Contagion and Optimal Portfolio Choice When Asset Returns are Self-Exciting

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based on joint work with

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1. Self-Exciting Jumps and Contagion

- Stock market crashes are very unlikely under standard Brownian-driven statistical models.

- Even more unlikely would be crashes that happen in many if not all markets around the world. Yet even more unlikely would be crashes that happen in close succession, like earthquake aftershocks.

- Despite the predictions of standard models, recurring crises happen every decade or so.

- These crises seldom have discernible economic causes or warnings, and they tend to propagate across the world with little regard for economic fundamentals in the affected markets.

- There is a large literature on contagion, both in economics and in finance.

- Theoretical rationales for the observed contagion
  - Krugman (1979); Gerlach and Smets (1995); Obstfeld (1996); Dornbusch, Park and Claessens (2000); Calvo and Mendoza (2000); Chang and Velasco (2001); Kodres and Pritsker (2002); Nikitin and Smith (2008); Pavlova and Rigobon (2008).

- Empirical measurements of contagion
  - Hamao, Masulis and Ng (1990); Becker, Finnerty and Gupta (1990); Eichengreen, Rose and Wyplosz (1996); Glick and Rose (1999); Kaminsky and Reinhart (2000); Van Rijckeghem and Weder (2001); Forbes and Rigobon (2002); Rigobon (2003); Bae, Karolyi and Stulz (2003); Caramazza, Ricci and Salgado (2004); Hartmann, Straetmans and de Vries (2004); Goetzmann, Li and Rouwenhorst (2005); Dungey and Gonzalez-Hermosillo (2005); Dungey, Fry and Martin (2007).

- In a crisis, a shock somewhere seems to increase the probability of successive shocks not only in the affected sector but also in other economic sectors or regions.

- To model this, we need jumps.
  - The observed clusters are too extreme to be explainable by volatility clusters.
  - With jumps, we need to leave the widely applied class of Lévy processes.
  - Lévy processes have independent increments and so do not allow for any type of serial dependence.
  - Whereas the propagation of jumps over time is a key component we wish to capture.

- So, in this paper, we propose a model for asset return dynamics that captures the cross-sectional and serial dependence observed across stock markets around the world

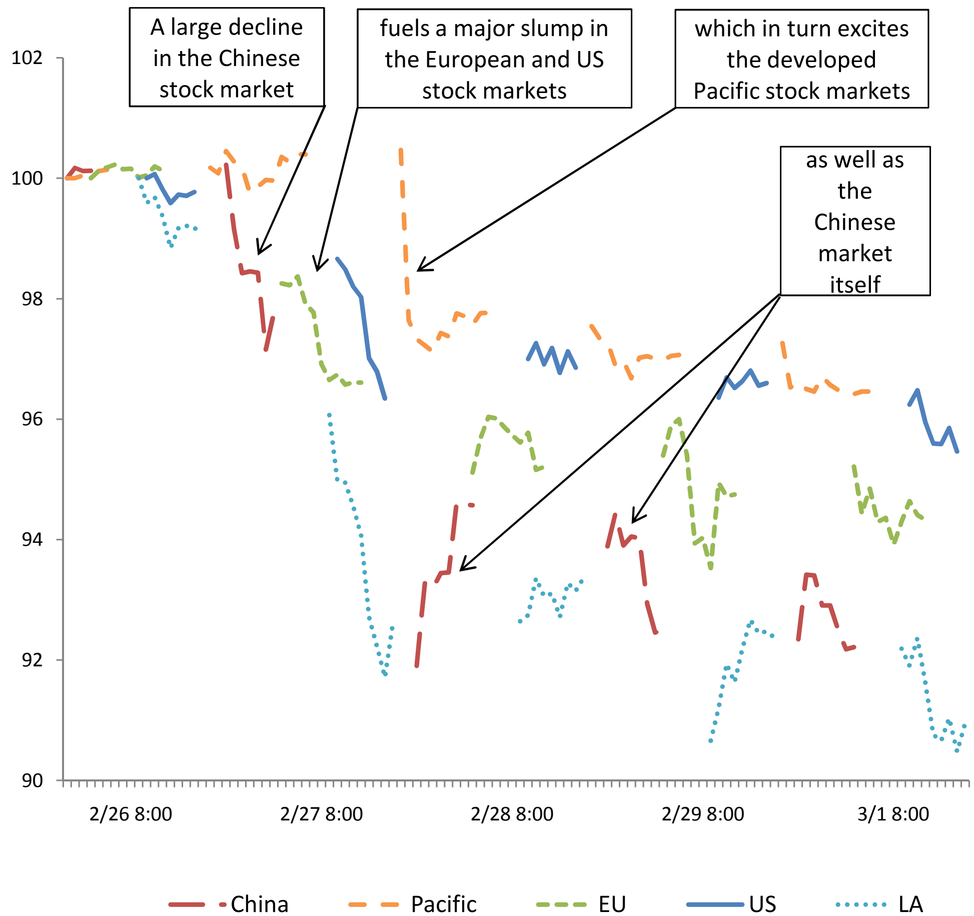
- We suggest to use mutually exciting jumps, known as Hawkes processes after Hawkes (1971).

- Hawkes processes were originally proposed to model epidemics. They have also been used to model earthquake occurrences and more recently joint defaults in credit derivatives (Giesecke (2008)).

