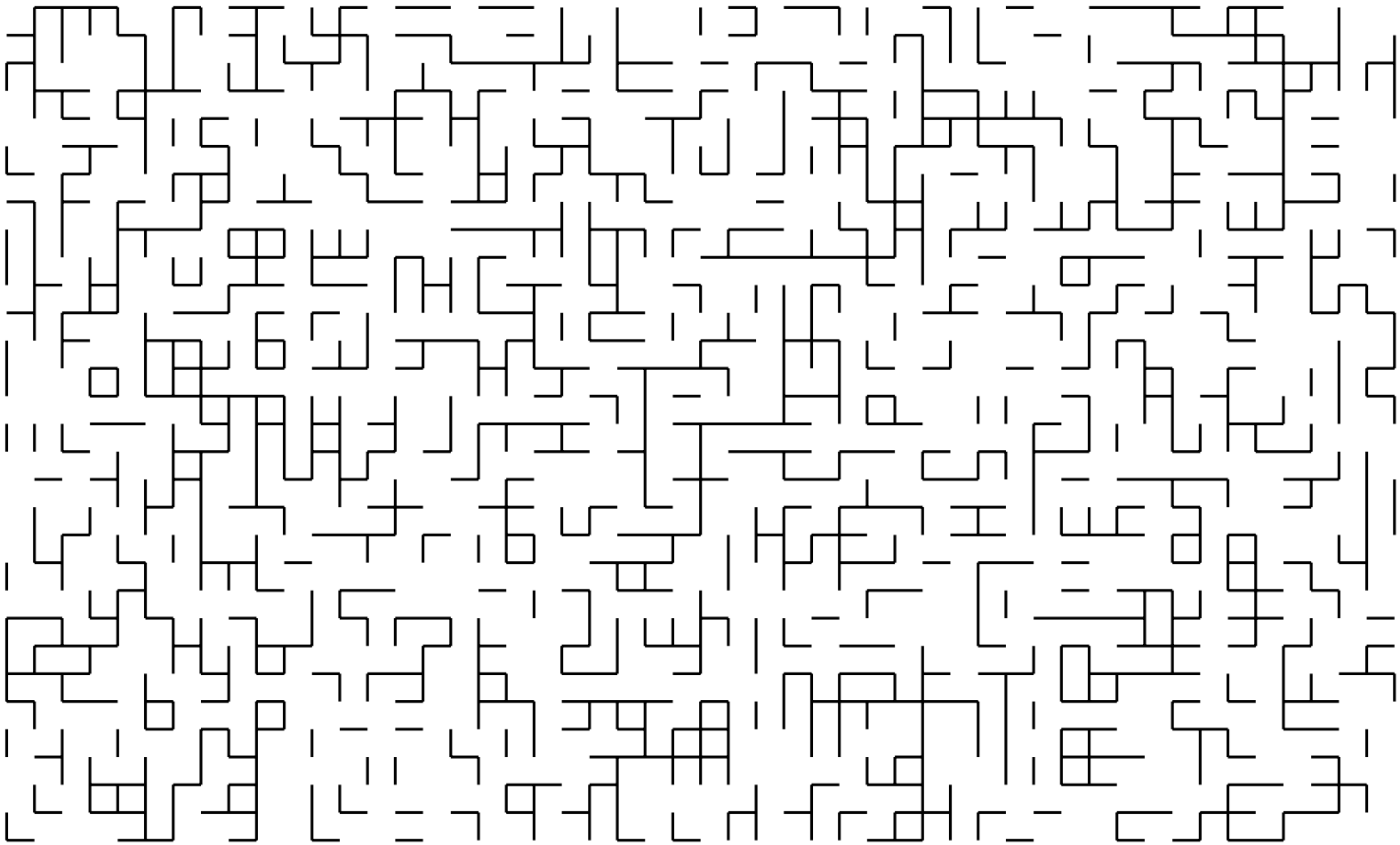

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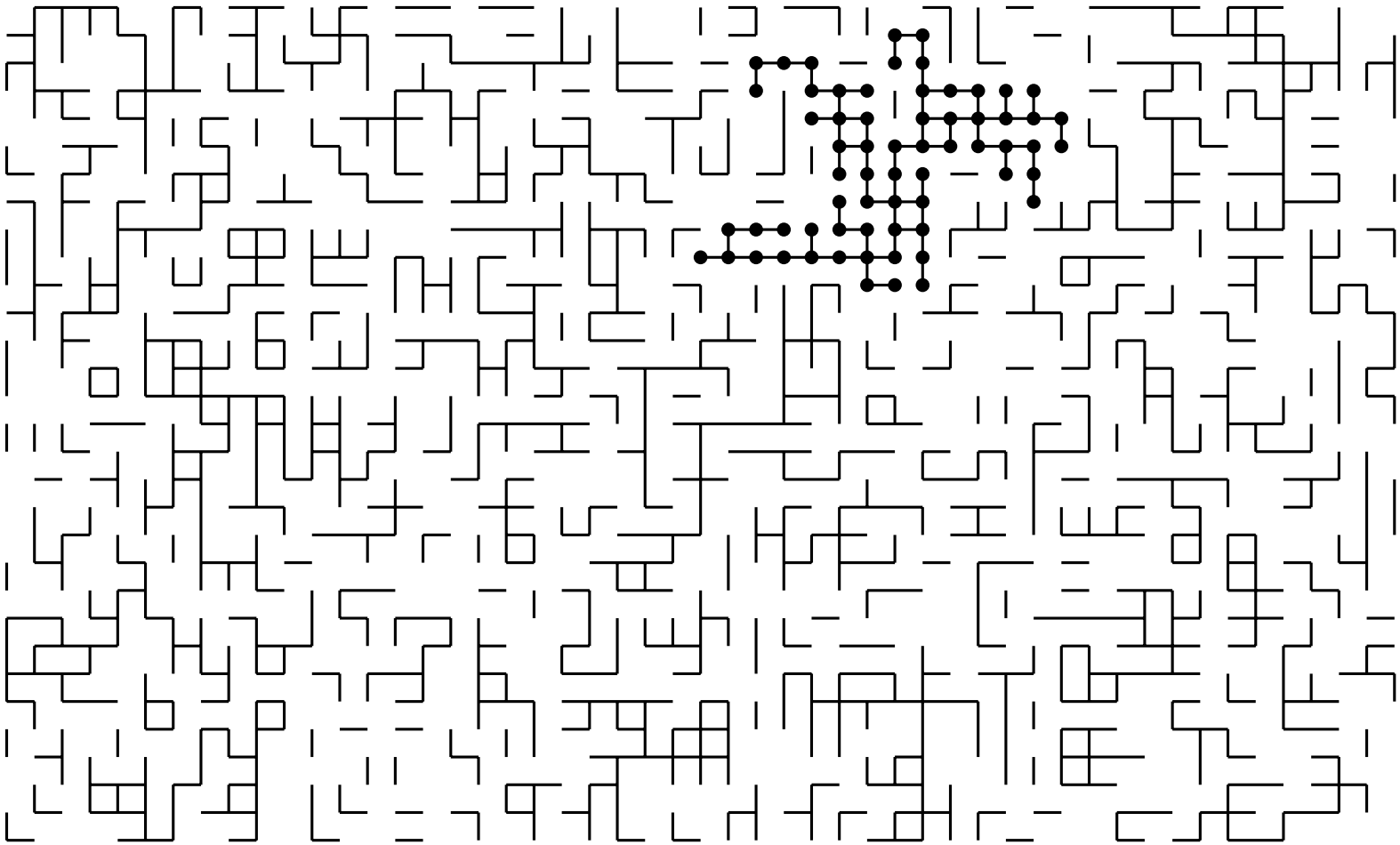

