Empirical Properties of a Diversified Index

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Benchmark Approach

Pl. & Heath (2006)

- Diversification Theorem
  well diversified portfolio approximates numeraire portfolio
- growth optimal portfolio equals numeraire portfolio, Long (1990)
- benchmark in portfolio optimization
- numeraire in derivative pricing
- statistical properties of non-diversifiable risk

Financial Market Model

- **observation times**

  $$0 = t_0 < t_1 < \ldots$$

  trades are discrete in time

$$\lim_{n \to \infty} t_n = \infty$$

$$t_{n+1} - t_n \ll 1 \quad \text{a.s.}$$

Supermartingale Property

Assume numeraire portfolio $S^{\delta_*}_{t_n}$ as benchmark s.t.

for all nonnegative portfolios $S^{\delta}_{t_n}$

$$\frac{S^{\delta}_{t_n}}{S^{\delta_*}_{t_n}} = \hat{S}^{\delta}_{t_n} \geq E_{t_n}\left(\hat{S}^{\delta}_{t_{n+1}}\right)$$

S^{δ_*} - best performing portfolio, numeraire portfolio, growth optimal

Kelly (1956), Long (1990), Becherer (2001), Pl. (2002),

Bühlmann & Pl. (2003), Pl. & Heath (2006), Karatzas & Kardaras (2007)

- V **fair** if

$$\frac{V_{t_n}}{S^{\delta_*}_{t_n}} = \hat{V}_{t_n} = E_{t_n}\left(\hat{V}_{t_{n+1}}\right)$$

for all $n \in \{0, 1, \ldots\}$
martingale

- **benchmarked primary security account**

$$\hat{S}^j_{t_n} = \frac{S^j_{t_n}}{S^{\delta_*}_{t_n}}$$

assume that

$$\frac{\hat{S}^j_{t_{n+1}} - \hat{S}^j_{t_n}}{\hat{S}^j_{t_n}} = \sum_{k=1}^{j} \sigma^{j,k}_{t_n} \, \Delta \, \xi^k_{t_{n+1}}$$

may be not fair

$$E_{t_n}\left(\Delta\,\xi^k_{t_{n+1}}\right) = 0,$$

$$E_{t_n}\left(\Delta\,\xi^k_{t_{n+1}}\,\Delta\,\xi^i_{t_{n+1}}\right) = \begin{cases} \Delta_{t_n,k} & \text{for} \quad k = i \\ 0 & \text{for} \quad k \neq i \end{cases}$$

uncorrelated, $0 < \Delta_{t_n,k} < t_{n+1} - t_n$ a.s.

- **Benchmarked numeraire portfolio**

$$\hat{S}^{\delta_*}_{t_n} = 1$$

$$\Longrightarrow$$

$$\frac{\hat{S}^{\delta_*}_{t_{n+1}} - \hat{S}^{\delta_*}_{t_n}}{\hat{S}^{\delta_*}_{t_n}} = 0 \quad \text{a.s.}$$

- approximate $\hat{S}^{\delta_*}$

  strictly positive

- sequence of strictly positive, benchmarked portfolios

$$\left(\hat{S}^{\delta_\ell}\right)_{\ell\in\{1,2,\dots\}}$$

with $\hat{S}_0^{\delta_\ell} = 1$

is a sequence of **approximate numeraire portfolios**

if for $\varepsilon > 0$

$$\lim_{\ell\to\infty} P_{t_n}\left(\frac{1}{\sqrt{t_{n+1}-t_n}}\left|\frac{\hat{S}^{\delta_\ell}_{t_{n+1}} - \hat{S}^{\delta_\ell}_{t_n}}{\hat{S}^{\delta_\ell}_{t_n}}\right| \geq \varepsilon\right) = 0$$

for all $n \in \{0, 1, \dots\}$

- financial market is **semi-regular** if for all $\varepsilon > 0$

$$\lim_{\ell \to \infty} P\left(\left| \frac{1}{\sqrt{\ell}} \sum_{j=1}^{\ell} \sigma_{t_n}^{j,k} \right| \geq \varepsilon \right) = 0$$

for all $k \in \{1, 2, \ldots\}$ and $n \in \{0, 1, \ldots\}$

Diversification Theorem:

In a semi-regular market each sequence of EWIs

is a sequence of approximate numeraire portfolios.

Pl. (2005), Pl. & Rendek (2009)

Numeraire portfolio has only non-diversifiable risk!