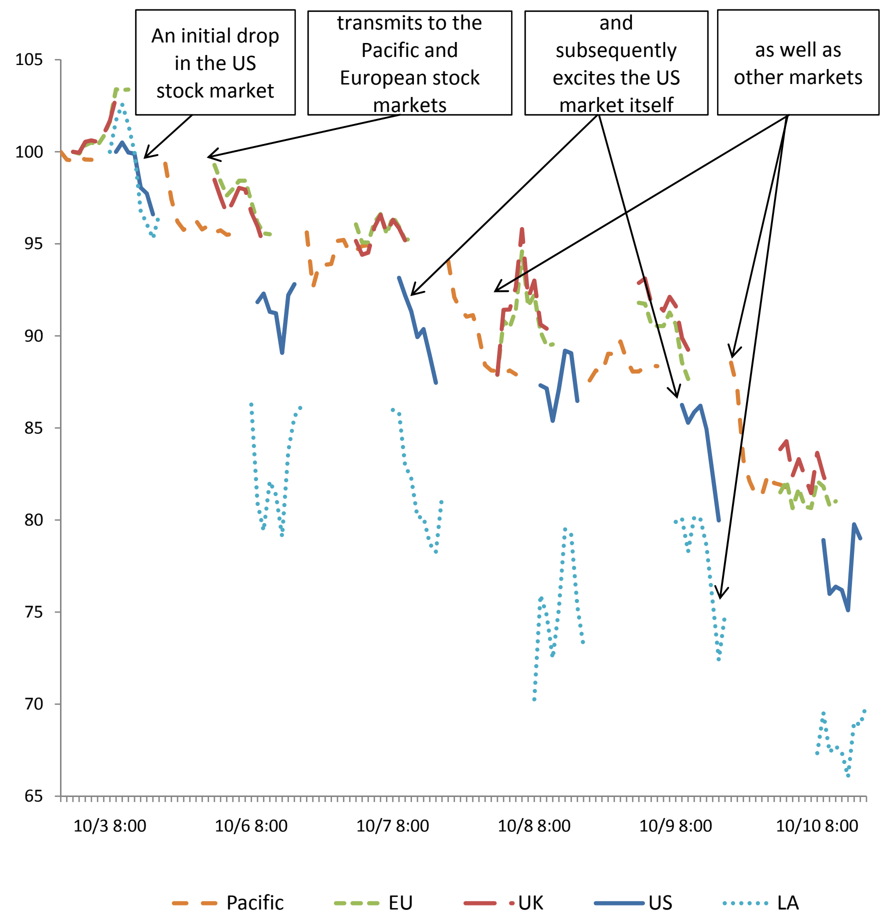
- We will use Hawkes processes to represent the jump part of our price processes, which remain semimartingales.

- Key features
  - A crisis is not just made up of jumps, it also needs them to be amplified across both markets and time.
  - The model generates the type of successive jumps across world markets that are often observed in economic crises.
  - And produces clusters of jumps over time: for example, what was frequently observed in 2008.

