Market Models of Forward CDS Spreads

Libo Li and Marek Rutkowski
School of Mathematics and Statistics
University of Sydney
Sydney, Australia

Workshop on Financial Derivatives and Risk Management
Fields Institute, Toronto, 24-28 May 2010

Outline

1. Credit Default Swaps
2. Credit Default Swaptions
3. Market Pricing Formula
4. Market Models for CDS Spreads

Modelling of CDS Spreads

- N. Bennani and D. Dahan: An extended market model for credit derivatives. Presented at the International Conference *Stochastic Finance*, Lisbon, 2004.
- D. Brigo: Candidate market models and the calibrated CIR++ stochastic intensity model for credit default swap options and callable floaters. In: *Proceedings of the 4th ICS Conference*, Tokyo, March 18-19, 2004.
- D. Brigo: Constant maturity credit default swap pricing with market models. Working paper, Banca IMI, 2004.
- L. Li and M. Rutkowski: Market models of forward CDS spreads. Working paper, 2009.
- L. Schlögl: Note on CDS market models. Working paper, Lehman Brothers, 2007.

Forward Credit Default Swaps

Hazard Process Set-up

Terminology and notation:

1. The default time is a strictly positive random variable τ defined on the underlying probability space $(\Omega, \mathcal{G}, \mathbb{P})$.
2. We define the default indicator process $H_t = \mathbb{1}_{\{\tau \leq t\}}$ and we denote by $\mathbb{H}$ its natural filtration.
3. We assume that we are given, in addition, some auxiliary filtration $\mathbb{F}$ and we write $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, meaning that $\mathcal{G}_t = \sigma(\mathcal{H}_t, \mathcal{F}_t)$ for every $t \in \mathbb{R}_+$.
4. The filtration $\mathbb{F}$ is termed the reference filtration.
5. The filtration $\mathbb{G}$ is called the full filtration.

Martingale Measure

The underlying market model is arbitrage-free, in the following sense:

1. Let the savings account B be given by

$$B_t = \exp\left(\int_0^t r_u \, du \right), \quad \forall \, t \in \mathbb{R}_+,$$

   where the short-term rate r follows an $\mathbb{F}$-adapted process.
2. A spot martingale measure $\mathbb{Q}$ is associated with the choice of the savings account B as a numéraire.
3. The underlying market model is arbitrage-free, meaning that it admits a spot martingale measure $\mathbb{Q}$ equivalent to $\mathbb{P}$.
4. Uniqueness of a martingale measure is not postulated.

Hazard Process

Let us summarize the main features of the hazard process approach:

1. Let us denote by
$$G_t = \mathbb{Q}(\tau > t \,|\, \mathcal{F}_t)$$
the survival process of τ with respect to the reference filtration $\mathbb{F}$. We postulate that $G_0 = 1$ and $G_t > 0$ for every $t \in [0, T]$.
2. For any $\mathbb{Q}$-integrable and $\mathcal{F}_T$-measurable random variable Y, the following classic formula is valid
$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{T<\tau\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} G_t^{-1} \mathbb{E}_{\mathbb{Q}}(G_T Y \,|\, \mathcal{F}_t).$$

Forward Credit Default Swap

Definition

A forward CDS issued at time s, with start date U, maturity T, and recovery δ at default is a defaultable claim $(0, A, Z, \tau)$ where

$$dA_t = -\kappa \mathbb{1}_{]U,T]}(t)\, dL_t, \quad Z_t = \delta_t \mathbb{1}_{[U,T]}(t).$$

- An $\mathcal{F}_s$-measurable rate κ is the CDS spread.
- An $\mathbb{F}$-adapted process L specifies the tenor structure of fee payments.
- An $\mathbb{F}$-adapted process $\delta : [U, T] \to \mathbb{R}$ represents the protection amount.

Lemma

The value of the forward CDS equals, for every $t \in [s, U]$,

$$\widehat{S}_t(\kappa) = B_t\, \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{U<\tau\leq T\}} B_\tau^{-1} Z_\tau \,\Big|\, \mathcal{G}_t\Big) - \kappa\, B_t\, \mathbb{E}_{\mathbb{Q}}\Big(\int_{]t\wedge U, \tau\wedge T]} B_u^{-1}\, dL_u \,\Big|\, \mathcal{G}_t\Big).$$

Valuation of a Forward CDS

Lemma

The value of a credit default swap started at s, equals, for every $t \in [s, U]$,

$$\widehat{S}_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G_t} \mathbb{E}_{\mathbb{Q}} \left(-\int_U^T B_u^{-1} \delta_u \, dG_u - \kappa \int_{]U,T]} B_u^{-1} G_u \, dL_u \,\Big|\, \mathcal{F}_t \right).$$

Note that $\widehat{S}_t(\kappa) = \mathbb{1}_{\{t<\tau\}} S_t(\kappa)$ where the $\mathbb{F}$-adapted process $S(\kappa)$ is the pre-default value. Moreover

$$S_t(\kappa) = P(t, U, T) - \kappa \, A(t, U, T)$$

where

- $P(t, U, T)$ is the pre-default value of the protection leg,
- $A(t, U, T)$ is the pre-default value of the fee leg per one unit of κ.

Forward CDS Spread

- The forward CDS spread is defined similarly as the forward swap rate for a default-free interest rate swap.

