Density Models for default risk

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In many models for credit risk, it is implicitly assumed that the intensity contains all the needed information. Our goal is to present a more general setting.

Gapeev, P. V., Jeanblanc, M., Li, L., and Rutkowski, M. (2009): Constructing Random Times with Given Survival Processes and Applications to Valuation of Credit Derivatives. Forthcoming in: *Contemporary Quantitative Finance*, C. Chiarella and A. Novikov, eds., Springer-Verlag, Berlin Heidelberg New York, 2010.

Jeanblanc, M. and Song, S. (2010) Explicit Model of Default Time with given Survival Probability
and Default times with given survival probability and their $\mathbb{F}$-martingale decomposition formula. *In preparation.*

El Karoui, N., Jeanblanc, M. and Jiao, Y. (2009). What happens after a default: the conditional density approach. *Forthcoming SPA* 120 (2010) 1011-1032.

Related works:

Brody D.C. and Hughston L.P. (2002): Entropy and information in the interest rate term structure, Quantitative Finance, 2, 70-80.

Papapantoleon, A. (2009): Old and new approaches to Libor modeling, preprint.

Filipovic, D., L. Overbeck and T. Schmidt (2009): Dynamic CDO term structure modelling, preprint

Mathematical Model

A filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ is given, as well as a random time τ. The default process is $H_t = 1\!\!1_{\tau \le t}$, the associated filtration is $\mathbb{H} = (\mathcal{H}_t = \sigma(t \wedge \tau), t \geq 0)$. The filtration $\mathbb{G}$ is defined as $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$. The $\mathbb{G}$-intensity of τ is the process $\lambda^{\mathbb{G}}$ such that

$$M_t = H_t - \int_0^t \lambda_s^{\mathbb{G}} ds$$

is a $\mathbb{G}$-martingale. There exists an $\mathbb{F}$-adapted process $\lambda^{\mathbb{F}}$ such that

$$M_t = H_t - \int_0^{t \wedge \tau} \lambda_s^{\mathbb{F}} ds$$

If $X \in \mathcal{F}_T$, and $G_t = \mathbb{P}(\tau > t | \mathcal{F}_t)$, then

$$\mathbb{E}(X 1\!\!1_{T < \tau} | \mathcal{G}_t) = 1\!\!1_{t < \tau} \frac{1}{G_t} \mathbb{E}(X G_T | \mathcal{F}_t)$$

One can think that the knowledge of λ and G will allow us to have the knowledge of the conditional law of τ. We shall show that this is not the case.

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Intensity models

Models with a given intensity are constructed as follows.
Let λ be a given $\mathbb{F}$-adapted positive process and Θ a random variable independent of $\mathcal{F}_\infty$, with unit exponential law. Then

$$\tau = \inf\{t : \int_0^t \lambda_s ds \geq \Theta\}$$

has intensity equal to λ.

In that model, $\mathbb{P}(\tau > u|\mathcal{F}_t) = \mathbb{E}(e^{-\Lambda_u}|\mathcal{F}_t)$ and immersion property holds:

$$\begin{aligned} \mathbb{P}(\tau > t|\mathcal{F}_t) &= \mathbb{P}(\tau > t|\mathcal{F}_\infty) = e^{\Lambda_t} \\ \mathbb{E}(X|\mathcal{F}_t) &= \mathbb{E}(X|\mathcal{G}_t), \forall X \in \mathcal{F}_\infty \end{aligned}$$

Under immersion property, one has

$$p_t(u)du := \mathbb{P}(\tau \in du|\mathcal{F}_t) = \mathbb{E}(\lambda_u e^{-\Lambda_u}|\mathcal{F}_t)du$$

and we note that $p_t(u) = p_u(u), \forall t \geq u$.

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We now construct probabilities $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that τ has intensity λ, and where immersion does not hold, hence, for $t > u$, the density $p_t^{\mathbb{Q}}(u)$ is not determined in terms of the intensity.