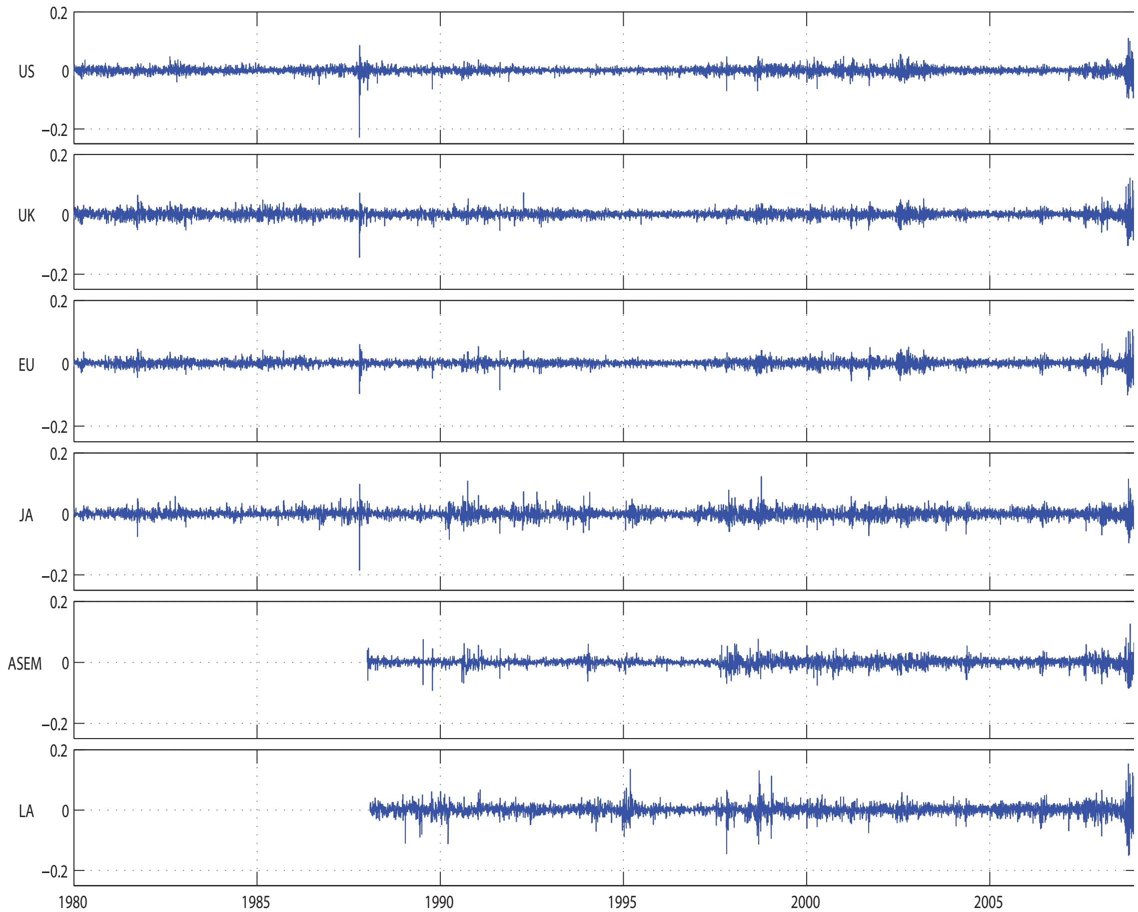
An example: February 26-29, 2007



Another example: October 3-10, 2008



Jump clustering in the full sample



- A self-exciting process is a special case of a point process, whose intensity depends on the path of the underlying process.

- Here, the jump intensity is going to increase in response to recent jumps.

- Consider m point processes $\mathbf{N}_t = \left[N_{1,t}, ..., N_{m,t}\right]$ such that

$$\Pr\left[N_{l,t+\Delta t} - N_{l,t} = 1 | \mathcal{F}_t\right] = \lambda_{l,t}\Delta t + o\left(\Delta t\right)$$
$$\Pr\left[N_{l,t+\Delta t} - N_{l,t} > 1 | \mathcal{F}_t\right] = o\left(\Delta t\right)$$

with jump intensities given by

$$\lambda_{l,t} = \lambda_{\infty,l,t} + \sum_{j=1}^{m} \int_0^t d_{j,l,t-s} dN_{j,s}$$

where $\lambda_{\infty,l,t}$ is a deterministic function.

- There will be one such jump process $N_{l,t}$ for each of the m sectors (or regions of the world).

- Each region's jump intensity is stochastic and depends upon the path of the past jumps:
  - in its own region
  - and in the other regions.

- A special case of particular interest is one where intensities depend on an exponentially weighted moving average of the recent jumps

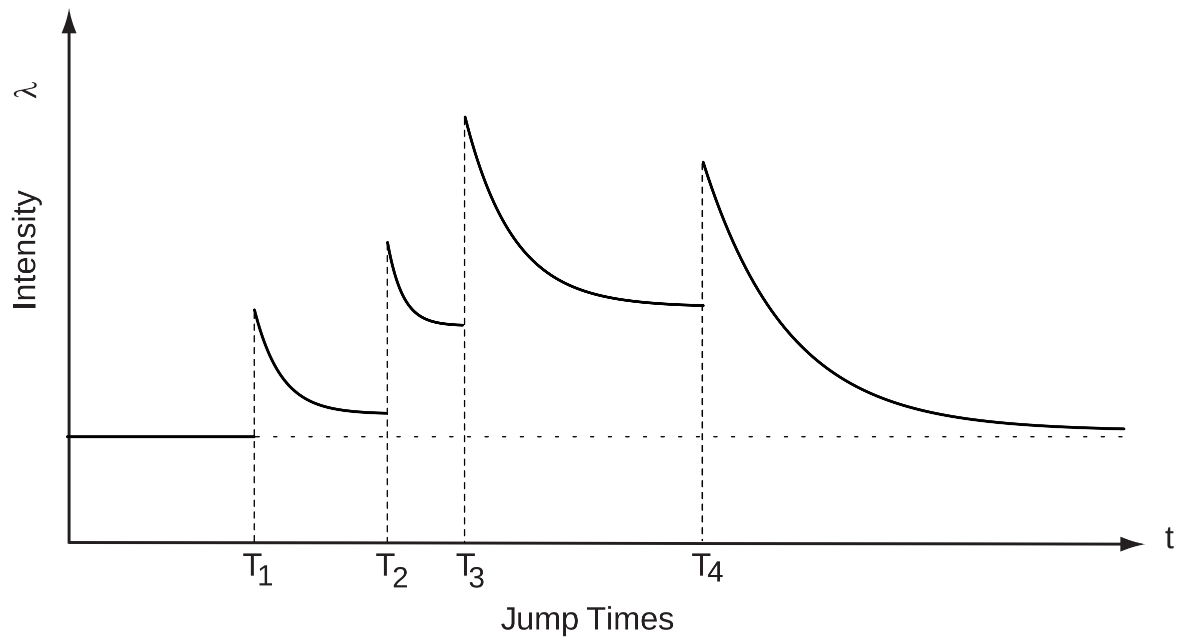
$$d_{j,l,s} = \delta_{j,l} e^{-\alpha_l s}$$

where α_l and $\delta_{l,j}$ are positive constant parameters such that $\alpha_l > \Sigma_{j=1}^{m} \delta_{l,j}$.

- The intensities then become stationary Markov processes with $\lambda_{l,t}$ jumping up by $\delta_{j,l}$ whenever a shock in sector j occurs, and then decaying exponentially back towards $\lambda_{l,\infty}$:

$$d\lambda_{l,t} = \alpha_l \left(\lambda_{l,\infty} - \lambda_{l,t} \right) dt + \sum_{j=1}^{m} \delta_{l,j} dN_{j,t}$$

Sample path of an intensity $\lambda_{l,t}$



- These jumps, by virtue of their self-excitation, introduce a feedback element.
  - This can be thought of as playing the same role for jumps as ARCH does for volatility.
  - Engle (1982)'s ARCH introduces feedback from returns to volatility and back: large returns $\longrightarrow$ large volatilities $\longrightarrow$ more likely to observe large returns. Then mean reversion.
  - In the absence of further excitation, volatility then reverts to its steady state level.
  - Here, similarly, jumps $\longrightarrow$ larger jump intensities$\longrightarrow$ more likely to observe further jumps. Then mean reversion.