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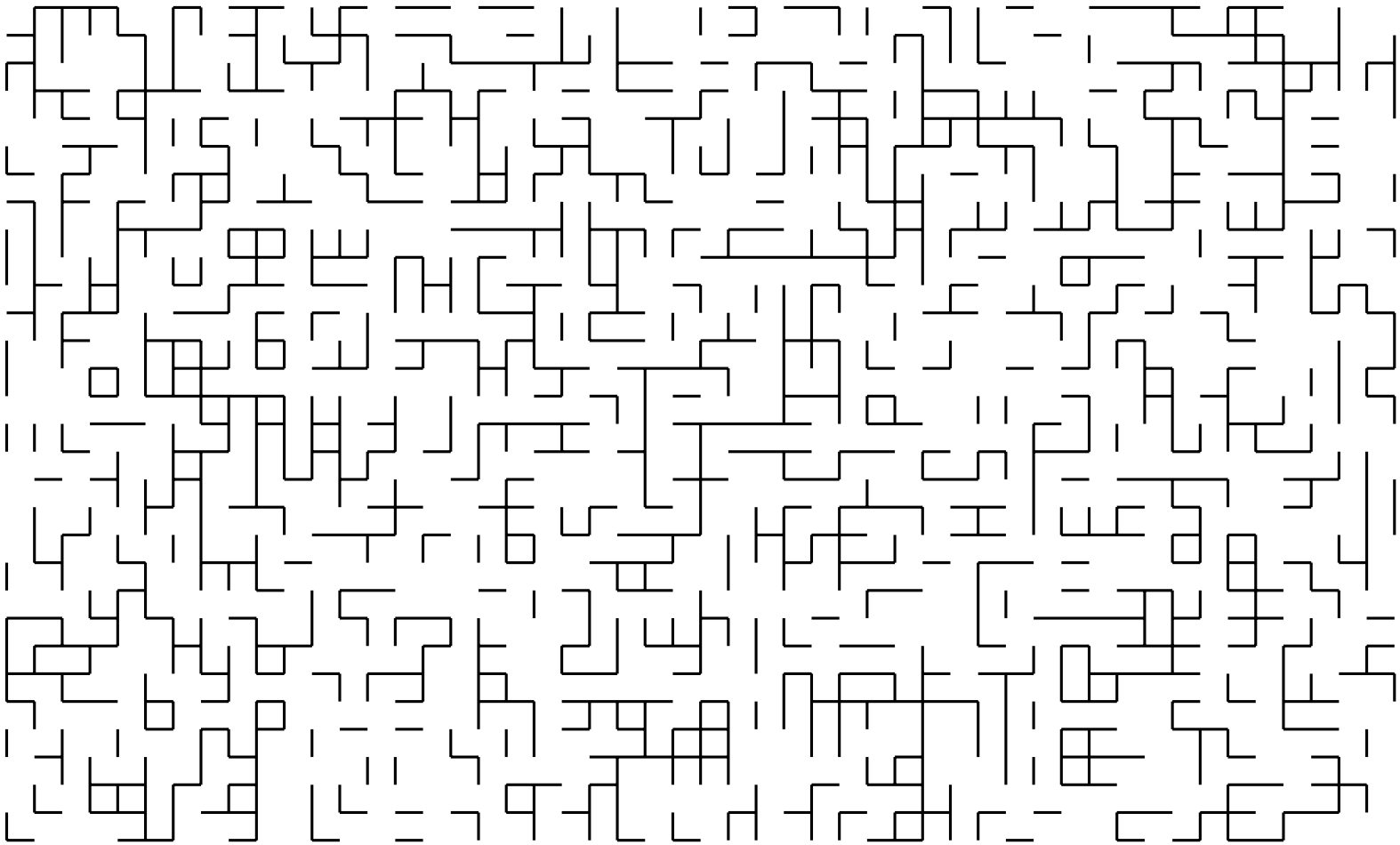

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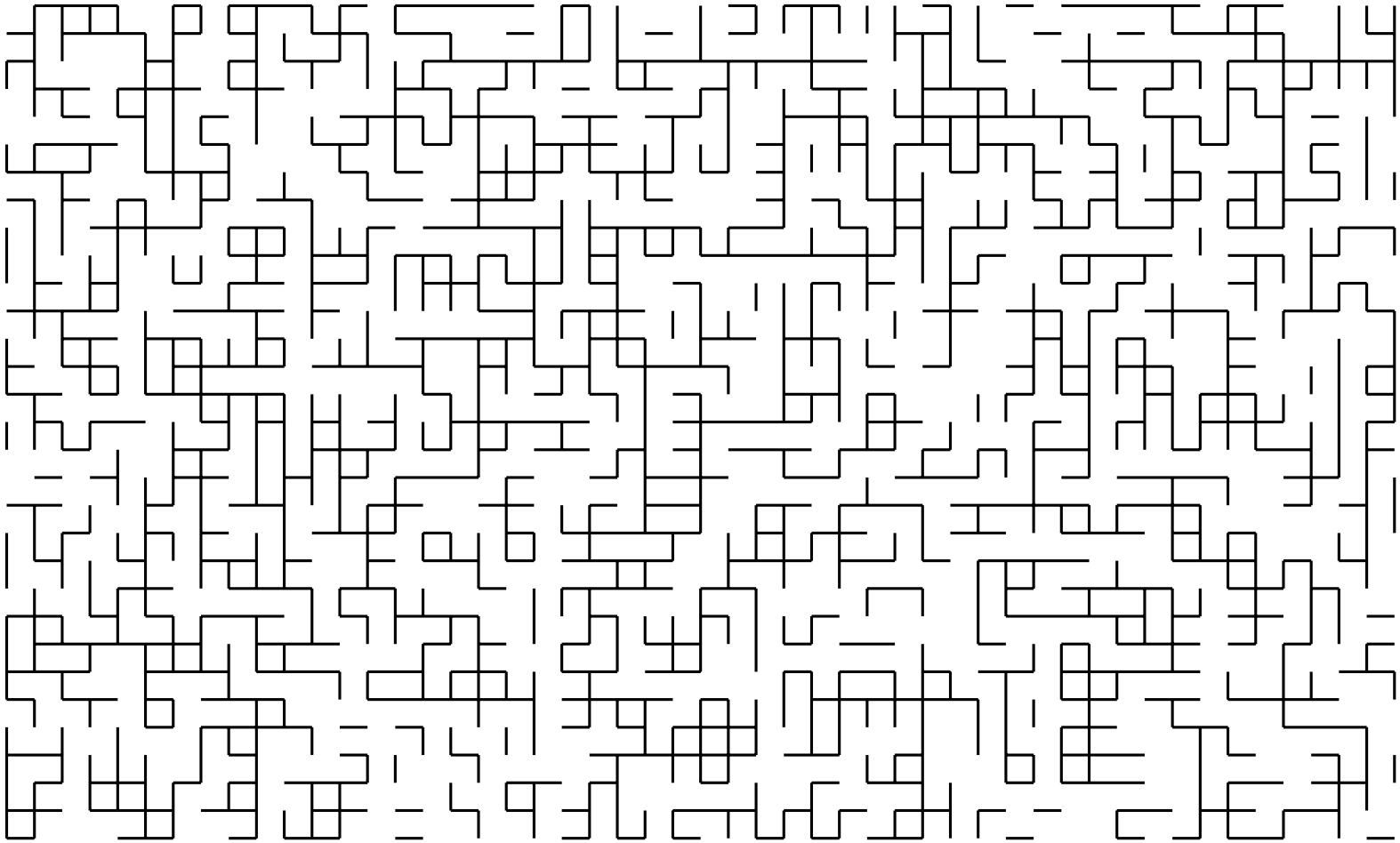
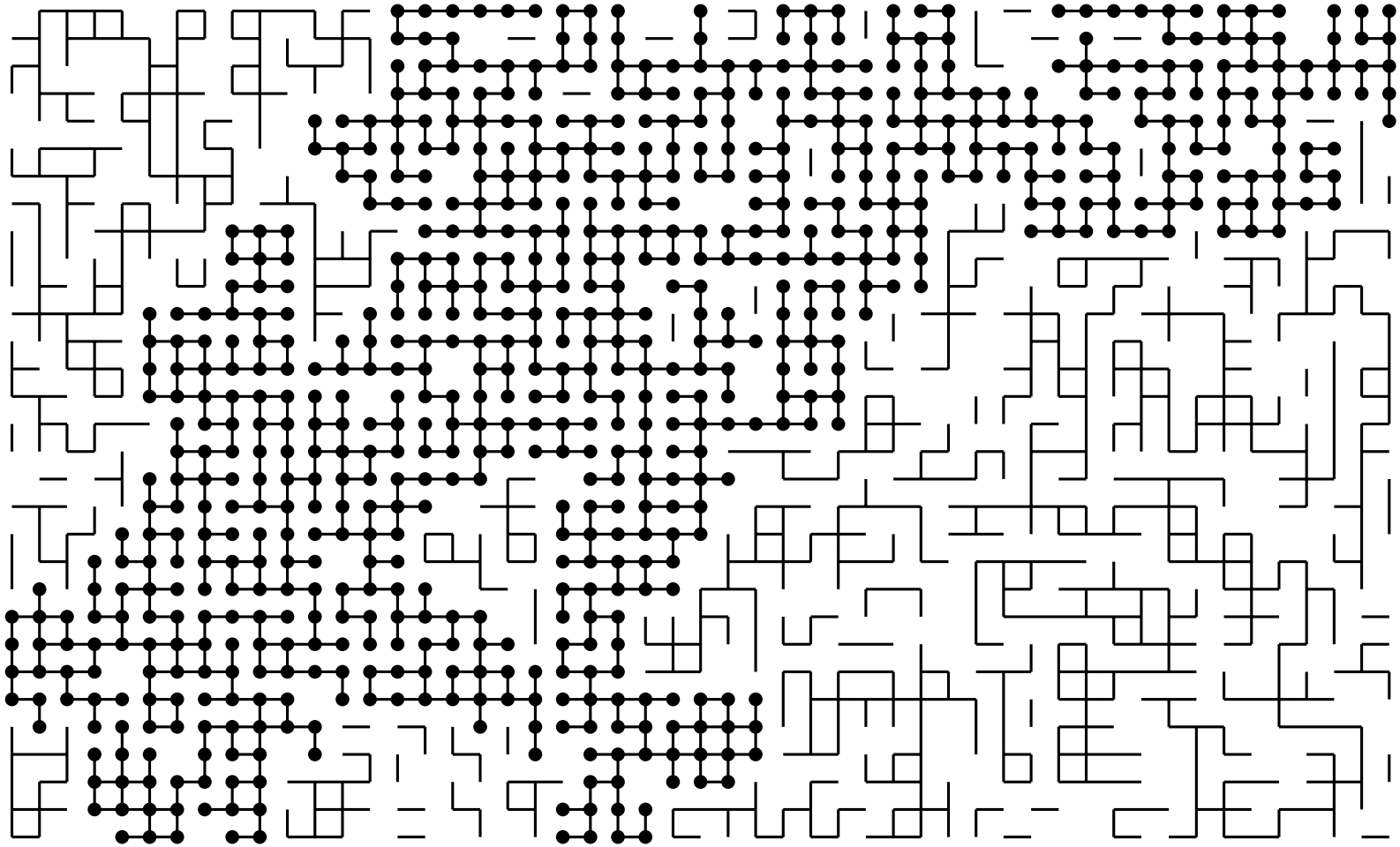