Let $p_t(u)du = \mathbb{P}(\tau \in du | \mathcal{F}_t)$ and $z(u)$ a family of processes such that
(i) $(z_t(u), t \geq u)$ are positive $\mathbb{F}$-martingales.
Define, for z positive $\mathbb{F}$-adapted process

$$Z_t^{\mathbb{G}} = z_t \mathbb{1}_{\{\tau > t\}} + z_t(\tau) \mathbb{1}_{\{\tau \leq t\}}$$

and let

$$Z_t^{\mathbb{F}} := \mathbb{E}(Z_t^{\mathbb{G}} | \mathcal{F}_t) = z_t G_t + \int_0^t z_t(u) p_t(u) du$$

be its $\mathbb{F}$-projection. Assume that
(ii) $Z^{\mathbb{F}}$ is a $\mathbb{F}$-martingale.
Then, $Z^{\mathbb{G}}$ is a positive $\mathbb{G}$-martingale.

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Proof: (we assume here that G is continuous.) Let $s < t$.

$$\mathbb{E}(Z_t^{\mathbb{G}}|\mathcal{G}_s) = \mathbb{E}(z_t 1\!\!1_{\tau>t}|\mathcal{G}_s) + \mathbb{E}(z_t(\tau) 1\!\!1_{s<\tau\leq t}|\mathcal{G}_s) + \mathbb{E}(z_t(\tau) 1\!\!1_{\tau\leq s}|\mathcal{G}_s) = I_1 + I_2.$$

For I_1, we apply the standard formula

$$I_1 = 1\!\!1_{\tau>s}\frac{1}{G_s}\mathbb{E}(Z_t G_t|\mathcal{F}_s) + 1\!\!1_{\tau>s}\frac{1}{G_s}\mathbb{E}(z_t(\tau) 1\!\!1_{s<\tau\leq t}|\mathcal{F}_s),$$

For I_2, we obtain

$$I_2 = \mathbb{E}(z_t(\tau) 1\!\!1_{\tau\leq s}|\mathcal{G}_s) = 1\!\!1_{\tau\leq s}\mathbb{E}(z_t(u)|\mathcal{F}_s)_{u=\tau} = 1\!\!1_{\tau\leq s}(z_s(u))_{u=\tau} = 1\!\!1_{\tau\leq s} z_s(\tau),$$

where the first equality holds under the immersion hypothesis and the second follows from (i). It thus suffices to show that $I_1 = Z_s 1\!\!1_{\tau>s}$.

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It thus suffices to show that $I_1 = z_s \mathbb{1}_{\tau > s}$ where

$$I_1 = \mathbb{1}_{\tau > s} \frac{1}{G_s} \mathbb{E}(z_t G_t | \mathcal{F}_s) + \mathbb{1}_{\tau > s} \frac{1}{G_s} \mathbb{E}(z_t(\tau) \mathbb{1}_{s < \tau \le t} | \mathcal{F}_s),$$

Condition (ii) yields

$$\mathbb{E}(z_t G_t | \mathcal{F}_s) + \mathbb{E}(z_t(\tau) \mathbb{1}_{\tau \le t} | \mathcal{F}_s) - \mathbb{E}(z_s(\tau) \mathbb{1}_{\tau \le s} | \mathcal{F}_s) = z_s G_s.$$

Therefore,

$$I_1 = \mathbb{1}_{\tau > s} \frac{1}{G_s} \Big(z_s G_s + \mathbb{E}((z_s(\tau) - z_t(\tau)) \mathbb{1}_{\tau \le s} | \mathcal{F}_s) \Big) = z_s \mathbb{1}_{\tau > s},$$

where the last equality holds since

$$\mathbb{E}((z_s(\tau) - z_t(\tau)) \mathbb{1}_{\tau \le s} | \mathcal{F}_s) = \mathbb{1}_{\tau \le s} \mathbb{E}((z_s(u) - z_t(u)) | \mathcal{F}_s)_{u=\tau} = 0.$$

For the last equality in the formula above, we have again used condition (i).