- Our model for asset prices: to these jumps, we add a drift and stochastic volatility to produce what me might call a "Hawkes-diffusion" process.
  - Basically, self-exciting jumps are there to capture crises
  - The rest of the model is there to represent the evolution of the asset returns the rest of the time.

- m sectors (or regions) with k firms (or countries) each

- $n = mk$ assets

- Model for asset prices $j = 1, \ldots, n$

$$dX_{i,t} = (r + R_i)\,dt + \sum_{j=1}^{n} \sigma_{i,j,t} dW_{j,t} + \sum_{l=1}^{m} \delta_{i,l} J_i Z_{l,t} dN_{l,t}$$

where N_l is a self-exciting process.

2. Inference for Self-Exciting Jumps

- We use Morgan Stanley Capital International (MSCI) equity index data, for six series: US; Latin America (LA); UK; Developed European countries (EU); Japan (JA); Emerging markets Asia (ASEM).

- Daily data are available from January 1, 1980 (for US, UK, EU and JA), from January 29, 1988 (for LA) and from January 1, 1988 (for ASEM).

- Inference is based on GMM

- We compute the first four moments explicitly

  - first the conditional moments using the full state vector: asset returns, stochastic volatilities, jump intensities

  - then condition down by taking expected values over the latent state variables: volatilities and jump intensities

- We use as moment functions, for each pair of assets

$$
\begin{aligned}
&\mathbb{E}\left[\Delta X_{j,t}\right] \\
&\mathbb{E}\left[(\Delta X_{j,t} - \mathbb{E}\left[\Delta X_{j,t}\right])^r\right], \qquad r = 2, \ldots, 4 \\
&\mathbb{E}\left[\Delta X_{j,t}\Delta X_{k,t} - \mathbb{E}\left[\Delta X_{j,t}\right]\mathbb{E}\left[\Delta X_{k,t}\right]\right], \qquad j \neq k \\
&\mathbb{E}\left[\Delta X_{j,t+\tau}\Delta X_{k,t} - \mathbb{E}\left[\Delta X_{j,t}\right]\mathbb{E}\left[\Delta X_{k,t}\right]\right], \qquad \tau > 0
\end{aligned}
$$

- Higher order moments ($r > 2$) separate jumps from volatility characteristics

- Auto- and cross-correlation moments isolate the self-excitation component of the model

- Example: Moments in the univariate case $m = 1$

- In this situation, there is a single asset with stochastic volatility and jumps that self-excite (meaning that future jump intensities depend upon the history of past jumps):

$$\begin{cases} dX_t = \mu dt + V_t^{1/2} dW_t^X + Z_t dN_t \\ dV_t = \kappa(\theta - V_t)dt + \eta V_t^{1/2} dW_t^V \\ d\lambda_t = \alpha \left(\lambda_\infty - \lambda_t\right) dt + \beta dN_t \end{cases}$$

with $\mathbb{E}\left[dW_t^X dW_t^V\right] := \rho dt$ and $\lambda := \mathbb{E}\left[\lambda_t\right] = \alpha\lambda_\infty/(\alpha - \beta)$.

- We can leave the distribution of the jump size essentially unrestricted, and provide expressions as functions of the moments of the jump size Z_t. Let $M[Z, k] := \mathbb{E}\left[Z_t^k\right]$.

Theorem 1: For the univariate model, the moments are given in closed-form up to order Δ^2 by the following expressions

$$\begin{aligned}
\mathbb{E}\left[\Delta X_t\right] &= (\mu + \lambda M[Z,1])\Delta + o(\Delta^2) \\
\mathbb{E}\left[(\Delta X_t - \mathbb{E}\left[\Delta X_t\right])^2\right] &= (\theta + \lambda M[Z,2])\Delta + \frac{\beta\lambda\,(2\alpha-\beta)}{2(\alpha-\beta)} M(Z,1)^2 \Delta^2 + o(\Delta^2) \\
\mathbb{E}\left[(\Delta X_t - \mathbb{E}\left[\Delta X_t\right])^3\right] &= \lambda M[Z,3]\Delta \\
&\quad + \frac{3}{2}\left(\eta\theta\rho + \frac{(2\alpha-\beta)\beta\lambda M\left[Z,1\right] M\left[Z,2\right]}{(\alpha-\beta)}\right)\Delta^2 + o(\Delta^2) \\
\mathbb{E}\left[(\Delta X_t - \mathbb{E}\left[\Delta X_t\right])^4\right] &= \lambda M[Z,4]\Delta + \left(\frac{3\theta\eta^2}{2\kappa} + 3\theta^2 + 6\theta\lambda M\left[Z,2\right]\right. \\
&\quad + 3\lambda\left(\lambda + \frac{(2\alpha-\beta)\beta}{2(\alpha-\beta)}\right) M\left[Z,2\right]^2 \\
&\quad \left. + \frac{2(2\alpha-\beta)\beta\lambda M\left[Z,1\right] M\left[Z,3\right]}{(\alpha-\beta)}\right)\Delta^2 + o(\Delta^2)
\end{aligned}$$

while the autocorrelation function of the process is given for all $\tau > 0$ by

$$\mathbb{E}\left[(\Delta X_t - \mathbb{E}\left[\Delta X_t\right])(\Delta X_{t+\tau} - \mathbb{E}\left[\Delta X_{t+\tau}\right])\right] = \frac{\beta\lambda(2\alpha-\beta)}{2(\alpha-\beta)} M(Z,1)^2 e^{-(\alpha-\beta)\tau}\Delta^2 + o(\Delta^2).$$