$p = 0.5 (= p_c)$



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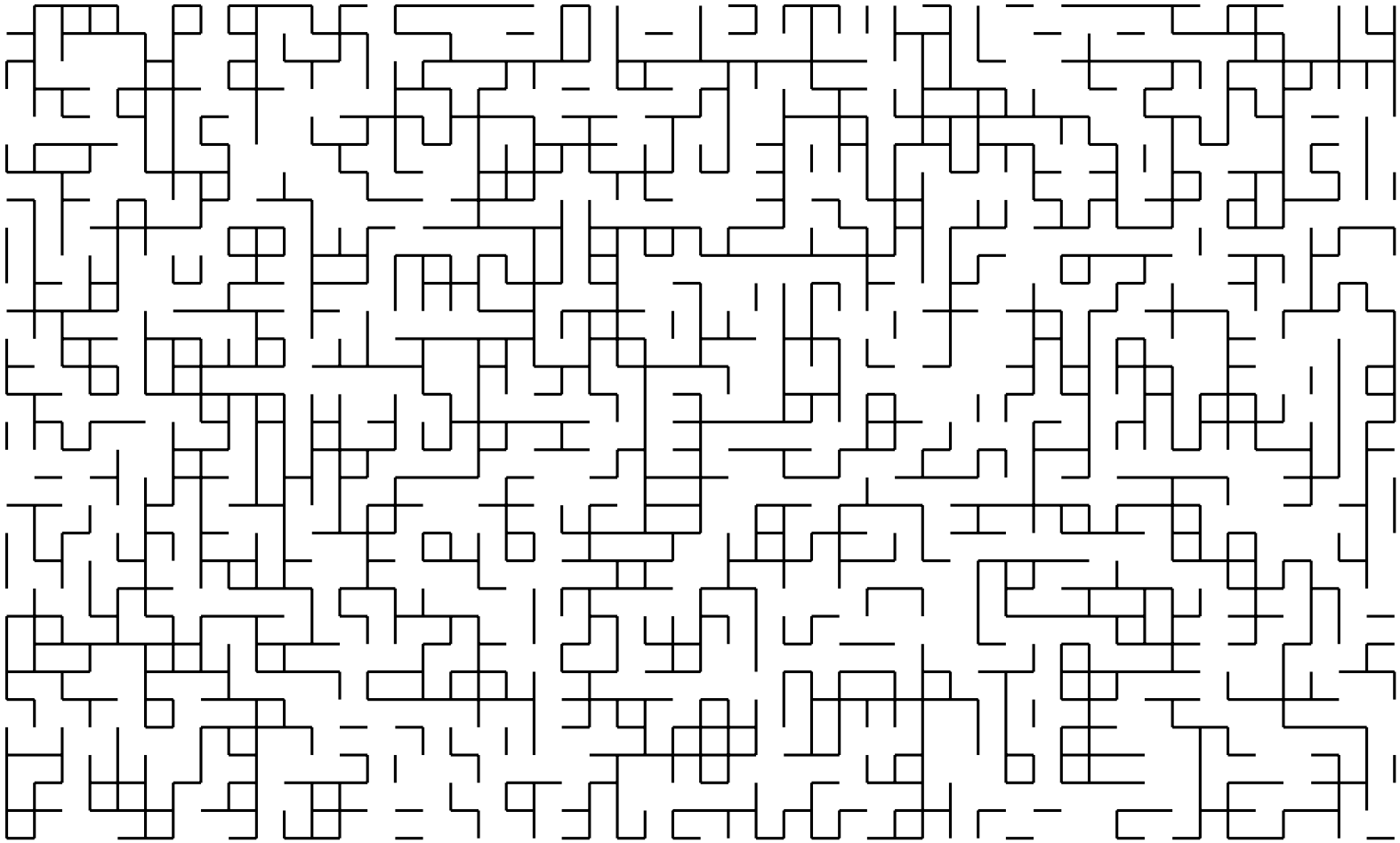

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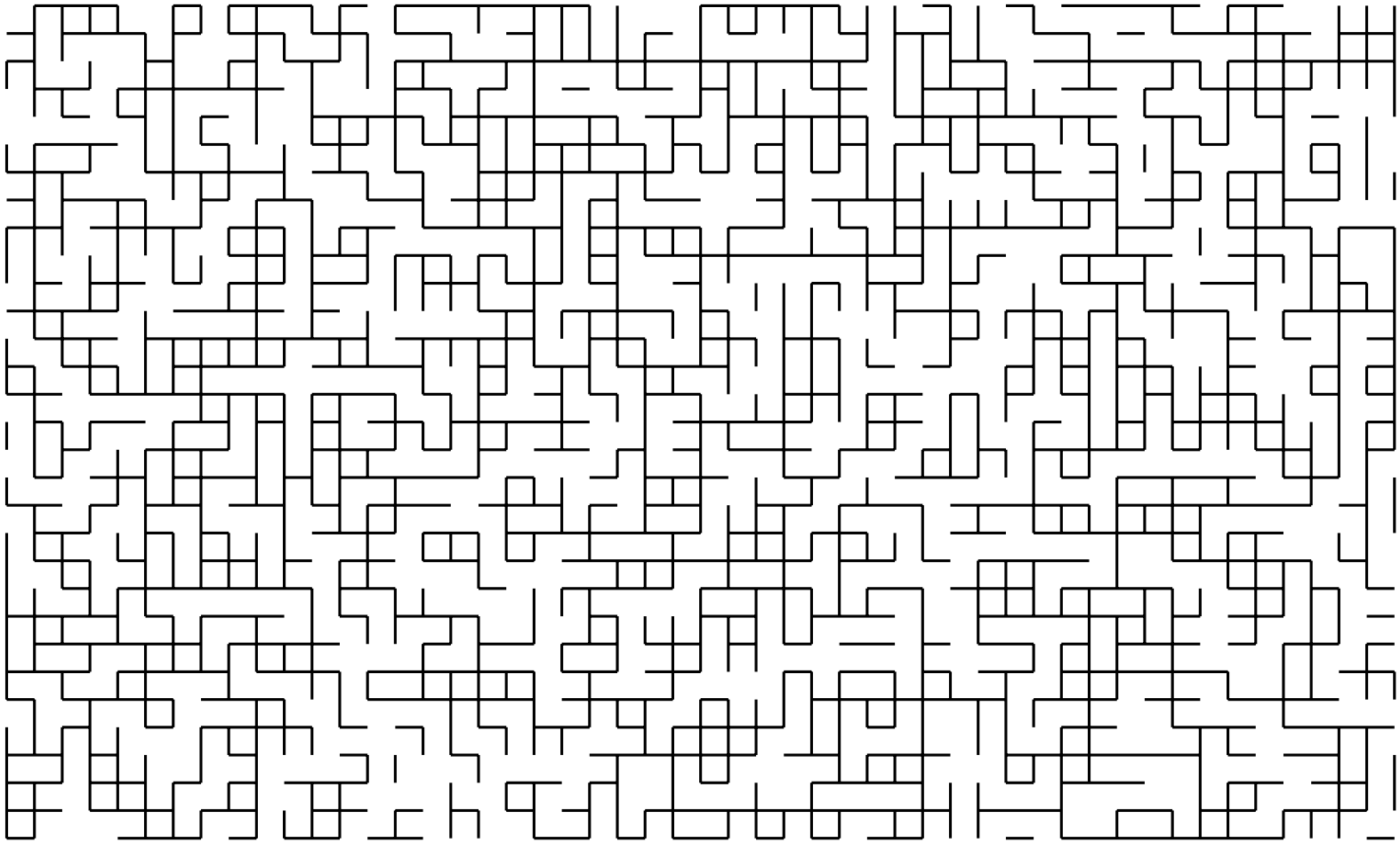