We assume (w.l.g.) that $Z_0^{\mathbb{G}} = 1$.
Let $\mathbb{Q}$ be the probability measure defined on $\mathcal{G}_t$ by $d\mathbb{Q} = Z_t^{\mathbb{G}} d\mathbb{P}$.
We assume that $z_t(t) = z_t$ (so that the RN density has no jump at time τ).
Then, for $t \geq \theta$,

$$p_t^{\mathbb{Q}}(\theta) = p_t(\theta) \frac{z_t(\theta)}{Z_t^{\mathbb{F}}},$$

and the $\mathbb{Q}$-conditional survival process is defined by

$$\mathbb{Q}(\tau > t | \mathcal{F}_t) = e^{-\Lambda_t} \frac{z_t}{Z_t^{\mathbb{F}}} = N_t^{\mathbb{Q}} e^{-\Lambda_t}$$

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Proof: For $t > u$,

$$\mathbb{Q}(\tau > u | \mathcal{F}_t) = \frac{1}{\mathbb{E}(Z_t^{\mathbb{G}} | \mathcal{F}_t)} \mathbb{E}(Z_t^{\mathbb{G}} 1\!\!1_{u<\tau} | \mathcal{F}_t)$$

$$\begin{aligned} \mathbb{E}(Z_t^{\mathbb{G}} 1\!\!1_{u<\tau} | \mathcal{F}_t) &= \mathbb{E}(Z_t^{\mathbb{G}} 1\!\!1_{t<\tau} | \mathcal{F}_t) + \mathbb{E}(Z_t^{\mathbb{G}} 1\!\!1_{u<\tau\leq t} | \mathcal{F}_t) = z_t G_t + \mathbb{E}(z_t(\tau) 1\!\!1_{u<\tau\leq t} | \mathcal{F}_t) \\ &= z_t G_t + \int_u^t z_t(v) p_t(v) dv \end{aligned}$$

and the density follows by differentiation. The form of the intensity ($\lambda_t^{\mathbb{G}} = \frac{p_t^{\mathbb{G}}(t)}{G_t^{\mathbb{G}}}$) follows. Indeed, if $G_t = \mu_t - A_t$ is the Doob-Meyer decomposition of G, $A_t = \int_0^t p_u(u) du$ and the intensity is $\lambda_t dt = \frac{dA_t}{G_t}$.

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Construction of a random time with given conditional law

Let $p(u)$ a family of positive $\mathbb{F}$-martingales such that

$$\int_0^\infty p_t(u)du = 1, \ \forall t$$

One can construct (on an extended space) a probability $\mathbb{Q}$ and a random time τ such that

$$\begin{aligned} \mathbb{Q}|_{\mathcal{F}_t} &= \mathbb{P}|_{\mathcal{F}_t} \\ \mathbb{Q}(\tau \in du|\mathcal{F}_t) &= p_t(u)du \end{aligned}$$

as follows:

- Construct $\mathbb{Q}^*$ and τ such that τ is independent from $\mathcal{F}_\infty$ and $\mathbb{Q}(\tau \in du) = p_0(u)du$
- Set $d\mathbb{Q}|_{\mathcal{F}_t \vee \sigma(\tau)} = (p_t(\tau))^{-1} d\mathbb{Q}^*$

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Construction of a random time with given Conditional Survival Probability

Construct (on an extended space) a probability $\mathbb{Q}$ and a random time τ such that

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where G is a given $\mathbb{F}$-supermartingale. One recall that any supermartingale admits a multiplicative decomposition decomposition as $G_t = N_t D_t = N_t e^{-\Lambda_t}$ where D (resp. Λ) is decreasing (resp. increasing) In what follows, we assume that G is continuous, and $0 \leq G_t < 1$ for $t > 0$.