- The identification of the parameters is achieved as follows:
  - The higher order moments (3 and 4) isolate the jump parameters at the leading order
  - While the variance puts them on an equal footing with the diffusive parameters
  - The autocovariances isolate the self-exciting jump parameters
    * Indeed, if the model had no jump component, then

$$\begin{aligned}\mathbb{E}\left[(\Delta X_t)\,(\Delta X_{t+\tau})\right] &= \mathbb{E}\left[(\Delta X_t)\,\mathbb{E}\left[(\Delta X_{t+\tau})\,|\mathcal{F}_{t+\tau}\right]\right] \\ &= \mu^2\Delta^2\end{aligned}$$

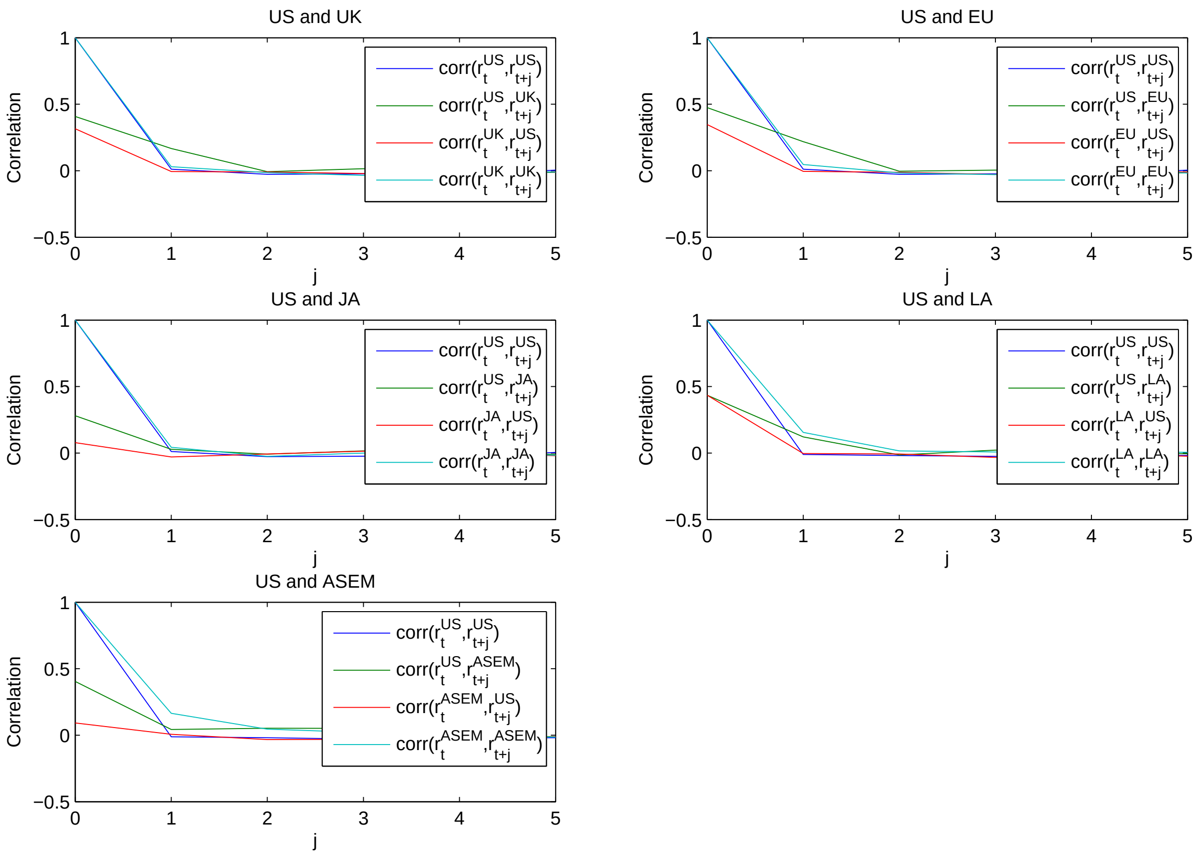
      and so $\mathbb{E}\left[(\Delta X_t - \mathbb{E}\left[\Delta X_t\right])(\Delta X_{t+\tau} - \mathbb{E}\left[\Delta X_{t+\tau}\right])\right] = 0.$
    * Thus any autocovariance is due to the jump component.

  - $*$ Further, if the jump component were Poissonian, then the increments would be independent.
  - $*$ Thus the observed autocovariances of the increments isolate the self-exciting component of the model.

- The same intuition holds in higher dimensions.

- The paper provides explicit expressions for the moments in dimension m.

Auto- and cross-correlations



- Empirically, we find relatively large values for δ_{21}, measuring the degree of transmission from the US to other regions of the world.
  - This implies that when the US jumps, the probability of a consecutive jump in another region of the world becomes large.
  - From the empirical cross-correlation plots, the effect seems to be mainly driven by transmission on the same day or the day following the day of occurrence of a US jump.

- There is little evidence for the reverse excitation (δ_{12} is approximately 0 in all cases).