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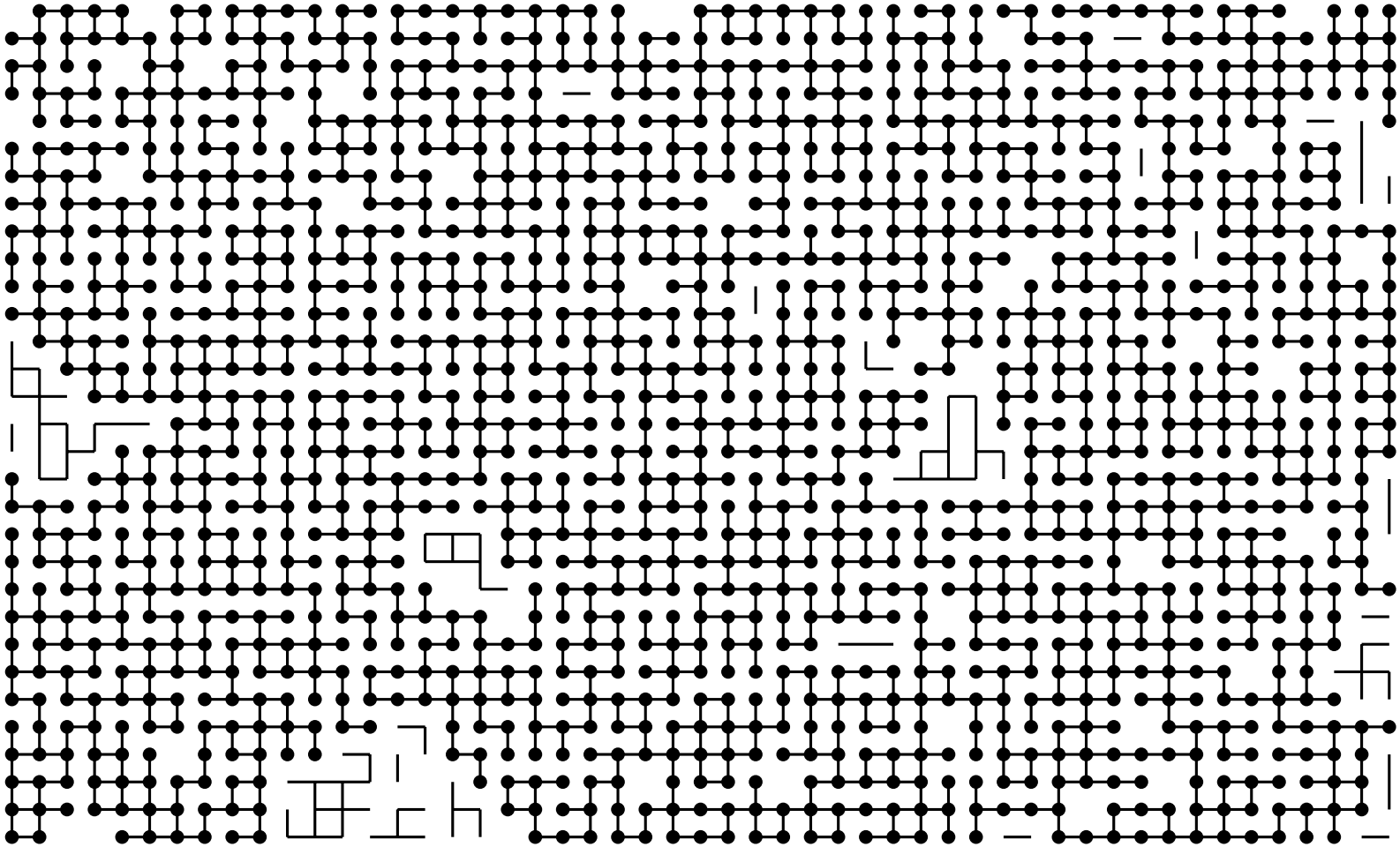

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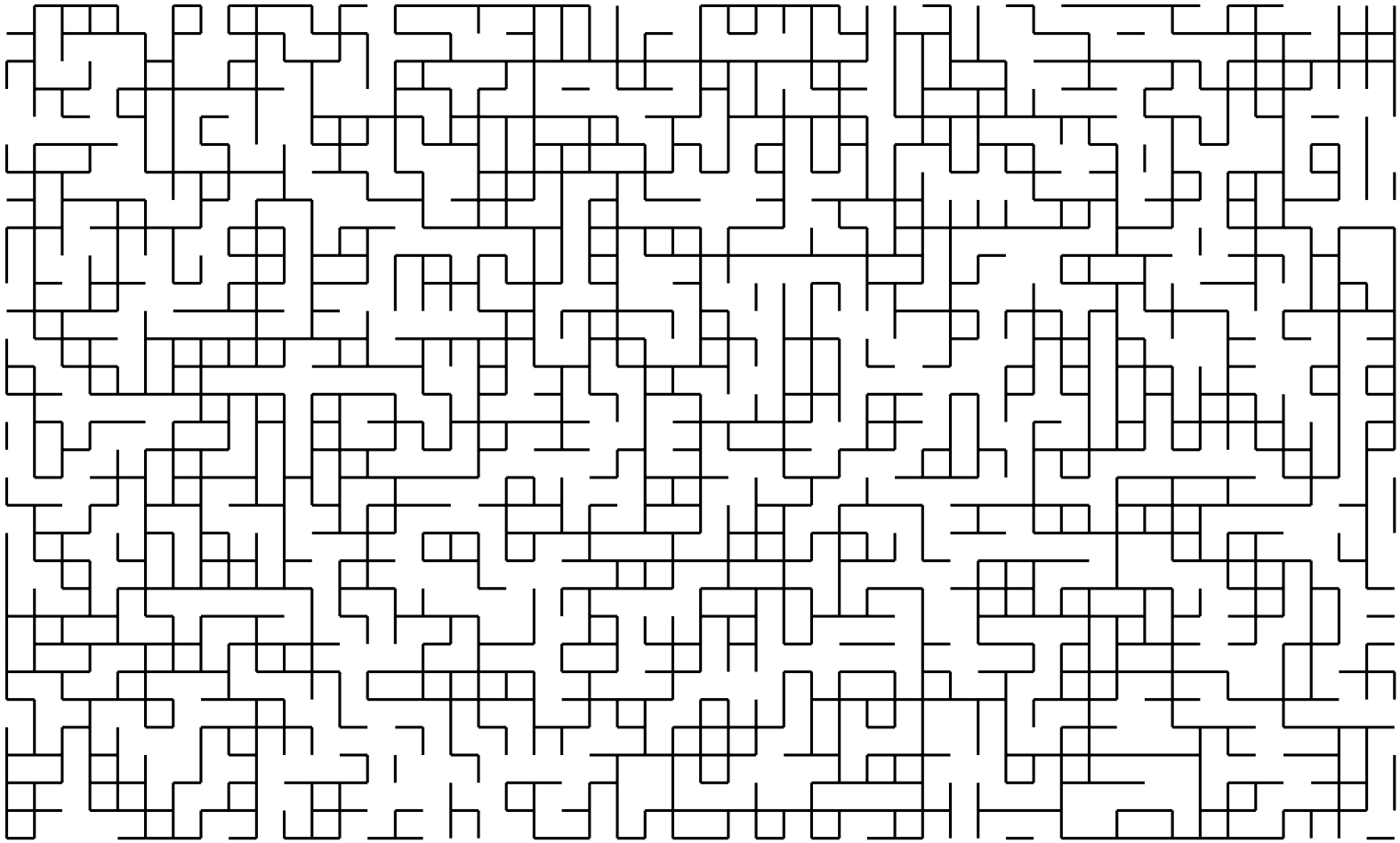