Definition

The forward market CDS at time $t \in [0, U]$ is the forward CDS in which the $\mathcal{F}_t$-measurable rate κ is such that the contract is valueless at time t.

The corresponding pre-default forward CDS spread at time t is the unique $\mathcal{F}_t$-measurable random variable $\kappa(t, U, T)$ that solves the equation

$$S_t(\kappa(t, U, T)) = 0.$$

- Recall that for any $\mathcal{F}_t$-measurable rate κ we have that

$$S_t(\kappa) = P(t, U, T) - \kappa A(t, U, T).$$

Forward CDS Spread

Lemma

For every $t \in [0, U]$,

$$\kappa(t,U,T) = \frac{P(t,U,T)}{A(t,U,T)} = -\frac{\mathbb{E}_{\mathbb{Q}}\Big(\int_U^T B_u^{-1} \delta_u \, dG_u \,\Big|\, \mathcal{F}_t \Big)}{\mathbb{E}_{\mathbb{Q}}\Big(\int_{]U,T]} B_u^{-1} G_u \, dL_u \,\Big|\, \mathcal{F}_t \Big)} = \frac{M_t^P}{M_t^A}$$

where the $(\mathbb{Q}, \mathbb{F})$*-martingales* M^P *and* M^A *are given by*

$$M_t^P = -\,\mathbb{E}_{\mathbb{Q}}\Big(\int_U^T B_u^{-1} \delta_u \, dG_u \,\Big|\, \mathcal{F}_t \Big)$$

and

$$M_t^A = \mathbb{E}_{\mathbb{Q}}\Big(\int_{]U,T]} B_u^{-1} G_u \, dL_u \,\Big|\, \mathcal{F}_t \Big).$$

Martingale Measure

Define an equivalent probability measure $\widehat{\mathbb{Q}}$ on $(\Omega, \mathcal{F}_U)$ by setting

$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{Q}} = \frac{M_U^A}{M_0^A}, \quad \mathbb{Q}\text{-a.s.}$$

Lemma

The forward CDS spread $(\kappa(t, U, T),\ t \leq R)$ is a $(\widehat{\mathbb{Q}}, \mathbb{F})$-martingale.

The forward swap measure corresponds to the choice of the pre-default swap annuity $A(t, U, T)$ as a numéraire.

Definition

The probability measure $\widehat{\mathbb{Q}}$ is called the forward swap measure.

Credit Default Swaptions

Credit Default Swaption

Definition

A credit default swaption is a call option with expiry date $R \leq U$ and zero strike written on the value of the forward CDS issued at time $0 \leq s < R$, with start date U, maturity T, and an $\mathcal{F}_s$-measurable rate κ.

The swaption's payoff $\widehat{C}_R$ at expiry equals $\widehat{C}_R = (\widehat{S}_R(\kappa))^+$.

Lemma

For a forward CDS with an $\mathcal{F}_s$-measurable rate κ, for every $t \in [s, U]$,

$$\widehat{S}_t(\kappa) = \mathbb{1}_{\{t<\tau\}} A(t, U, T)(\kappa(t, U, T) - \kappa).$$

Hence

$$\widehat{C}_R = \mathbb{1}_{\{R<\tau\}} A(R, U, T)(\kappa(R, U, T) - \kappa)^+.$$

A credit default swaption is equivalent to a call option on the forward CDS spread with strike κ. This option is knocked out if default occurs prior to R.

Credit Default Swaption

Lemma

The price at time $t \in [s, R]$ of a credit default swaption equals

$$\widehat{C}_t = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G_t} \mathbb{E}_{\mathbb{Q}} \left(\frac{G_R}{B_R} A(R, U, T)(\kappa(R, U, T) - \kappa)^+ \,\Big|\, \mathcal{F}_t \right).$$

Recall that the probability measure $\widehat{\mathbb{Q}}$ on $(\Omega, \mathcal{F}_R)$ is given by

$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{Q}} = \frac{M_R^A}{M_0^A} = \frac{1}{M_0^A} \frac{G_R}{B_R} A(R, U, T), \quad \mathbb{Q}\text{-a.s.}$$

Proposition

The price of the credit default swaption equals, for every $t \in [s, R]$,

$$\widehat{C}_t = \mathbb{1}_{\{t<\tau\}} A(t, U, T) \, \mathbb{E}_{\widehat{\mathbb{Q}}} \left((\kappa(R, U, T) - \kappa)^+ \,\big|\, \mathcal{F}_t \right) = \mathbb{1}_{\{t<\tau\}} C_t.$$

The forward CDS spread $(\kappa(t, U, T),\ t \le R)$ is a $(\widehat{\mathbb{Q}}, \mathbb{F})$-martingale.

Brownian Case

- Let the filtration $\mathbb{F}$ be generated by a Brownian motion W under $\mathbb{Q}$.
- Since M^P and M^A are strictly positive $(\mathbb{Q}, \mathbb{F})$-martingales, we have that

$$dM_t^P = M_t^P \sigma_t^P \, dW_t, \quad dM_t^A = M_t^A \sigma_t^A \, dW_t,$$

for some $\mathbb{F}$-adapted processes σ^P and σ^A.