- Also, we find relatively large values for δ_{11} and δ_{22}, measuring the degree of self-excitation, implying that jumps tend to be clustered in time.

- Testing for the presence of contagion

- In the context of the model, this boils down to testing the join hypothesis that all δ_{ij}'s are 0.

- We can separate between:
  - Self- or time-series contagion: diagonal $\delta_{ii} = 0$
  - Cross-sectional contagion: off-diagonal $\delta_{ij} = 0$, $i \neq j$

3. Optimal Portfolio Choice When Jumps Are Self-Exciting

- The model is reduced-form: it cannot explain the source(s) of the contagion that is observed in the data, or get at the channels of transmission of that contagion, whether trade linkages, financial linkages, financial constraints, outflows of capital, herding behaviors, the fragility of the system, lack of coordinated responses, etc.

- But the model can be employed as a description of the process driving the asset returns, as input for other purposes.

- Asset return dynamics

$$
\begin{aligned}
\frac{dS_{0,t}}{S_{0,t}} &= r dt, \\
\frac{dS_{i,t}}{S_{i,t-}} &= (r + R_i)\, dt + \sum_{j=1}^{n} \sigma_{i,j} dW_{j,t} + \sum_{l=1}^{m} \delta_{i,l} J_i Z_{l,t} dN_{l,t}
\end{aligned}
$$

where $\mathbf{N}_t = \left[N_{1,t}, \ldots, N_{m,t}\right]'$ is an $m-$dimensional mutually exciting process

- The intensity $\lambda_{l,t}$ of $N_{l,t}$ is self-exciting with mean-reversion

$$d\lambda_{l,t} = \alpha_l \left(\lambda_{\infty,l} - \lambda_{l,t}\right) dt + \sum_{j=1}^{m} \delta_{l,j} dN_{j,t}.$$

- The Brownian motions, jump processes and the random jump amplitude variables Z_l are all mutually independent.

- Let $\omega_{0,t}$ denote the weight invested in the riskless asset and $\omega_t = \left[\omega_{1,t}, \ldots, \omega_{n,t}\right]'$ denote the vector of portfolio weights invested in each of the n risky assets.

- The investor's problem at time t is to pick $\{C_s, \omega_s\}_{t \leq s}$ to maximize expected utility

$$V\left(X_t, t\right) = \max_{\{C_s, \omega_s;\ t \leq s\}} E_t \left[\int_t^{\infty} e^{-\beta s} U(C_s) ds\right]$$

subject to the wealth dynamics

$$dX_t = -C_t dt + \sum\nolimits_{i=0}^{n} \omega_{i,t} X_t \frac{dS_{i,t}}{S_{i,t-}}.$$

- We can determine $\omega^*_{n,t}$ in closed form in some specific situations.
  - For example, suppose that the investor has logarithmic utility: $U(x) = \log(x)$ and that the jump sizes are equally distributed: $\nu_l(dz) = \delta(z = \bar{z})$.
  - There is a unique solution $\omega^*_{n,t}$ satisfying the solvency constraint $\omega_{n,t} j_l \bar{z} > -1$:

$$\omega^*_{n,t} = \frac{-\kappa_{1l}/k + j_l \bar{z} r_l + \left((j_l \bar{z} r_l + \kappa_{1l}/k)^2 + 4\lambda_{l,t} j_l^2 \bar{z}^2 \kappa_{1l}/k \right)^{1/2}}{2 j_l \bar{z} \kappa_{1l}/k}$$

- The optimal solution is now time-varying, with the investor reacting to changes in the intensity of the jumps.
- Even though this is a log-utility investor, the solution is not myopic.

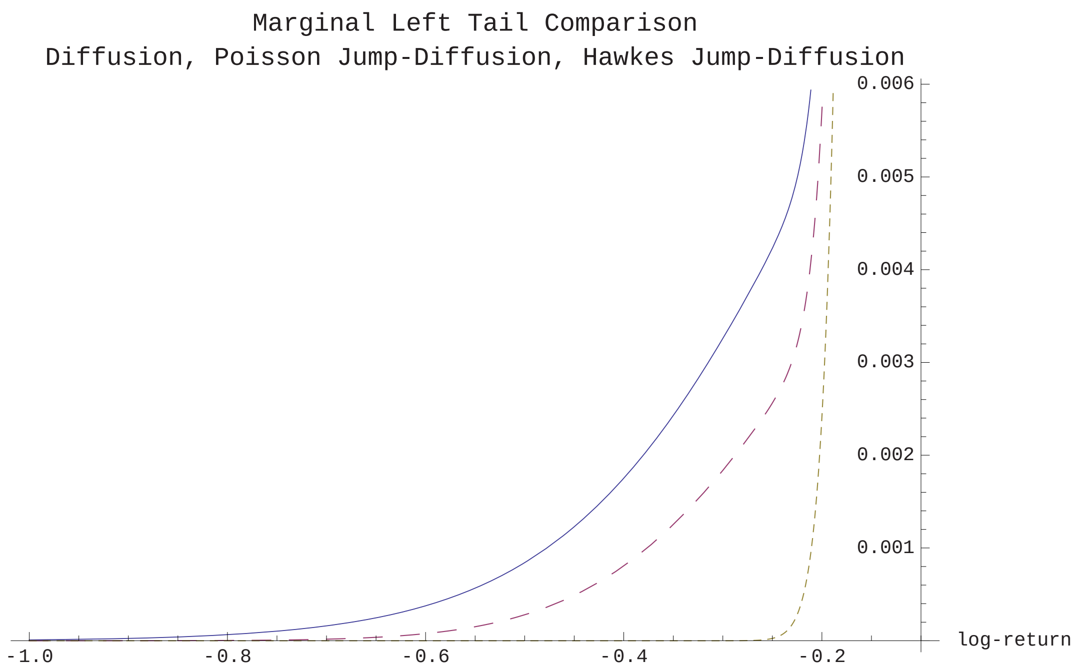
4. Tails and VaR

- The model generates fat tails with specific roles for the different parameters.