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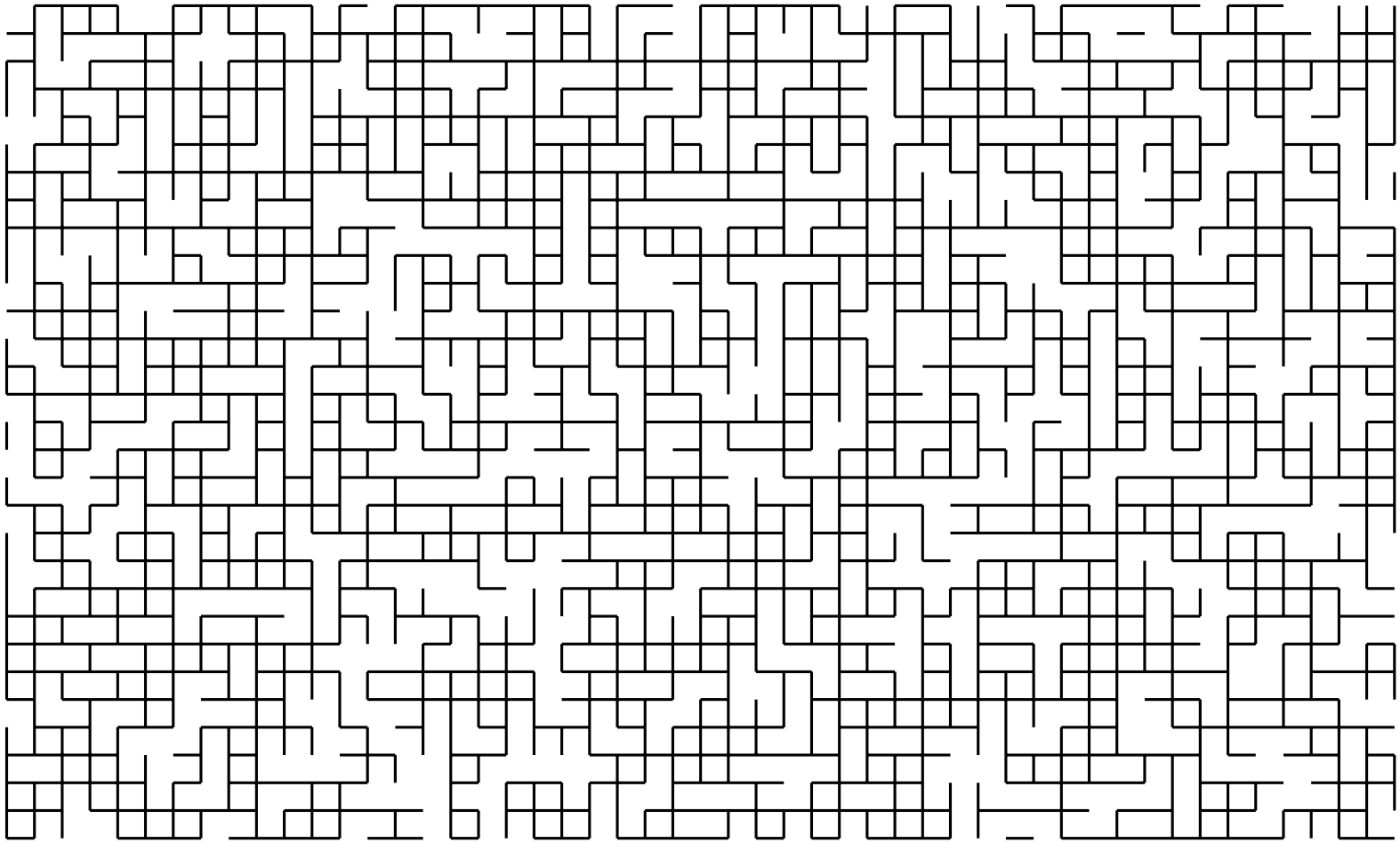

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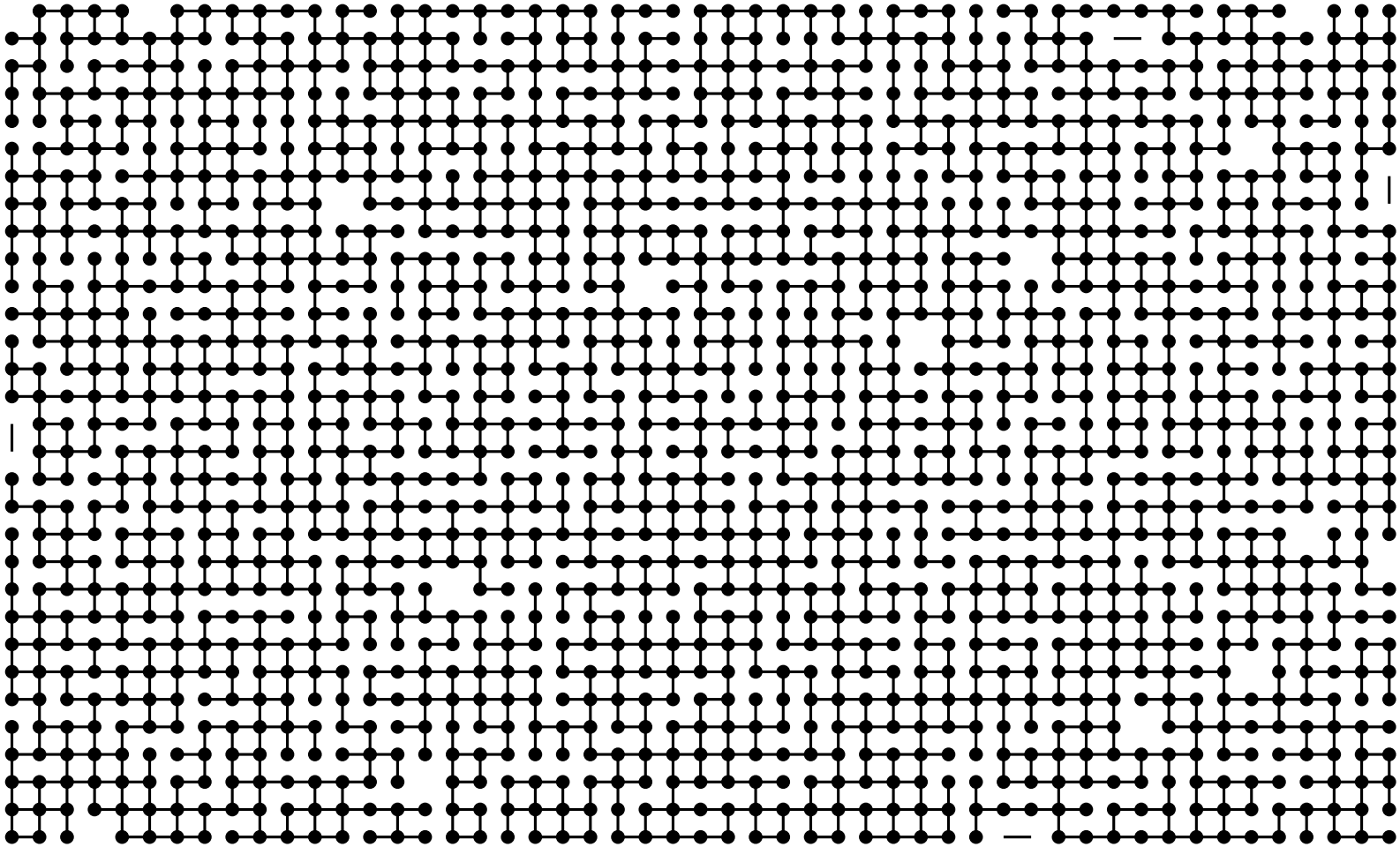

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Mathematical Implementation

Let $\Omega = \{0,1\}^{E_d}$; $\omega \in \Omega$ is a *configuration*. If $\omega \in \Omega$,

$$\omega(e) = 1 \quad \Leftrightarrow \quad e \text{ is open}$$
$$\omega(e) = 0 \quad \Leftrightarrow \quad e \text{ is not open}$$

So $\mathcal{O}(\omega) = \{e : \omega(e) = 1\}$.
Let $\mathbb{P}_p$ be the probability measure on Ω which makes the edges independent with

$$\mathbb{P}_p(\omega(e) = 1) = p \text{ for all } e \in E_d.$$

The statement "if $p < p_c$ all clusters are finite" means that if

$$A = \{\omega : \text{ all connected components of } (\mathbb{Z}^d, \mathcal{O}(\omega)) \text{ are finite}\},$$

then $\mathbb{P}_p(A) = 1$ if $p < p_c$.

Random walks on graphs

Let $G = (V, E)$ be a graph: V is the set of vertices (finite or countable) and E the edges (unoriented). For $x \in V$ let μ_x be the number of edges containing x:

$$\mu_x = \mu(\{x\}) = \#\{y : \{x, y\} \in E\} = \#\{y : y \sim x\}.$$

Assume G is *locally finite*, i.e. $\mu_x < \infty$ for all x.
The simple random walk (SRW) on G is the random walk on the vertex set V which moves 'at random' from $x \in V$ along an edge e to a neighbour y of x. So

$$P(X^o_{n+1} = y | X^o_n = x) = \frac{1}{\mu_x} \quad \text{if } y \sim x. \tag{1}$$

On a bipartite graph, such as $\mathbb{Z}^d$, X^o can only return to x after an even number of steps. From now on I will consider the *lazy walk* X, which stays where it is with probability $\frac{1}{2}$ and jumps (according to (1)) with probability $\frac{1}{2}$.