Lemma

The forward CDS spread $(\kappa(t, U, T),\ t \in [0, R])$ *is* $(\widehat{\mathbb{Q}}, \mathbb{F})$*-martingale and*

$$d\kappa(t, U, T) = \kappa(t, U, T)\sigma_t^\kappa \, d\widehat{W}_t$$

where $\sigma^\kappa = \sigma^P - \sigma^A$ *and the* $(\widehat{\mathbb{Q}}, \mathbb{F})$*-Brownian motion* $\widehat{W}$ *equals*

$$\widehat{W}_t = W_t - \int_0^t \sigma_u^A \, du, \quad \forall\, t \in [0, R].$$

Market Formula for Credit Default Swaptions

Proposition

Assume that the volatility $\sigma^\kappa = \sigma^P - \sigma^A$ *of the forward CDS spread is deterministic. Then the pre-default value of the credit default swaption with strike level* κ *and expiry date* R *equals, for every* $t \in [0, U]$,

$$C_t = A_t \Big(\kappa_t \, N\big(d_+(\kappa_t, U - t)\big) - \kappa \, N\big(d_-(\kappa_t, U - t)\big) \Big)$$

where $\kappa_t = \kappa(t, U, T)$ *and* $A_t = A(t, U, T)$. *Equivalently,*

$$C_t = P_t \, N\big(d_+(\kappa_t, t, R)\big) - \kappa \, A_t \, N\big(d_-(\kappa_t, t, R)\big)$$

where $P_t = P(t, U, T)$ *and*

$$d_\pm(\kappa_t, t, R) = \frac{\ln(\kappa_t/\kappa) \pm \frac{1}{2} \int_t^R (\sigma^\kappa(u))^2 \, du}{\sqrt{\int_t^R (\sigma^\kappa(u))^2 \, du}} \, .$$

Market Models for CDS Spreads

Notation

- Let $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{Q})$ be a filtered probability space, where $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ is a filtration such that $\mathcal{F}_0$ is trivial.
- We assume that the random time τ defined on this space is such that the $\mathbb{F}$-*survival process* $G_t = \mathbb{Q}(\tau > t \,|\, \mathcal{F}_t)$ is positive.
- The probability measure $\mathbb{Q}$ is interpreted as the risk-neutral measure.
- Let $\mathcal{T} = \{T_0 < T_1 < \cdots < T_n\}$ and let $a_i = T_i - T_{i-1}$.
- We denote $\widehat{a}_i = a_i/(1 - \delta_i)$ where δ_i is the recovery rate if default occurs between T_{i-1} and T_i.
- We denote by $\beta(t, T)$ the default-free discount factor for $[t, T]$.

Notation

- We consider a stylized forward CDS starting at T_i and maturing at T_k.
- The pre-default value at time $t \in [0, T_i]$ of the *defaultable annuity* equals

$$A_t^{i,k} := \sum_{j=i+1}^{k} a_j G_t^{-1} \, \mathbb{E}_{\mathbb{Q}} \left(\beta(t, T_j) \mathbb{1}_{\{\tau > T_j\}} \,\middle|\, \mathcal{F}_t \right).$$

- The pre-default value at time $t \in [0, T_i]$ of the *protection leg* equals

$$P_t^{i,k} := \sum_{j=i+1}^{k} \delta_j G_t^{-1} \, \mathbb{E}_{\mathbb{Q}} \left(\beta(t, T_j) \mathbb{1}_{\{T_{j-1} < \tau \leq T_j\}} \,\middle|\, \mathcal{F}_t \right),$$

  where $\delta_j \in [0, 1)$ is the constant protection payment if default occurs between T_{j-1} and T_j.
- The pre-default forward CDS spread equals

$$\kappa_t^{i,k} = \frac{P_t^{i,k}}{A_t^{i,k}}, \qquad t \in [0, T_i].$$

One-Period and Co-Terminal CDS Spreads

Bottom-Up Approach: Forward CDS Spreads

- For $i = 0, \dots, n-1$, the one-period forward CDS spread with start date T_i and maturity T_{i+1} equals

$$\kappa_t^i := \kappa_t^{i,i+1} = \frac{P_t^{i,i+1}}{A_t^{i,i+1}}, \quad \forall\, t \in [0, T_i].$$

- For $i = 0, \dots, n-1$, the co-terminal forward CDS spread with start date T_i and maturity T_n equals

$$\widehat{\kappa}_t^i := \kappa_t^{i,n} = \frac{P_t^{i,n}}{A_t^{i,n}}, \quad \forall\, t \in [0, T_i].$$

Note that $\kappa_t^{n-1} = \widehat{\kappa}_t^{n-1}$.