- Over short horizons (e.g., 10-day for VaR), these expressions are explicit.

- Self- and cross-excitation parameters have asymmetrical influences on the tails.

Univariate Tails



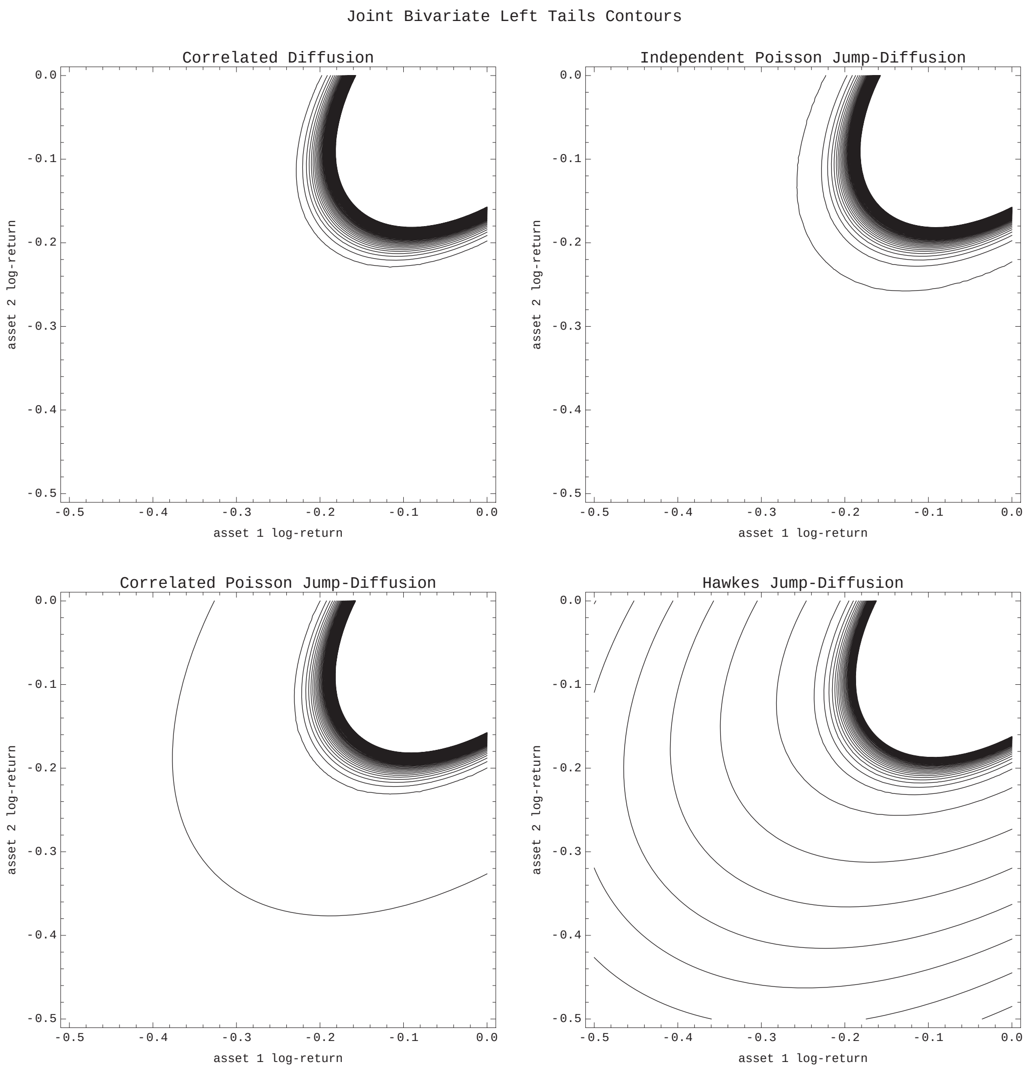
- Typical VaR calculation over a time horizon of $\Delta = 10$ days
  - from the perspective of a regulator concerned with the probability of joint individual losses L_1 and L_2 in two firms under scrutiny:

$$\mathbb{P}\left(\Delta X_1 \leq -L_1, \Delta X_2 \leq -L_2\right)$$

  - or that of a portfolio manager concerned with losses exceeding a level L in a portfolio invested in the two assets in proportions ω_1 and ω_2 :