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Let us start with a model in which $\mathbb{P}(\tau > t|\mathcal{F}_t) = e^{-\Lambda_t}$, where $\Lambda_t = \int_0^t \lambda_s ds$ and let N be an $\mathbb{F}$-local martingale such that $0 \leq N_t e^{-\Lambda_t} \leq 1$.

There exists a $\mathbb{G}$-martingale L such that, setting $d\mathbb{Q} = Ld\mathbb{P}$

(i) $\mathbb{Q}|_{\mathcal{F}_\infty} = \mathbb{P}|_{\mathcal{F}_\infty}$

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The $\mathbb{G}$-adapted process L

$$L_t = \ell_t 1\!\!1_{t<\tau} + \ell_t(\tau) 1\!\!1_{\tau \leq t}$$

is a martingale if for any u, $(\ell_t(u), t \geq u)$ is a martingale and if $\mathbb{E}(L_t|\mathcal{F}_t)$ is a $\mathbb{F}$-martingale. Then, (i) is satisfied if

$$1 = \mathbb{E}(L_t|\mathcal{F}_t) = \ell_t G_t + \int_0^t \ell_t(u) \lambda_u e^{-\Lambda_u} du$$

and (ii) implies that $\ell = N$ and $\ell_t(t) = \ell_t$.

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$$1 = \mathbb{E}(L_t|\mathcal{F}_t) = \ell_t e^{-\Lambda_t} + \int_0^t \ell_t(u) \lambda_u e^{-\Lambda_u} du$$

and (ii) implies that $\ell = N$ and $\ell_t(t) = \ell_t$.

It remains to find a family of martingales $\ell(u)$ such that

$$\begin{aligned} \ell_u(u) &= N_u \\ 1 &= N_t e^{-\Lambda_t} + \int_0^t \ell_t(u)\lambda_u e^{-\Lambda_u} du \end{aligned}$$

We choose

$$\ell_t(u) = \frac{N_u}{1-G_u}(1-G_t)\exp\left(-\int_u^t \frac{G_s}{1-G_s}\lambda_s ds\right)$$

Then, $\mathbb{Q}[\tau \le u|\mathcal{F}_t] = M_t^u$ for $0 \le u \le t \le \infty$ where

$$M_t^u = (1-G_t)\exp\left(-\int_u^t \frac{G_s}{1-G_s}\lambda_s ds\right) \quad 0 \le u \le t \le \infty,$$

One can also construct other martingales M^u which give a solution (i.e., families of $[0,1]$-valued martingales such that $u \to M_t^u$ is decreasing and $M_t^t = 1 - G_t$).

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Cox processes

Let λ be a strictly positive $\mathbb{F}$-adapted process, and $\Lambda_t = \int_0^t \lambda_s ds$.

Let Θ be a strictly positive random variable **whose conditional distribution w.r.t. $\mathbb{F}$ admits a density** w.r.t. the Lebesgue measure, i.e., there exists a family of $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable functions $\gamma_t(u)$ such that $\mathbb{P}(\Theta > \theta|\mathcal{F}_t) = \int_\theta^\infty \gamma_t(u)du$.

Let $\tau = \inf\{t > 0 : \Lambda_t \geq \Theta\}$.

Then τ admits the density

$$p_t(\theta) = \lambda_\theta \gamma_t(\Lambda_\theta) \text{ if } t \geq \theta \qquad \text{and} \qquad p_t(\theta) = \mathbb{E}[\lambda_\theta \gamma_\theta(\Lambda_\theta)|\mathcal{F}_t] \text{ if } t < \theta.$$

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Proof: By definition and by the fact that Λ is strictly increasing and absolutely continuous, we have for $t \geq \theta$,

$$\mathbb{P}(\tau > \theta | \mathcal{F}_t) = \mathbb{P}(\Theta > \Lambda_\theta | \mathcal{F}_t) = \int_{\Lambda_\theta}^{\infty} \gamma_t(u) du = \int_{\theta}^{\infty} \gamma_t(\Lambda_u) d\Lambda_u = \int_{\theta}^{\infty} \gamma_t(\Lambda_u) \lambda_u du,$$

which implies $p_t(\theta) = \lambda_\theta \gamma_t(\Lambda_\theta)$. The martingale property of p gives the whole density.