Lemma

The following equalities hold, for $i = 0, \dots, n-1$ and $t \in [0, T_i]$,

$$\frac{A_t^{i+1,n}}{A_t^{i,n}} = \frac{\widehat{\kappa}_t^i - \kappa_t^i}{\widehat{\kappa}_t^{i+1} - \kappa_t^i}, \qquad \frac{A_t^{i,i+1}}{A_t^{i,n}} = \frac{\widehat{\kappa}_t^{i+1} - \widehat{\kappa}_t^i}{\widehat{\kappa}_t^{i+1} - \kappa_t^i}.$$

Bottom-Up Approach: Martingale Measures

1. The process $\widehat{A}^{0,n}_t := G_t B_t^{-1} A^{0,n}_t$ is a positive $(\mathbb{Q}, \mathbb{F})$-martingale and thus it defines the probability measure $\widehat{\mathbb{P}}^0$ on $(\Omega, \mathcal{F}_{T_n})$.
2. We define a family of probability measures $\mathbb{P}^i$ for $i = 0, \dots, n-2$ and $\widehat{\mathbb{P}}^i$ for $i = 0, \dots, n-1$, such that κ^i is a $(\mathbb{P}^i, \mathbb{F})$-martingale and $\widehat{\kappa}^i$ is a $(\widehat{\mathbb{P}}^i, \mathbb{F})$-martingale.

$$
\begin{array}{ccccccccccc}
\mathbb{Q} & \xrightarrow{\frac{d\widehat{\mathbb{P}}^0}{d\mathbb{Q}}} & \widehat{\mathbb{P}}^0 & \xrightarrow{\frac{d\widehat{\mathbb{P}}^1}{d\widehat{\mathbb{P}}^0}} & \widehat{\mathbb{P}}^1 & \xrightarrow{\frac{d\widehat{\mathbb{P}}^2}{d\widehat{\mathbb{P}}^1}} & \dots & \xrightarrow{\frac{d\widehat{\mathbb{P}}^{n-2}}{d\widehat{\mathbb{P}}^{n-3}}} & \widehat{\mathbb{P}}^{n-2} & \xrightarrow{\frac{d\widehat{P}^{n-1}}{d\widehat{\mathbb{P}}^{n-2}}} & \widehat{\mathbb{P}}^{n-1} = \mathbb{P}^{n-1} \\
 & & \frac{d\mathbb{P}^0}{d\widehat{\mathbb{P}}^0}\Big\downarrow & & \frac{d\mathbb{P}^1}{d\widehat{\mathbb{P}}^1}\Big\downarrow & & \Big\downarrow & & \frac{d\mathbb{P}^{n-2}}{d\widehat{\mathbb{P}}^{n-2}}\Big\downarrow & & \\
 & & \mathbb{P}^0 & \xrightarrow{\frac{d\mathbb{P}^1}{d\mathbb{P}^0}} & \mathbb{P}^1 & \xrightarrow{\frac{d\mathbb{P}^2}{d\mathbb{P}^1}} & \dots & \xrightarrow{\frac{d\mathbb{P}^{n-2}}{d\mathbb{P}^{n-3}}} & \mathbb{P}^{n-2} & \xrightarrow{\frac{d\mathbb{P}^{n-1}}{d\mathbb{P}^{n-2}}} & \mathbb{P}^{n-1}
\end{array}
$$

Martingale Measures

1. We obtain the following family of the Radon-Nikodým densities, for every $i = 0, \dots n-1$ and every $t \in [0, T_0]$,

$$\frac{d\widehat{\mathbb{P}}^i}{d\widehat{\mathbb{P}}^0}\Big|_{\mathcal{F}_t} := \frac{A_t^{i,n}}{A_t^{0,n}} = \prod_{j=0}^{i-1} \frac{\widehat{\kappa}_t^j - \kappa_t^j}{\widehat{\kappa}_t^{j+1} - \kappa_t^j},$$

$$\frac{d\mathbb{P}^i}{d\widehat{\mathbb{P}}^0}\Big|_{\mathcal{F}_t} := \frac{A_t^{i,i+1}}{A_t^{0,n}} = \frac{A_t^{i,i+1}}{A_t^{i,n}} \frac{A_t^{i,n}}{A_t^{0,n}} = \frac{\widehat{\kappa}_t^{i+1} - \widehat{\kappa}_t^i}{\widehat{\kappa}_t^{i+1} - \kappa_t^i} \prod_{j=0}^{i-1} \frac{\widehat{\kappa}_t^j - \kappa_t^j}{\widehat{\kappa}_t^{j+1} - \kappa_t^j}.$$

2. It is now not difficult to derive the joint dynamics under $\widehat{\mathbb{P}}^0$ of processes $\kappa^0, \dots, \kappa^{n-1}$ and $\widehat{\kappa}^0, \dots, \widehat{\kappa}^{n-2}$.

Top-Down Approach: Postulates

We are given a filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ and we postulate that:

1. The initial values of processes $\kappa^0, \dots, \kappa^{n-2}$ and $\widehat{\kappa}^0, \dots, \widehat{\kappa}^{n-1}$ are given.
2. For every $i = 0, \dots, n-1$, the process $Z^{\widehat{\kappa}, i}$, which is given by the formula (by the usual convention, $Z^{\widehat{\kappa}, 0} = 1$)

$$Z_t^{\widehat{\kappa}, i} = \widehat{c}_i \prod_{j=0}^{i-1} \frac{\widehat{\kappa}_t^j - \kappa_t^j}{\widehat{\kappa}_t^{j+1} - \kappa_t^j}, \quad \forall\, t \in [0, T_0],$$

   is a positive $(\mathbb{P}, \mathbb{F})$-martingale and $\widehat{c}_i$ is a constant such that $Z_0^{\widehat{\kappa}, i} = 1$.
3. For every $i = 0, \dots, n-2$, the process $Z^{\kappa, i}$, which is given by the formula

$$Z_t^{\kappa, i} = c_i \frac{\widehat{\kappa}_t^{i+1} - \widehat{\kappa}_t^i}{\widehat{\kappa}_t^{i+1} - \kappa_t^i} Z_t^{\widehat{\kappa}, i}, \quad \forall\, t \in [0, T_0],$$

   is a positive $(\mathbb{P}, \mathbb{F})$-martingale and c_i is a constant such that $Z_0^{\kappa, i} = 1$.
4. For every $i = 0, \dots, n-2$, the process κ^i is a $(\mathbb{P}^i, \mathbb{F})$-martingale, where the Radon-Nikodým density of $\mathbb{P}^i$ with respect to $\mathbb{P}$ equals $Z^{\kappa, i}$.