Laplacian and heat equation on a graph G

Discrete Laplacian on G:

$$\Delta f(x) = \frac{1}{\mu_x} \sum_{y \sim x} (f(y) - f(x)).$$

Let $p_n(x, y)$ be the transition density (w.r.t. μ) of X started with $X_0 = x$ so that

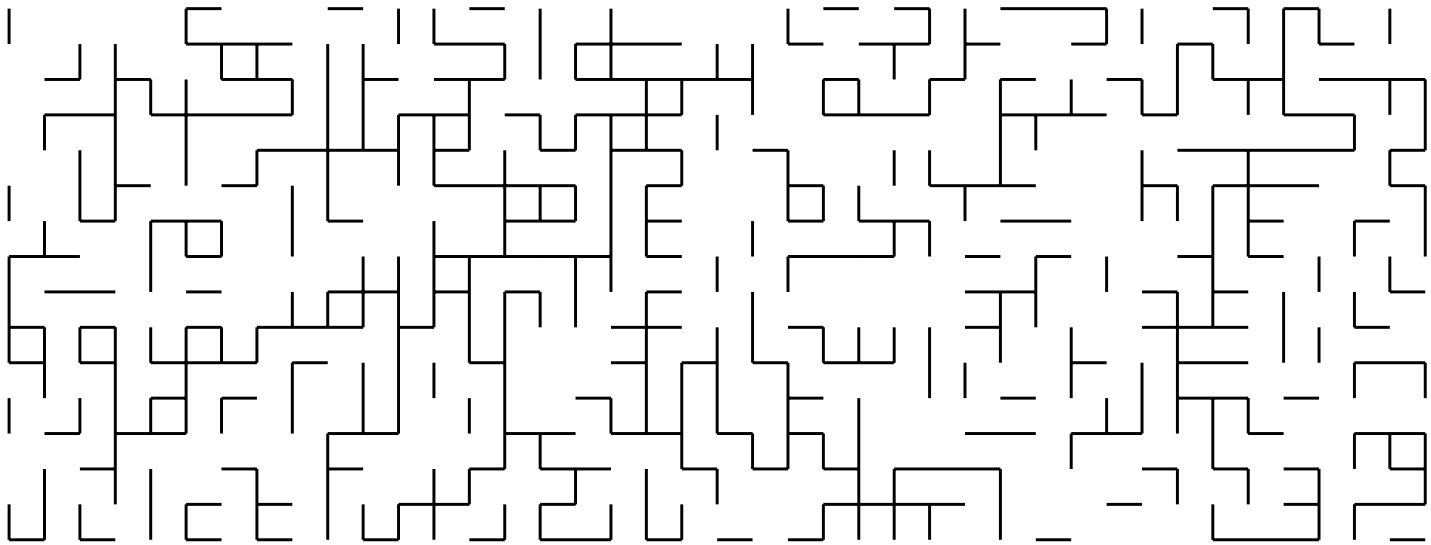
$$P^x(X_n = y) = P^x(X_n = y | X_0 = x) = p_n(x, y)\mu_y.$$

p_n is also the (discrete time, lazy) heat kernel on the graph G:

$$p_{n+1}(x, y) - p_n(x, y) = (\tfrac{1}{2}\Delta + \tfrac{1}{2})p_n(x, y),$$
$$p_0(x, y)\mu_y = \delta_x(y).$$

SRW on percolation clusters

For a percolation configuration ω look at the the (lazy) SRW X on the graph $\mathcal{G}(\omega) = (\mathbb{Z}^d, \mathcal{O}(\omega))$. We are interested in the long time behaviour of X and its heat kernel $p_n^\omega(x, y)$.
Note the probability that X leaves the connected component of $\mathcal{G}$ that it starts in is zero, so we will usually restrict X to the infinite cluster $\mathcal{C}_\infty$ (if it exists).
This SRW was called the 'ant in the labyrinth' by De Gennes 1976.



Three cases

1. Supercritical: $p > p_c$. There is (with probability one) a unique infinite cluster, denoted $\mathcal{C}_\infty$. We start X at a point in $\mathcal{C}_\infty$.
Questions:

1. How does the SRW X_n behave for large n? Are there significant differences from the behaviour of the SRW on $\mathbb{Z}^d$?
2. Behaviour of the heat kernel $p_n^\omega(x, y)$.

2. Subcritical: $p < p_c$. All clusters are finite; there seems to be little to say.

3. Critical: $p = p_c$ (or $p \to p_c$). In high dimensions substantial recent progress. In low dimensions we don't even know what the right conjectures are.

Early papers:

Kesten (1986) – work on the critical case, $d = 2$.
Grimmett, Kesten, Zhang, 1993: X is transient iff $d \geq 3$.
Benjamini, Lyons, Schramm, 1999: transitive graphs, Liouville property $+\ldots$

PDE

Heat equation for a divergence form operator in $\mathbb{R}^d$:

$$\frac{\partial u}{\partial t} = \mathcal{L}u = (\nabla a \nabla)u. \tag{2}$$

Here $u = u(x,t)$, $x \in \mathbb{R}^d$, $t \geq 0$, $u(x,0) = u_0(x)$. $a = (a_{ij}(x))$ is symmetric, bounded, measurable, and uniformly elliptic:

$$C^{-1}|\xi|^2 \leq \sum_{ij} \xi_i a_{ij}(x) \xi_j \leq C|\xi|^2.$$

Note. If $a_{ij}(x)$ are (say) C^2 then classical estimates give regularity of the solution $u(x,t)$.

For various reasons, particularly for applications to non-linear PDE, one wants regularity of u without any additional assumptions on a.

Work of de Giorgi, Moser, Nash

This problem was solved by de Giorgi, Moser, Nash in the late 1950s.

Nash (1958) proved that $u(x,t)$ is Hölder continuous.

Moser (1961-71) proved an (elliptic) Harnack inequality for $\mathcal{L}$-harmonic functions, and then a parabolic Harnack inequality (PHI) for solutions of (2) (This easily gives Hölder continuity of $u(x,t)$.)

Aronsen (1967): used Moser's PHI to obtain bounds on the fundamental solutions of (2). These bounds are of the same form as the Gaussian distribution. Let $p_t(x,y)$ be solution of (2) with $p_0(x,y) = \delta_x(y)$. Then

$$c_1 t^{-d/2} e^{-c_2|x-y|^2/t} \le p_t(x,y) \le c_3 t^{-d/2} e^{-c_4|x-y|^2/t}. \qquad (GB)$$

Note. On more general spaces the terms $c_i t^{-d/2}$ above have to be replaced by $c_i |B(x,t^{1/2})|^{-1}$, where $|B(x,r)|$ is the volume of the ball centre x and radius r.

Extensions to manifolds

Bombieri and Giusti (1972) generalised Moser's argument to manifolds. They showed it works given three facts about the space:

A Sobolev inequality (later found to be unnecessary)

A Poincaré inequality (PI), or more strictly, a family of PI.

Good control of the volume growth, i.e. the behaviour of

$$|B(x,r)|, \quad x \in M, r \geq 0.$$

Fabes and Stroock (1986) ("A new proof of Moser's PHI via the old ideas of Nash") gave a proof of PHI via (GB).

Theorem *(Grigoryan, Saloff-Coste (1992)). Let M be a complete manifold. The following are equivalent:*

(a) Solutions of the heat equation on M satisfy a PHI.

(b) M satisfies two conditions: (VD) and (PI).

(c) (Fabes – Stroock ...) $p_t(x,y)$ satisfies (GB).

So, what are ‘VD’ and ‘PI’ ?

VD = ‘volume doubling’: there exists C_0 such that

$$|B(x,2r)| \leq C_0 |B(x,r)|, \quad x \in M,\ r \geq 0.$$

Hence $|B(x,2^n r)| \leq C_0^n |B(x,r)|$, and M has polynomial volume growth.

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PI = Poincaré Inequality: there exists C_P such that if $B = B(x,r)$ is a ball in M, and $f : B \to \mathbb{R}$ then

$$\int_B (f(x) - \overline{f}_B)^2 d\mu \leq C_P r^2 \int_B |\nabla f|^2 d\mu. \qquad (PI)$$

$\overline{f}_B$ is the real number which minimises the LHS.

Globally, PI restricts the extent to which the space can have ‘bottlenecks’.

The Graph case: Delmotte's Theorem

These ideas also work for random walks on graphs:

Theorem *(T. Delmotte, 1999). Let G be a (locally finite) graph. The following are equivalent:*
(a) Solutions of the heat equation on G satisfy a PHI.
(b) G satisfies (VD) and (PI).
(c) The heat kernel $p_n(x, y)$ satisfies (GB).

The proof used Moser's ideas.

Notes. 1. SRW on $\mathbb{Z}^d$ satisfies (a)–(c).
2. On a general graph, one uses the usual (shortest path) graph distance.

PI for graphs

For every ball $B = B(x, r)$, and $f : B \to \mathbb{R}$,

$$\sum_{x \in B} (f(x) - \overline{f}_B)^2 \mu_x \leq C_P r^2 \sum_{x \sim y, x, y \in B} (f(y) - f(x))^2$$

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An example of a graph for which the PI fails is two copies of $\mathbb{Z}^d$ ($d \geq 2$) connected at their origins.
If $d \geq 3$ it is easy to see PI fails: consider $B(0, r)$ and let $f = 1$ on one copy, and $f = -1$ on the other. Then LHS $\approx r^d$ while the RHS $\approx r^2$.