Conversely, if we are given a density p, and hence an associated process $\Lambda_t = \int_0^t \lambda_s ds$ with $\lambda_s = \frac{p_s(s)}{G_s}$, then it is possible to find a threshold Θ such that τ has p as density.

We denote by Λ^{-1} the inverse of the strictly increasing process λ.

We let $\Lambda_t = \int_0^t \frac{p_s(s)}{G_s} ds$ and $\Theta = \Lambda_\tau$. Then Θ has a density γ with respect to $\mathbb{F}$ given by

$$\gamma_t(\theta) = \mathbb{E}\Big[p_{t \vee \Lambda_\theta^{-1}}(\Lambda_\theta^{-1}) \frac{1}{\lambda_{\Lambda_\theta^{-1}}} | \mathcal{F}_t\Big].$$

Proof: We set $\Theta = \Lambda_\tau$ and compute the density of Θ w.r.t. $\mathbb{F}$

$$\begin{aligned}\mathbb{P}(\Theta > \theta|\mathcal{F}_t) &= \mathbb{P}(\Lambda_\tau > \theta|\mathcal{F}_t) = \mathbb{P}(\tau > t, \Lambda_\tau > \theta|\mathcal{F}_t) + \mathbb{P}(\tau \le t, \Lambda_\tau > \theta|\mathcal{F}_t) \\ &= \mathbb{E}[-\int_t^\infty 1\!\!1_{\{\Lambda_u > \theta\}} dG_u|\mathcal{F}_t] + \int_0^t 1\!\!1_{\{\Lambda_u > \theta\}} p_t(u)du \\ &= \mathbb{E}[\int_t^\infty 1\!\!1_{\{\Lambda_u > \theta\}} p_u(u)du|\mathcal{F}_t] + \int_0^t 1\!\!1_{\{\Lambda_u > \theta\}} p_t(u)du\end{aligned}$$

where the last equality comes from the fact that $(G_t + \int_0^t p_u(u)du, t \ge 0)$ is an $\mathbb{F}$-martingale. Note that since the process Λ is continuous and strictly increasing, also is its inverse. Hence

$$\mathbb{E}\big[\int_\theta^\infty p_{\Lambda_s^{-1} \vee t}(\Lambda_s^{-1}) \frac{1}{\lambda_{\Lambda_s^{-1}}} ds|\mathcal{F}_t\big] = \mathbb{E}\big[\int_{\Lambda_\theta^{-1}}^\infty p_{s \vee t}(s) \frac{1}{\lambda_s} d\Lambda_s|\mathcal{F}_t\big]$$

$$= \mathbb{E}\big[\int_0^\infty 1\!\!1_{\{s > \Lambda_\theta^{-1}\}} p_{s\vee t}(s) ds|\mathcal{F}_t\big] = \mathbb{E}\big[\int_0^\infty 1\!\!1_{\{\Lambda_s > \theta\}} p_{s \vee t}(s) ds|\mathcal{F}_t\big],$$

which equals $\mathbb{P}(\Theta > \theta|\mathcal{F}_t)$.