Top-Down Approach: Postulates

1. For every $i = 0, \ldots, n-2$, the process κ^i satisfies

$$\kappa^i_t = \kappa^i_0 + \int_{(0,t]} \kappa^i_{s-} \widetilde{\sigma}^i_s \cdot d\widetilde{N}^i_s,$$

   where $\widetilde{N}^i$ is an $\mathbb{R}^{k_i}$-valued $(\mathbb{P}^i, \mathbb{F})$-martingale and $\widetilde{\sigma}^i$ is an $\mathbb{R}^{k_i}$-valued, $\mathbb{F}$-predictable, $\widetilde{N}^i$-integrable process.

2. For every $i = 0, \ldots, n-1$, the process $\widehat{\kappa}^i$ is a $(\widehat{\mathbb{P}}^i, \mathbb{F})$-martingale, where the Radon-Nikodým density of $\widehat{\mathbb{P}}^i$ with respect to $\mathbb{P}$ equals $Z^{\widehat{\kappa},i}$.

3. For every $i = 0, \ldots, n-1$, the process $\widehat{\kappa}^i$ satisfies

$$\widehat{\kappa}^i_t = \widehat{\kappa}^i_0 + \int_{(0,t]} \widehat{\kappa}^i_{s-} \widehat{\zeta}^i_s \cdot d\widehat{N}^i_s,$$

   where $\widehat{N}^i$ is an $\mathbb{R}^{l_i}$-valued $(\widehat{\mathbb{P}}^i, \mathbb{F})$-martingale and $\widehat{\zeta}^i$ is an $\mathbb{R}^{l_i}$-valued, $\mathbb{F}$-predictable, $\widehat{N}^i$-integrable process.

Top-Down Approach: Joint Dynamics

- Let $k = k_0 + \cdots + k_{n-2} + l_0 + \cdots + l_{n-1}$. There exists an $\mathbb{R}^k$-valued $(\mathbb{P}, \mathbb{F})$-martingale M such that for every $i = 0, \ldots, n-2$, the $(\mathbb{P}^i, \mathbb{F})$-martingale κ^i admits the following representation under $\mathbb{P}^i$

$$\kappa^i_t = \kappa^i_0 + \int_{(0,t]} \kappa^i_{s-} \sigma^i_s \cdot d\Psi^i(M)_s,$$

  where $\sigma^i = (\sigma^{i,1}, \ldots, \sigma^{i,k})$ is an $\mathbb{R}^k$-valued, $\mathbb{F}$-predictable process extending $\widetilde{\sigma}^i$.

- The $(\widehat{\mathbb{P}}^i, \mathbb{F})$-martingale $\widehat{\kappa}^i$ has the following representation under $\widehat{\mathbb{P}}^i$, for every $i = 0, \ldots, n-1$,

$$\widehat{\kappa}^i_t = \widehat{\kappa}^i_0 + \int_{(0,t]} \widehat{\kappa}^i_{s-} \zeta^i_s \cdot d\widehat{\Psi}^i(M)_s,$$

  where $\zeta^i = (\zeta^{i,1}, \ldots, \zeta^{i,k})$ is an $\mathbb{R}^k$-valued, $\mathbb{F}$-predictable process, which extends $\widehat{\zeta}^i$.

Top-Down Approach: Joint Dynamics

- The $\mathbb{R}^k$-valued $(\mathbb{P}^i, \mathbb{F})$-martingale $\Psi^i(M)$ satisfies, for every $l = 1, \dots, k$,

$$\Psi^i(M^l)_t = M^l_t - \left[\mathcal{L}(Z^{\kappa,i})^c, M^{l,c}\right]_t - \sum_{0<s\le t} \frac{1}{Z^{\kappa,i}_s} \Delta Z^{\kappa,i}_s \Delta M^l_s.$$

- The $\mathbb{R}^k$-valued $(\widehat{\mathbb{P}}^i, \mathbb{F})$-martingale $\widehat{\Psi}^i(M)$ satisfies, for every $l = 1, \dots, k$,

$$\widehat{\Psi}^i(M^l)_t = M^l_t - \left[\mathcal{L}(Z^{\widehat{\kappa},i})^c, M^{l,c}\right]_t - \sum_{0<s\le t} \frac{1}{Z^{\widehat{\kappa},i}_s} \Delta Z^{\widehat{\kappa},i}_s \Delta M^l_s.$$

- Note that $\mathbb{P} = \widehat{\mathbb{P}}^0$ and thus $\widehat{\Psi}^0(M) = M$. Consequently, the forward CDS spread $\widehat{\kappa}^0$ satisfies

$$\widehat{\kappa}^0_t = \widehat{\kappa}^0_0 + \int_{(0,t]} \widehat{\kappa}^0_{s-} \zeta^0_s \cdot dM_s.$$

This agrees with the fact that $\widehat{\mathbb{P}}^0$ is the martingale measure for $\widehat{\kappa}^0$.