PI follows from an isoperimetric inequality

Let $B = B(x, r)$, and write

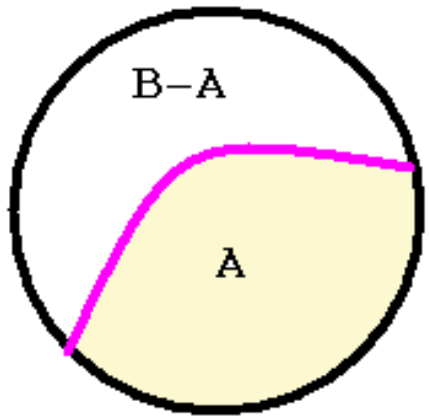
$$N(A, B - A) = |\{e = \{x, y\} \in E : x \in A, y \in B - A\}|$$

for the number of edges between A and $B - A$.
If for all balls $B = B(x, r)$ one has

$$N(A, B - A) \geq \frac{c|A|}{r}, \quad \text{whenever } A \subset B,\ |A| \leq \tfrac{1}{2}|B|,$$

then G satisfies PI.

B=B(x,r)



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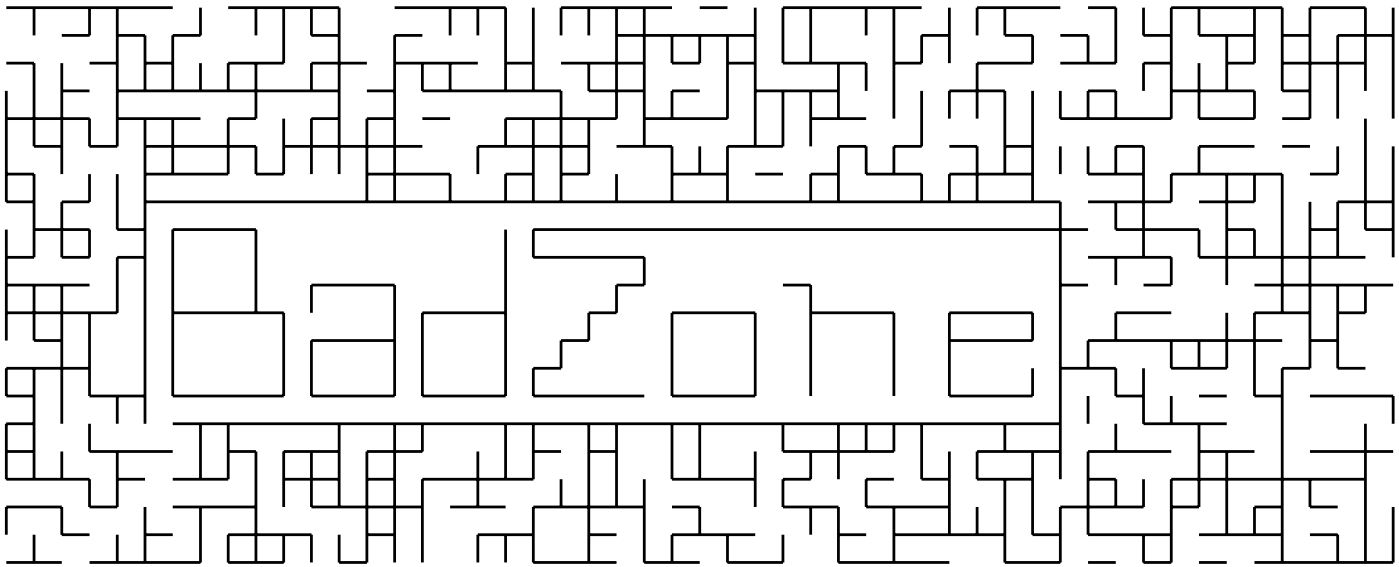

Big bad regions are a long way away

Suppose we are looking for a specific bad configuration of volume r. This has probability of order e^{-cr}.
So to find it in $B(0,R)$ we need $R^d e^{-cr} \approx 1$, or $r \approx \log R$.
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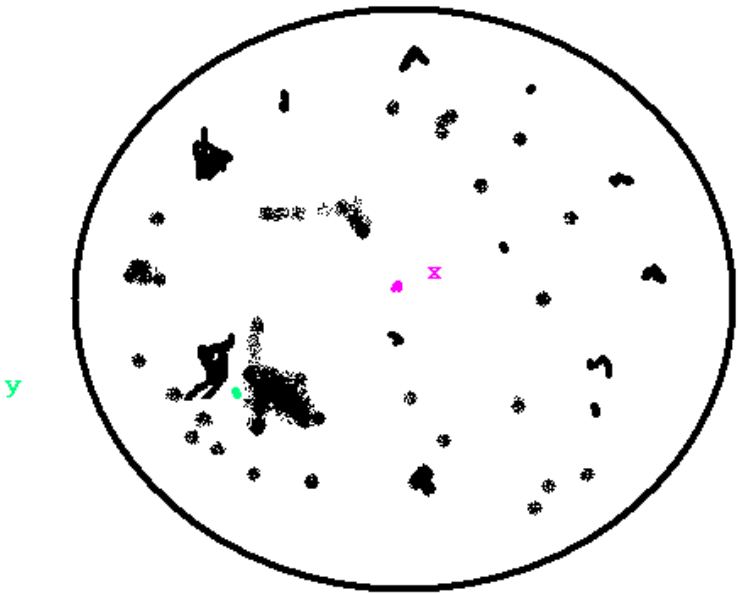
No. The time to leave a bad region is $\leq (\log R)^{2c_0}$, and this is much less than the time to leave $B(0,R)$, which is R^2.



Local inequalities

Fix (non random) constants C_1, C_2, C_3. Call a ball $B(x,r)$ *good* if:

$$C_1 r^d \leq |B(x,r)| \leq C_2 r^d, \qquad \text{and PI holds (with constant } C_3) \text{ for } B(x,r)\ .$$

Theorem. (Benjamini-Mossel, Mathieu-Remy, MB.) *If $p > p_c$ then*

$$\mathbb{P}_p(B(x,R) \textit{ is good }) \geq 1 - e^{-R^\delta}.$$

Hence with high probability:

$$\textit{every ball } B(y,r) \subset B(x,R) \textit{ with } R^\varepsilon \leq r \leq R \textit{ is good.} \qquad (*)$$

Call a ball satisfying (*) *very good*.

Natural guess: if $B(x,r)$ is 'good', so that VD and PI hold for it, then $p_n(x,y)$ should satisfy (GB) when

$$n \approx r^2, \qquad x, y \in B(x,r).$$

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This is the right general idea.

However 'good' is not enough.
The proofs of (GB) all use iterative methods or differential inequalities, which rely on the space being regular over a substantial range of length scales.
To control $p_n(x,y)$ one roughly needs to take $R = (n \log n)^{1/2}$ and have $B(y,r)$ 'good' for $y \in B(x,R)$, and $R^\epsilon < r \leq R$.

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So the current proofs need 'very good' not good.

Gaussian bounds

Theorem 1. *Let $p > p_c$. For each $x \in \mathbb{Z}^d$ there exist r.v. $T_x(\omega) \geq 1$ with*

$$\mathbb{P}_p(T_x \geq n, x \in \mathcal{C}_\infty) \leq c \exp(-n^{\varepsilon_d}) \tag{3}$$

and (non-random) constants $c_i = c_i(d,p)$ such that the transition density of X satisfies,

$$\frac{c_1}{n^{d/2}} e^{-c_2|x-y|^2/n} \leq p_n^\omega(x,y) \leq \frac{c_3}{n^{d/2}} e^{-c_4|x-y|^2/n}, \tag{GB}$$

for $x, y \in \mathcal{C}_\infty(\omega)$, $n \geq \max(T_x(\omega), c|x-y|)$.

1. The randomness of the environment is taken care of by the $T_x(\omega)$, which will be small for most points, and large for the rare points in large 'bad regions'.
2. Good control of the tails of the r.v. T_x, as in (3), is essential for applications.
3. The proof used 'Nash' rather than 'Moser'.

Nash's idea (PDE setting)

The key hard step in Nash's 1958 paper was to prove that if $M(x,t) = \int |x-y| p_t(x,y) dy$ then

$$c_1 t^{1/2} \leq M(x,t) \leq c_2 t^{1/2}. \tag{4}$$

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He considered the entropy $Q(t) = -\int p_t(x,y) \log p_t(x,y) dy$, and found an ingenious, but not very transparent argument using three inequalities between M and Q:

$$Q(x,t) \geq c + \tfrac{1}{2} d \log t, \tag{5}$$

$$M(x,t) \geq c e^{Q(x,t)/d}, \tag{6}$$

$$Q'(x,t) \geq c M'(x,t)^2. \tag{7}$$

Theorem. *(Nash (1958).) If functions Q, M satisfy (5)–(7) (and $M(0) = 0$) then M satisfies (4).*

Nash-Bass method

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These techniques also work for graphs. It is useful for percolation clusters, because if we fix a base point x then 'distant bad regions' have little effect on $M(x,t)$ and $Q(x,t)$.
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(7) $Q'(x,t) \geq c M'(x,t)^2$ holds in general.

Functional CLT

Theorem 2. *(Sidoravicius and Sznitman, 2004 ($d \geq 4$), Berger and Biskup, 2005, Mathieu and Piatnitski, 2005). Let $p > p_c$. For a set of ω with probability one, a functional CLT holds for X.*

This means that the rescaled SRW

$$X_t^{(n)} = n^{-1} X_{n^2 t}$$

converges to (a constant multiple of) Brownian motion.