Defaultable Zero-Coupon Bonds

A defaultable zero-coupon with maturity T associated with the default time τ is an asset which pays one monetary unit at time T if (and only if) the default has not occurred before T. We assume that $\mathbb{P}$ is the pricing measure.

$$D(t,T) := \mathbb{P}(\tau > T \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{P}(\tau > T \,|\, \mathcal{F}_t)}{G_t} = \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{E}_{\mathbb{P}}\left(N_T e^{-\Lambda_T} \,|\, \mathcal{F}_t\right)}{G_t}$$

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Let $\mathbb{P}^*$ be defined as

$$d\mathbb{P}^*|_{\mathcal{G}_t} = Z_t^* d\mathbb{P}|_{\mathcal{G}_t}$$

where Z^* is the $(\mathbb{P}, \mathbb{G})$-martingale defined as

$$Z_t^* = 1\!\!1_{\{t<\tau\}} + 1\!\!1_{\{t\geq\tau\}} \lambda_\tau e^{-\Lambda_\tau} \frac{N_t}{p_t(\tau)}$$

Then,

(a) Immersion property holds under $\mathbb{P}^*$,

(b) $d\mathbb{P}^*|_{\mathcal{F}_t} = N_t d\mathbb{P}|_{\mathcal{F}_t}$

(c) $\mathbb{P}^*$ and $\mathbb{P}$ coincide on $\mathcal{G}_\tau$.

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Proof: We prove first that $d\mathbb{P}^*|_{\mathcal{F}_t} = N_t d\mathbb{P}|_{\mathcal{F}_t}$

$$\mathbb{E}_{\mathbb{P}}(Z_t^*|\mathcal{F}_t) = G_t + \int_0^t \lambda_u e^{-\Lambda_u} \frac{N_t}{p_t(u)} p_t(u)du = N_t e^{-\Lambda_t} + N_t(1 - e^{-\Lambda_t}) = N_t$$

We compute the $\mathbb{P}^*$ conditional law of τ. For $t > \theta$,

$$\begin{aligned}\mathbb{P}^*(\theta < \tau|\mathcal{F}_t) &= \frac{1}{N_t}\mathbb{E}_{\mathbb{P}}(Z_t^* \mathbb{1}_{\{\theta<\tau\}}|\mathcal{F}_t) = \frac{1}{N_t}\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{t<\tau\}} + \mathbb{1}_{\{t\geq\tau>\theta\}}\lambda_\tau e^{-\Lambda_\tau}\frac{N_t}{p_t(\tau)}|\mathcal{F}_t) \\ &= \frac{1}{N_t}\left(N_t e^{-\Lambda_t} + \int_\theta^t \lambda_u e^{-\Lambda_u}\frac{N_t}{p_t(u)}p_t(u)du\right) = e^{-\Lambda_\theta}\end{aligned}$$

which proves that immersion holds true under $\mathbb{P}^*$, and the intensity of τ is the same under $\mathbb{P}$ and $\mathbb{P}^*$. It follows that

$$\mathbb{E}^*(X\mathbb{1}_{\{T<\tau\}}|\mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}}\frac{1}{e^{-\Lambda_t}}\mathbb{E}^*(e^{-\Lambda_T}X|\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(X\mathbb{1}_{\{T<\tau\}}|\mathcal{G}_t)$$

Note that, if the intensity is the same under $\mathbb{P}$ and $\mathbb{P}^*$, its dynamics under $\mathbb{P}^*$ will involve a change of driving process, since $\mathbb{P}$ and $\mathbb{P}^*$ do not coincide on $\mathcal{F}_\infty$.

Let us now study the pricing of a recovery. Let Z be an $\mathbb{F}$-predictable bounded process.

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}}(Z_\tau \mathbb{1}_{\{t<\tau\leq T\}}|\mathcal{G}_t) &= \mathbb{1}_{\{t<\tau\}}\frac{1}{G_t}\mathbb{E}_{\mathbb{P}}(-\int_t^T Z_u dG_u|\mathcal{F}_t) \\
&= \mathbb{1}_{\{t<\tau\}}\frac{1}{G_t}\mathbb{E}_{\mathbb{P}}(\int_t^T Z_u N_u \lambda_u e^{-\Lambda_u} du|\mathcal{F}_t) \\
&= \mathbb{1}_{\{t<\tau\}}\frac{1}{e^{-\Lambda_t}}\mathbb{E}^*(\int_t^T Z_u \lambda_u e^{-\Lambda_u} du|\mathcal{F}_t) \\
&= \mathbb{E}^*(Z_\tau \mathbb{1}_{\{t<\tau\leq T\}}|\mathcal{G}_t)
\end{aligned}
$$