Top-Down Approach: Joint Dynamics

Proposition

The semi-martingale decomposition of the spanning $(\widehat{\mathbb{P}}^i, \mathbb{F})$-martingale $\widehat{\Psi}^i(M)$ under the probability measure $\mathbb{P} = \widehat{\mathbb{P}}^0$ is given by, for every $i = 0, \ldots, n-2$,

$$
\begin{aligned}
\widehat{\Psi}^i(M)_t &= M_t - \sum_{j=0}^{i-1} \int_{(0,t]} \frac{(\widehat{\kappa}^j_s - \widehat{\kappa}^{j+1}_s)\, \kappa^j_s \sigma^j_s \cdot d[M^c]_s}{(\widehat{\kappa}^j_s - \kappa^j_s)(\widehat{\kappa}^{j+1}_s - \kappa^j_s)} - \sum_{j=0}^{i-1} \int_{(0,t]} \frac{\widehat{\kappa}^j_s \zeta^j_s \cdot d[M^c]_s}{\widehat{\kappa}^j_s - \kappa^j_s} \\
&\quad + \sum_{j=0}^{i-1} \int_{(0,t]} \frac{\widehat{\kappa}^{j+1}_s \zeta^{j+1}_s \cdot d[M^c]_s}{\widehat{\kappa}^{j+1}_s - \kappa^j_s} - \sum_{0<s\le t} \frac{1}{Z^{\widehat{\kappa},i}_s} \Delta Z^{\widehat{\kappa},i}_s \Delta M_s.
\end{aligned}
$$

Top-Down Approach: Joint Dynamics

Proposition

The semi-martingale decomposition of the spanning $(\mathbb{P}^i, \mathbb{F})$*-martingale* $\Psi^i(M)$ *under the probability measure* $\mathbb{P} = \widehat{\mathbb{P}}^0$ *is given by, for every* $i = 0, \ldots, n-1$,

$$\begin{aligned}\Psi^i(M)_t &= M_t - \int_{(0,t]} \frac{(\kappa_s^i - \widehat{\kappa}_s^i)\,\widehat{\kappa}_s^{i+1}\zeta_s^{i+1} \cdot d[M^c]_s}{(\widehat{\kappa}_s^{i+1} - \kappa_s^i)(\widehat{\kappa}_s^{i+1} - \widehat{\kappa}_s^i)} + \int_{(0,t]} \frac{\kappa_s^i \sigma_s^i \cdot d[M^c]_s}{(\widehat{\kappa}_s^{i+1} - \kappa_s^i)} \\ &\quad - \int_{(0,t]} \frac{\widehat{\kappa}_s^i \zeta_s^i \cdot d[M^c]_s}{\widehat{\kappa}_s^{i+1} - \widehat{\kappa}_s^i} - \sum_{j=i+1}^{n} \int_{(0,t]} \frac{(\widehat{\kappa}_s^j - \widehat{\kappa}_s^{j+1})\,\kappa_s^j \sigma_s^j \cdot d[M^c]_s}{(\widehat{\kappa}_s^j - \kappa_s^j)(\widehat{\kappa}_s^{j+1} - \kappa_s^j)} \\ &\quad - \sum_{j=i+1}^{n} \int_{(0,t]} \frac{\widehat{\kappa}_s^j \zeta_s^j \cdot d[M^c]_s}{\widehat{\kappa}_s^j - \kappa_s^j} + \sum_{j=i+1}^{n} \int_{(0,t]} \frac{\widehat{\kappa}_s^{j+1}\zeta_s^{j+1} \cdot d[M^c]_s}{\widehat{\kappa}_s^{j+1} - \kappa_s^j} \\ &\quad - \sum_{0<s\le t} \frac{1}{Z_s^{\kappa,i}} \Delta Z_s^{\kappa,i} \Delta M_s.\end{aligned}$$

Top-Down Approach: Joint Dynamics

Under the martingale measure $\mathbb{P} = \widehat{\mathbb{P}}^0$, for every $i = 0, \dots, n-2$