The proofs for $d \geq 3$ use the upper bounds in Theorem 1.

Theorem 3. *(MB-Hambly). A local limit theorem also holds:*

$$p_t^{(n,\omega)}(0,x) = n^d p^\omega_{n^2 t}(\lfloor 0 \rfloor, \lfloor nx \rfloor) \to (2\pi D)^{-d/2} \exp(-|x|^2/2Dt).$$

Summary: for 'fixed' $p > p_c$, SRW on $\mathcal{C}_\infty$ looks very much like SRW on $\mathbb{Z}^d$.

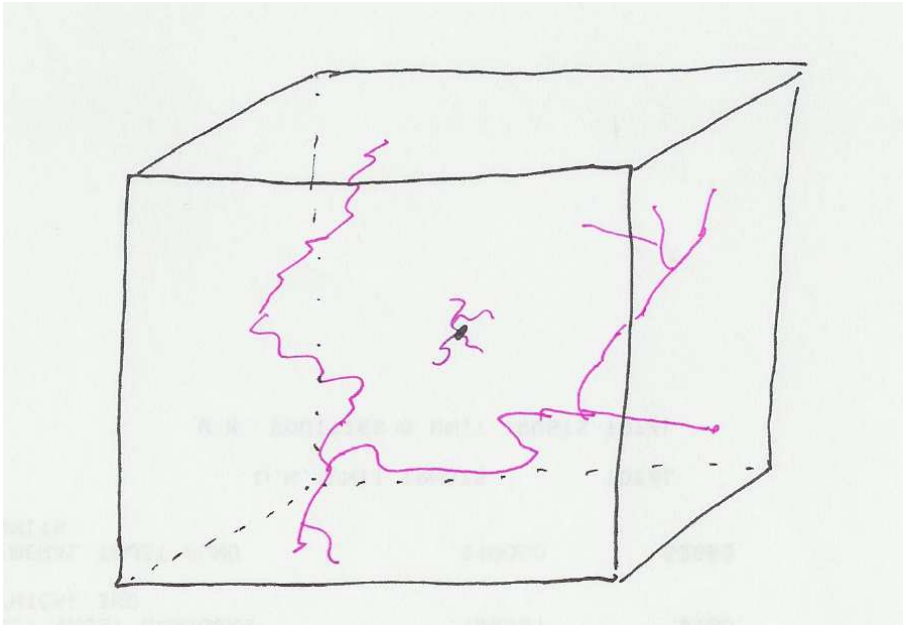
Critical Percolation: $p = p_c$

Conjecture. No infinite clusters for critical percolation in $\mathbb{Z}^d$, for $d \geq 2$.

Proved for $d = 2$ (Kesten) and $d \geq 19$ (Hara – Slade).

What can one do if there is no infinite cluster?

It is known that in any large box $Q_n = [-n, n]^d$ then (with high probability) there exists a cluster with diameter $O(n)$.



Incipient Infinite Cluster (IIC)

This is an infinite connected random subset $\widetilde{C}_d$ of $\mathbb{Z}^d$ which locally 'looks like' the large finite clusters which occur at $p = p_c$.
Constructed when $d = 2$ (Kesten) and for large d (van Hofstad, Jarai).
For any infinite graph $G = (V, E)$ one can define the *spectral dimension* by

$$d_s = d_s(G) = -2 \lim_{n\to\infty} \frac{\log p_n(x, x)}{\log n} \qquad \text{(if this limit exists).}$$

Note that $d_s(\mathbb{Z}^d) = d$. By Theorem 1 we have, for $p > p_c$, $d_s(\mathcal{C}_\infty) = d$.

Alexander–Orbach Conjecture (1983). (Restated for mathematicians). For all $d \geq 2$,

$$d_s(\widetilde{C}_d) = \tfrac{2}{3}.$$

AO conjecture

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For percolation $d_c = 6$. At $p_c(d)$ there is just enough probability of an edge being open to allow large scale connected structures of open edges. However, these structures are 'thin' and when $d > d_c$ they don't self-intersect except locally – hence they don't 'see' the true dimension of the space they are in. In fact, they are close to being 'fractal trees', and should be similar to those for percolation on the binary tree $\mathbb{B}$.

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Techniques developed for random walks on fractal sets turn out to be useful for critical clusters. (Not a surprise, since this was one original motivation to look at SRW on fractals.) One needs 'volume' estimates, but instead of 'PI' one can use 'electrical resistance'.

SRW on critical percolation clusters

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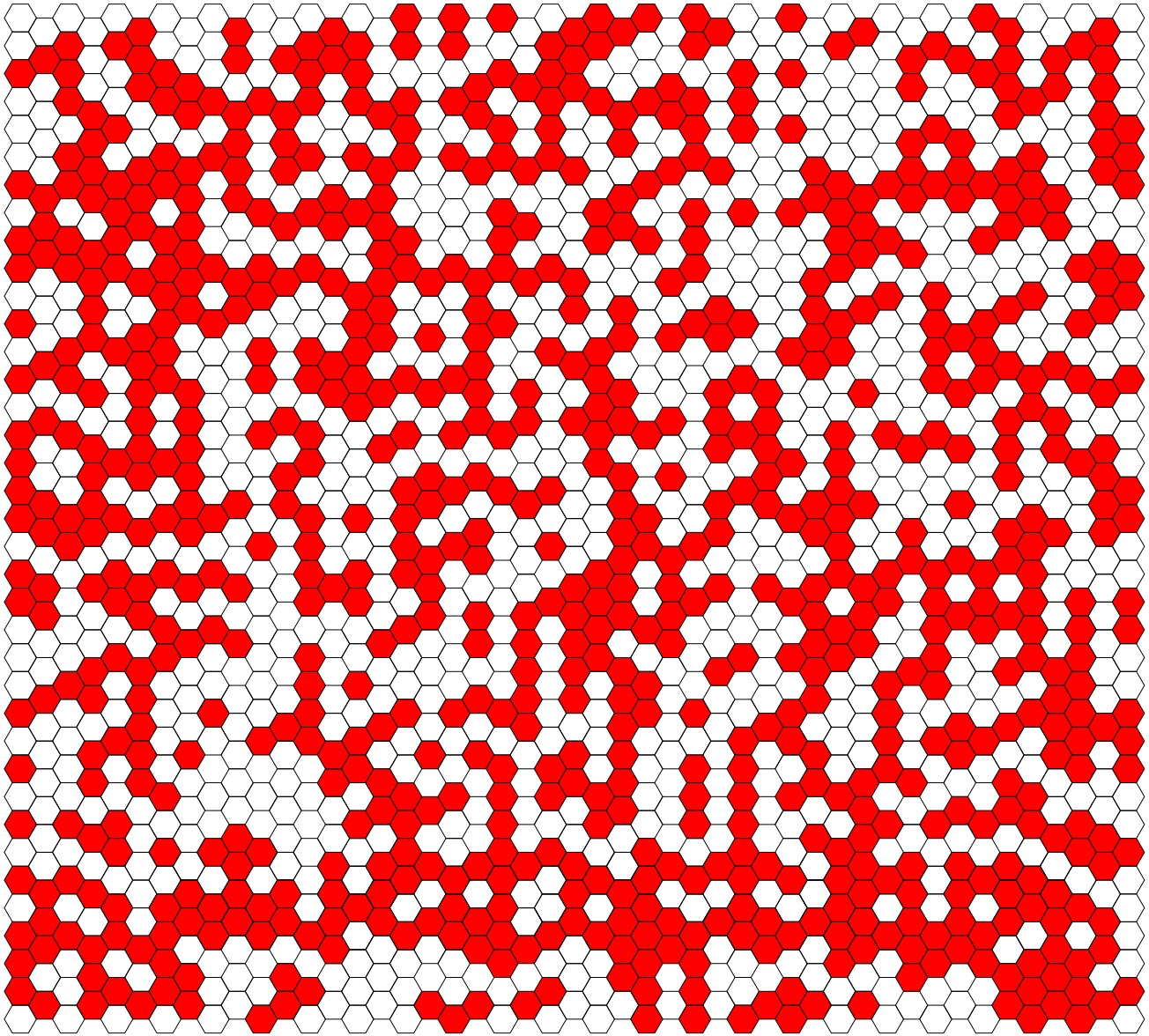
For $d = 2$ one percolation model ('site percolation on the triangular lattice') has been proved to have a conformally invariant scaling limit, which is SLE_6. (Smirnov; Lawler-Schramm-Werner.)

Critical site percolation, $d = 2$



Critical site percolation, $d = 2$

Using the SLE limit, many exponents have been calculated for the IIC for this percolation process. For example, the dimension of the cluster is 91/48, and the dimension of the boundary of the 'holes' is 4/3.

I do not know of any generally accepted conjecture on what d_s should be.

To calculate d_s one needs 'electrical resistance' properties of the IIC, and these are harder to obtain than the geometric properties which have been obtained from the SLE theory.

It is also likely that one needs to know the length of the shortest path in the cluster across a box of side R – another quantity that the SLE theory does not seem to give.