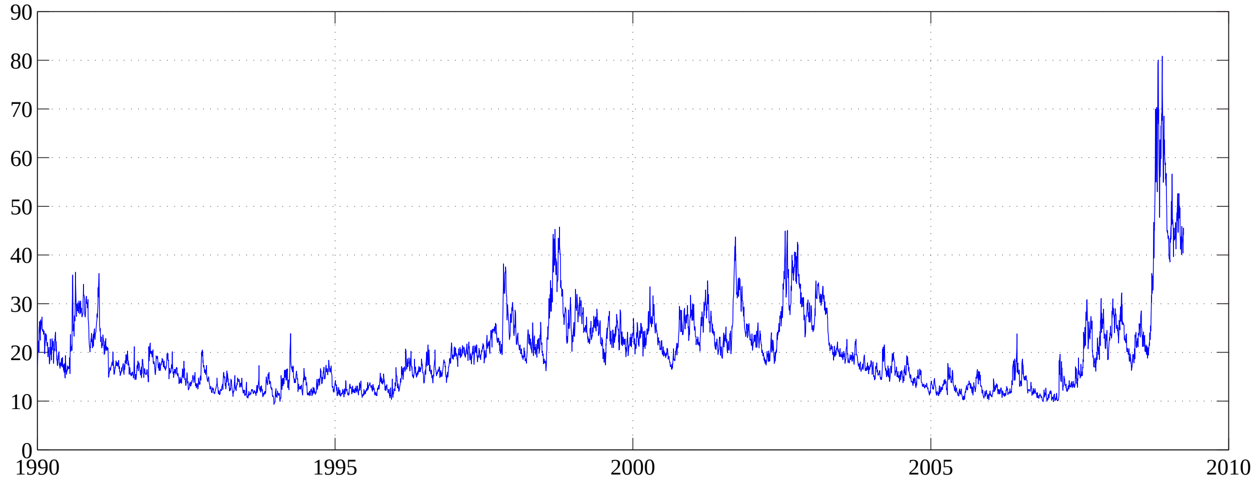
$$\mathbb{P}\left(\omega_1 \Delta X_1 + \omega_2 \Delta X_2 \leq -L\right).$$

- Depend on the tails of the joint distribution.

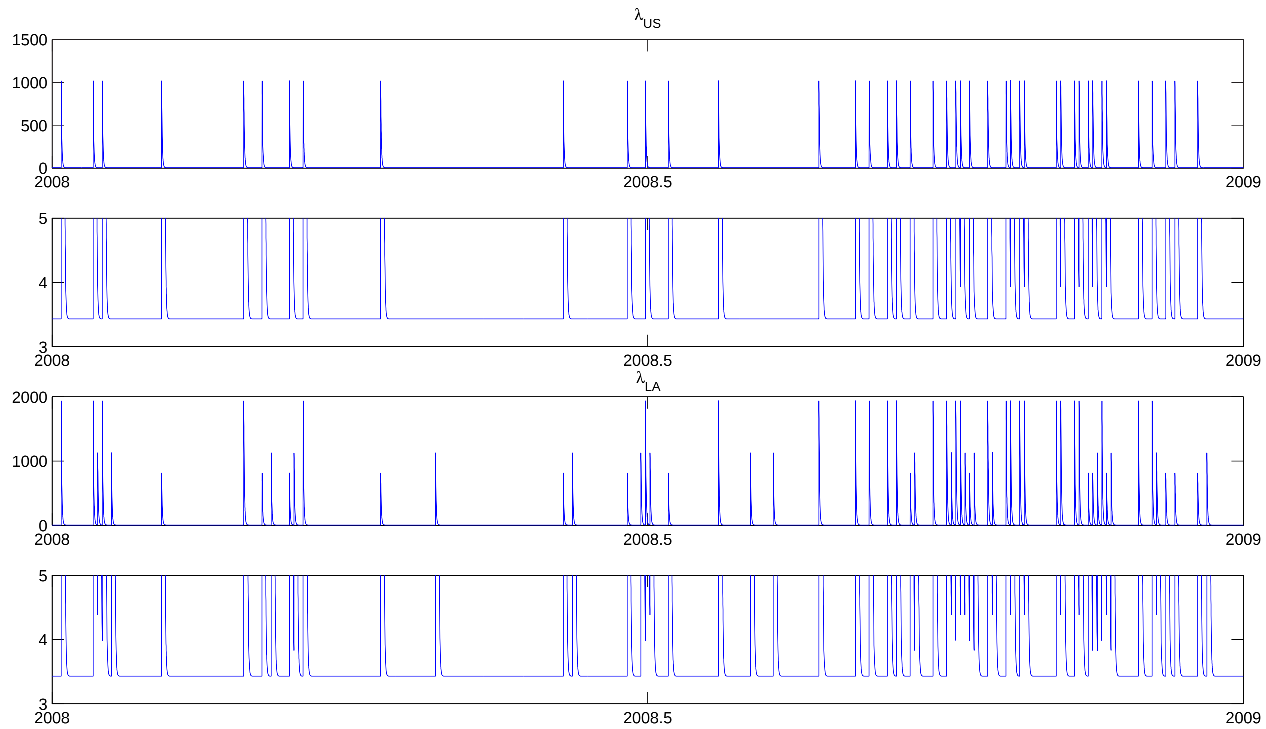

Joint Bivariate Left Tails Contours
Correlated Diffusion
Independent Poisson Jump-Diffusion
Correlated Poisson Jump-Diffusion
Hawkes Jump-Diffusion
asset 1 log-return
asset 2 log-return
0.0
-0.1
-0.2
-0.3
-0.4
-0.5

5. Measuring Market Stress Using Filtered Values of the Jump Intensities

VIX: Total Volatility



Real-Time Estimated Jump Intensities



6. Derivative Pricing

- The model can be restricted to fit the rich class of affine jump-diffusion models, in their generalized version allowing for multiple jump types defined in Duffie, Pan and Singleton (2000, Appendix B).

- An affine special case of our model would therefore share in the tractable pricing implementation that results from an affine structure.