$$
\begin{aligned}
d\kappa_t^i &= \sum_{l=1}^{k} \kappa_{t-}^i \sigma_t^{i,l}\, dM_t^l - \frac{\kappa_t^i \widehat{\kappa}_t^{i+1}}{\widehat{\kappa}_t^{i+1} - \widehat{\kappa}_t^i} \sum_{l,m=1}^{k} \sigma_t^{i,l} \zeta_t^{i+1,m}\, d[M^{l,c}, M^{m,c}]_t \\
&+ \frac{\kappa_t^i \widehat{\kappa}_t^i}{\widehat{\kappa}_t^{i+1} - \widehat{\kappa}_t^i} \sum_{l,m=1}^{k} \sigma_t^{i,l} \zeta_t^{i,m}\, d[M^{l,c}, M^{m,c}]_t + \frac{\kappa_t^i \widehat{\kappa}_t^{i+1}}{\widehat{\kappa}_t^{i+1} - \kappa_t^i} \sum_{l,m=1}^{k} \sigma_t^{i,l} \zeta_t^{i+1,m}\, d[M^{l,c}, M^{m,c}]_t \\
&- \frac{\kappa_t^i \kappa_t^i}{\widehat{\kappa}_t^{i+1} - \kappa_t^i} \sum_{l,m=1}^{k} \sigma_t^{i,l} \sigma_t^{i,m}\, d[M^{l,c}, M^{m,c}]_t - \sum_{j=0}^{i-1} \frac{\kappa_t^i \widehat{\kappa}_t^j}{\widehat{\kappa}_t^j - \kappa_t^j} \sum_{l,m=1}^{k} \sigma_t^{i,l} \zeta_t^{j,m}\, d[M^{l,c}, M^{m,c}]_t \\
&+ \sum_{j=0}^{i-1} \frac{\kappa_t^i \kappa_t^j}{\widehat{\kappa}_t^j - \kappa_t^j} \sum_{l,m=1}^{k} \sigma_t^{i,l} \sigma_t^{j,m}\, d[M^{l,c}, M^{m,c}]_t + \sum_{j=0}^{i-1} \frac{\kappa_t^i \widehat{\kappa}_t^{j+1}}{\widehat{\kappa}_t^{j+1} - \kappa_t^j} \sum_{l,m=1}^{k} \sigma_t^{i,l} \zeta_t^{j+1,m}\, d[M^{l,c}, M^{m,c}]_t \\
&- \sum_{j=0}^{i-1} \frac{\kappa_t^i \kappa_t^j}{\widehat{\kappa}_t^{j+1} - \kappa_t^j} \sum_{l,m=1}^{k} \sigma_t^{i,l} \sigma_t^{j,m}\, d[M^{l,c}, M^{m,c}]_t - \frac{\kappa_{t-}^i}{Z_t^{\widehat{\kappa},i}} \Delta Z_t^{\widehat{\kappa},i} \sum_{l=1}^{k} \sigma_t^{i,l} \Delta M_t^l
\end{aligned}
$$

Top-Down Approach: Joint Dynamics

and for every $i = 0, \ldots, n-1$

$$
\begin{aligned}
d\widehat{\kappa}^i_t &= \sum_{l=1}^{k} \widehat{\kappa}^i_{t-} \zeta^{i,l}_t \, dM^l_t - \sum_{j=0}^{i-1} \frac{\widehat{\kappa}^i_t \widehat{\kappa}^j_t}{\widehat{\kappa}^j_t - \kappa^j_t} \sum_{l,m=1}^{k} \zeta^{i,l}_t \zeta^{j,m}_t \, d[M^{l,c}, M^{m,c}]_t \\
&+ \sum_{j=0}^{i-1} \frac{\widehat{\kappa}^i_t \kappa^j_t}{\widehat{\kappa}^j_t - \kappa^j_t} \sum_{l,m=1}^{k} \zeta^{i,l}_t \sigma^{j,m}_t \, d[M^{l,c}, M^{m,c}]_t \\
&+ \sum_{j=0}^{i-1} \frac{\widehat{\kappa}^i_t \widehat{\kappa}^{j+1}_t}{\widehat{\kappa}^{j+1}_t - \widehat{\kappa}^j_t} \sum_{l,m=1}^{k} \zeta^{i,l}_t \zeta^{j+1,m}_t \, d[M^{l,c}, M^{m,c}]_t \\
&- \sum_{j=0}^{i-1} \frac{\widehat{\kappa}^i_t \kappa^j_t}{\widehat{\kappa}^{j+1}_t - \widehat{\kappa}^j_t} \sum_{l,m=1}^{k} \zeta^{i,l}_t \sigma^{j,m}_t \, d[M^{l,c}, M^{m,c}]_t \\
&- \frac{\widehat{\kappa}^i_{t-}}{Z^{\kappa,i}_t} \Delta Z^{\kappa,i}_t \sum_{l=1}^{k} \zeta^{i,l}_t \Delta M^l_t .
\end{aligned}
$$

Construction of Default Time

We set, for $i = 1, \dots, n-1$,

$$M_t^{T_i} := \prod_{j=1}^{i} \frac{1}{1 + \widehat{a}_j \kappa_t^j}$$

The process $a_i \kappa^i$ is positive and stopped at T_{i-1} for all i and thus $M_{T_i}^{T_i}$ is decreasing in i

$$M_{T_i}^{T_i} = \frac{M_{T_{i-1}}^{T_{i-1}}}{1 + a_i \kappa_{T_i}^i} \leq M_{T_{i-1}}^{T_{i-1}}, \qquad \forall i = 1, \dots, n$$

This allows for canonical construction of τ with values in $T_1, \dots, T_n$ such that

$$\mathbb{P}^{n-1}(\tau > T_i \,|\, \mathcal{F}_{T_i}) = M_{T_i}^{T_i} := \prod_{j=1}^{i} \frac{1}{1 + \widehat{a}_j \kappa_{T_i}^j}, \quad \forall i = 1, \dots, n.$$