The problem is different for pricing a recovery paid at maturity, i.e. for $X \in \mathcal{F}_T$

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}(X \mathbb{1}_{\tau<T} | \mathcal{G}_t) &= \mathbb{E}_{\mathbb{P}}(X|\mathcal{G}_t) - \mathbb{E}_{\mathbb{P}}(X \mathbb{1}_{\tau>T} | \mathcal{G}_t) \\ &= \mathbb{E}_{\mathbb{P}}(X|\mathcal{G}_t) - \mathbb{1}_{\{\tau>t\}} \frac{1}{e^{-\Lambda_t}} \mathbb{E}^*\big(X e^{-\Lambda_T} \,|\, \mathcal{F}_t\big)\end{aligned}$$

If both quantities $\mathbb{E}_{\mathbb{P}}(X \mathbb{1}_{\tau<T} | \mathcal{G}_t)$ and $\mathbb{E}^*(X \mathbb{1}_{\tau<T} | \mathcal{G}_t)$ are the same, this would imply that $\mathbb{E}_{\mathbb{P}}(X|\mathcal{G}_t) = \mathbb{E}^*(X|\mathcal{F}_t)$ i.e., immersion holds under $\mathbb{P}$.

Misspecification of the Information

In this section, we point out that the price of a derivative product written on a default τ depends strongly on the other default and the hedging strategies have to be constructed with the full observation. Let us study the following toy model

Two default times τ_1, τ_2 let $G(t,s) = \mathbb{P}(\tau_1 > t, tau_2 > s)$

We consider two filtrations $\mathbb{H}^1$ an $\mathbb{H} = \mathbb{H}^1 \vee \mathbb{H}^2$

The price of a DZC is

$$\begin{aligned}
\bar{D}^1(t,T) &= \mathbb{P}(\tau_1 > T | \mathcal{H}^1_t) = (1 - H^1_t)\frac{G(T,0)}{G(t,0)} \\
D^1(t,T) &= \mathbb{P}(\tau_1 > T | \mathcal{H}_t) = (1 - H^1_t)\left((1 - H^2_t)\frac{G(T,t)}{G(t,t)} + H^2_t \frac{\partial_2(T,\tau_2}{\partial_2 G(t,\tau_2)}\right)
\end{aligned}$$

The un-informed agent knows only $\mathbb{H}^1$. He will hedge the contingent claim $C = h(\tau_1)\mathbb{1}_{\tau_1 > T} + k\mathbb{1}_{\tau_1 > T}$ thinking the market is complete, with an initial wealth $x = \mathbb{E}(C)$ buying ζ_t DZC, so that

$$\widehat{X}_T := x + \int_0^T \zeta_s d\widehat{D}(t,T) = C$$

and he will invest $\zeta_t^0 = X_t - \zeta_t \widehat{D}(t,T)$ in the savings account in a self financing way. However, his "real" wealth will be $X_t = \zeta_t^0 + \zeta_t D(t,T)$ and the strategy is not self-financing. The cost of refinancing is

$$dX_t - \zeta_t dD(t,T) = d\zeta_t^0 + D(t,T)d\zeta_t^0 + d\langle \zeta, D(\cdot,T)\rangle_t$$

If he uses a self financing strategy, his terminal wealth will be $X_T^* = x + \int_0^T \zeta_t dD(t,T)$ and the associated cost is $C - X_T^* = \int_0^T \zeta_t (d\widehat{D}(t,T) - dD(t,T))$. One has $\mathbb{E}(C - X_T^*) = 0$

Thank you for your attention