Since M^{T_i} is a martingale under $\mathbb{P}^{i-1}$

$$\mathbb{P}^{i-1}(\tau > T_i \,|\, \mathcal{F}_t) = M_t^{T_i}, \quad \forall i = 1, \dots, n.$$

Towards Generic Swap Models

Assumptions

Let $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ be a filtered probability space. We postulate that the $\mathbb{F}$-adapted processes $\kappa^1, \dots, \kappa^n$ and $Z^1, \dots, Z^n$ satisfy the following conditions, for every $i = 1, \dots, n$

1. the process Z^i is a positive $(\mathbb{P}, \mathbb{F})$-martingale with $\mathbb{E}_{\mathbb{P}}(Z^i_0) = 1$,
2. the process $\kappa^i Z^i$ is a $(\mathbb{P}, \mathbb{F})$-martingale,
3. the process Z^i is given as a function of some subset of the family $\kappa^1, \dots, \kappa^n$; specifically, there exists a subset $\{\kappa^{n_1}, \dots, \kappa^{n_{l_i}}\}$ of $\{\kappa^1, \dots, \kappa^n\}$ and a function $f_i : \mathbb{R}^{l_i} \to \mathbb{R}$ of class C^2 such that

$$Z^i = f_i(\kappa^{n_1}, \dots, \kappa^{n_{l_i}}).$$

Comments

1. Assumptions 1 and 2 yield the existence of a family of probability measures $\mathbb{P}^1, \dots, \mathbb{P}^n$, equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ for some fixed $T > 0$, such that the process κ^i is a $(\mathbb{P}^i, \mathbb{F})$-martingale for every $i = 1, \dots, n$. This in turn implies that κ^i is a $(\mathbb{P}, \mathbb{F})$-semimartingale.
2. Assumption 3 implies that the continuous martingale part of Z^i has the following integral representation

$$Z_t^{i,c} = Z_0^i + \sum_{j=1}^{l_i} \int_0^t \frac{\partial f_i}{\partial x_j}(\kappa_s^{n_1}, \dots, \kappa_s^{n_{l_i}})\, d\kappa_s^{n_j,c}$$

where $\kappa^{j,c}$ stands for the continuous martingale part of κ^j.

Volatilities

1. The semimartingale decomposition of κ^j can be uniquely specified under $\mathbb{P}$ by the choice of the initial values, the volatility processes and the driving martingale, which is, as usual, denoted by M.
2. For the purpose of an explicit construction of the model for processes $\kappa^1, \ldots, \kappa^n$, we select an $\mathbb{R}^k$-valued $(\mathbb{P}, \mathbb{F})$-martingale $M = (M^1, \ldots, M^k)$ and we define the process κ^i under $\mathbb{P}^i$ as follows, for every $i = 1, \ldots, n$

$$\kappa^i_t = \kappa^i_0 + \int_{(0,t]} \kappa^i_{s-} \sigma^i_s \cdot d\Psi^i(M)_s,$$

where σ^i is the $\mathbb{R}^k$-valued volatility process and the $(\mathbb{P}^i, \mathbb{F})$-martingale $\Psi^i(M)$ equals

$$\Psi^i(M)_t = M_t - \int_{(0,t]} \frac{1}{Z^i_s} \, d[Z^{i,c}, M^c]_s - \sum_{0<s\leq t} \frac{1}{Z^i_s} \Delta Z^i_s \Delta M_s.$$

Volatility-Based Modelling

Proposition

For every $i = 1, \dots, n$ the dynamics of the forward CDS spread κ^i are

$$d\kappa^i_t = \sum_{l=1}^{k} \kappa^i_{t-} \sigma^{i,l}_t \, dM^l_t$$

$$- \frac{1}{f_i(\kappa^{n_1}_t, \dots, \kappa^{n_{l_i}}_t)} \sum_{j=1}^{l_i} \frac{\partial f_i}{\partial x_j}(\kappa^{n_1}_t, \dots, \kappa^{n_{l_i}}_t) \sum_{l,m=1}^{k} \kappa^i_t \kappa^{n_j}_t \sigma^{i,l}_t \sigma^{n_j,m}_t \, d[M^{l,c}, M^{m,c}]_t$$

$$- \frac{\kappa^i_{t-}}{Z^i_t} \Delta Z^i_t \sum_{l=1}^{k} \sigma^{i,l}_t \Delta M^l_t.$$

Admissibility of Market Models

Further issues that should be addressed in the context of top-down models:

1. construction of a default time consistent with the dynamics of a given family of forward CDS spreads,
2. positivity of forward CDS spreads given by the model,
3. positivity of prices of zero-coupon defaultable bonds implicit in forward CDS spreads,
4. monotonicity of prices of zero-coupon defaultable bonds with respect to maturity date,
5. positivity of other forward CDS spreads computed within a model.

Admissibility of Market Models

- S. Galluccio, J.-M. Ly, Z. Huang, and O. Scaillet: Theory and calibration of swap market models. *Mathematical Finance* 17 (2007), 111–141.
- R. Pietersz and M. van Regenmortel: Generic market models. *Finance and Stochastics* 10 (2006), 507–528.
- L. Li and M. Rutkowski: Admissibility of generic market models for forward swap rates. Submitted, 2010.
- L. Li and M. Rutkowski: Admissibility of market models for forward CDS spreads. Work in progress.