Equally Weighted Index

EWI

$$\pi^{j}_{\delta_{\mathrm{EWI}},t} = \frac{1}{d}$$

$j \in \{1, 2, \ldots, d\}$

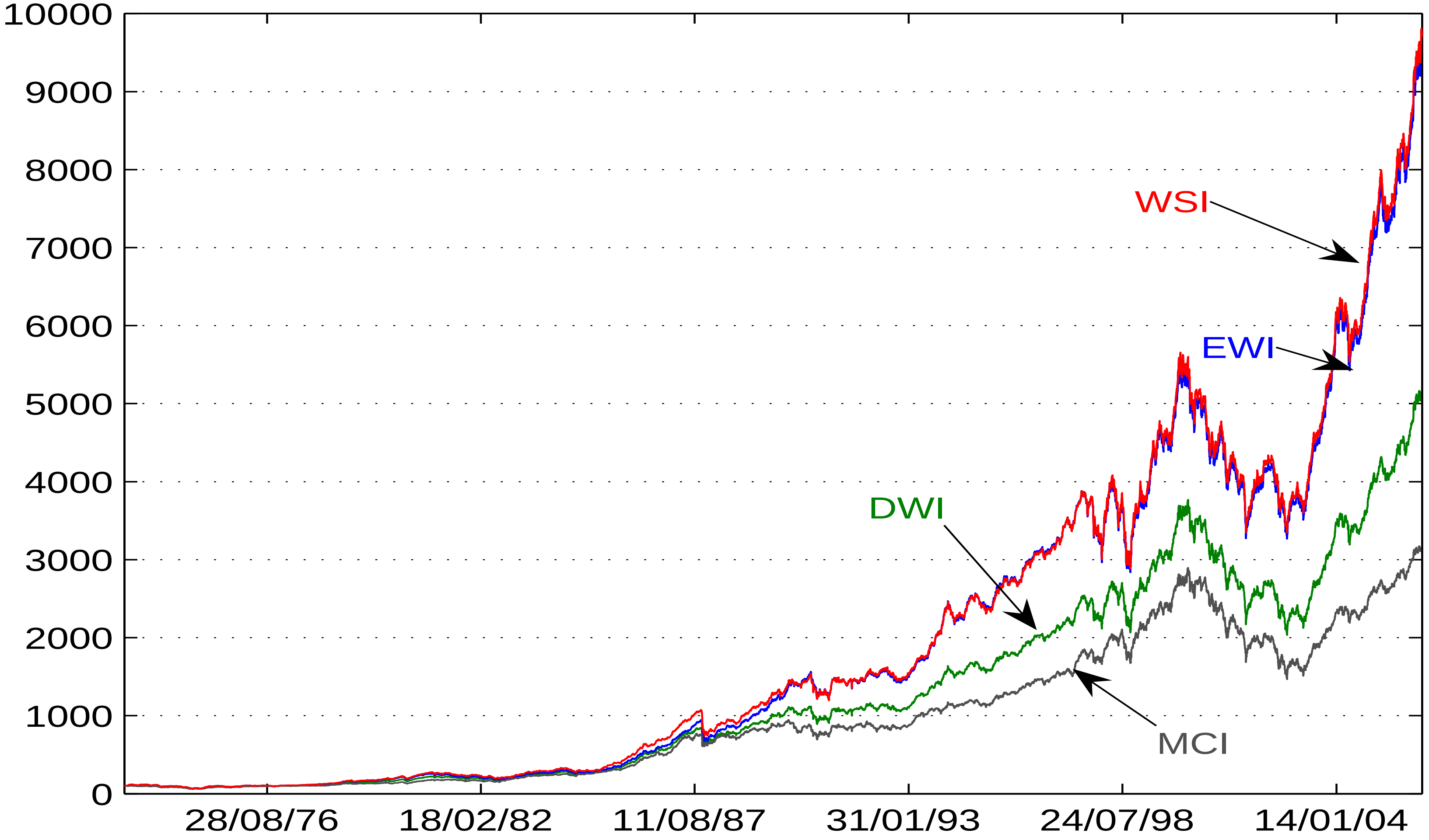

Figure 1: Indices constructed from regional stock market indices.

Log-return Distributions

Barndorff-Nielsen (1978)

Hurst & Pl. (1997)

McNeil, Frey & Embrechts (2005)

Pl. & Rendek (2008)

- **symmetric generalized hyperbolic density**

$$f_X(x) = \frac{1}{\delta\sigma K_\lambda(\bar{\alpha})}\sqrt{\frac{\bar{\alpha}}{2\pi}}\left(1+\frac{x^2}{(\delta\sigma)^2}\right)^{\frac{1}{2}(\lambda-\frac{1}{2})} K_{\lambda-\frac{1}{2}}\left(\bar{\alpha}\sqrt{1+\frac{x^2}{(\delta\sigma)^2}}\right)$$

$$\lambda \in \Re, \alpha, \delta \geq 0, \quad \alpha \neq 0 \text{ if } \lambda \geq 0, \quad \delta \neq 0 \text{ if } \lambda \leq 0$$
$$\bar{\alpha} = \alpha\delta$$

unique scale parameter

$$c^2 = \begin{cases} \frac{(\delta\sigma)^2}{-2(\lambda+1)} & \text{if } \alpha = 0 \text{ for } \lambda < 0 \text{ and } \bar{\alpha} = 0, \\ \frac{2\lambda\sigma^2}{\alpha^2}, & \text{if } \delta = 0 \text{ for } \lambda > 0 \text{ and } \bar{\alpha} = 0, \\ \frac{(\delta\sigma)^2 K_{\lambda+1}(\bar{\alpha})}{\bar{\alpha}K_\lambda(\bar{\alpha})} & \text{otherwise} \end{cases}$$

Likelihood Ratio Test

- **likelihood ratio**

$$\Lambda = \frac{\mathcal{L}^*{}_{model}}{\mathcal{L}^*{}_{nesting\ model}}$$

$\mathcal{L}^*{}_{model}$ maximized likelihood function

- **test statistic**

$$L_n = -2\ln(\Lambda)$$

$$P(L_n < \chi^2_{1-\alpha,1}) \approx F_{\chi^2(1)}(\chi^2_{1-\alpha,1}) = 1 - \alpha$$

$$L_n < \chi^2_{0.01,1} \approx 0.000157$$

$$L_n < \chi^2_{0.001,1} \approx 0.000002$$

not rejected at the $\mathbf{99.9\%}$ level

Fitted Log-return Distributions

daily log-returns 1973 – 2006

EWI104s

denominated in 27 currencies

$> \mathbf{200.000}$ observations

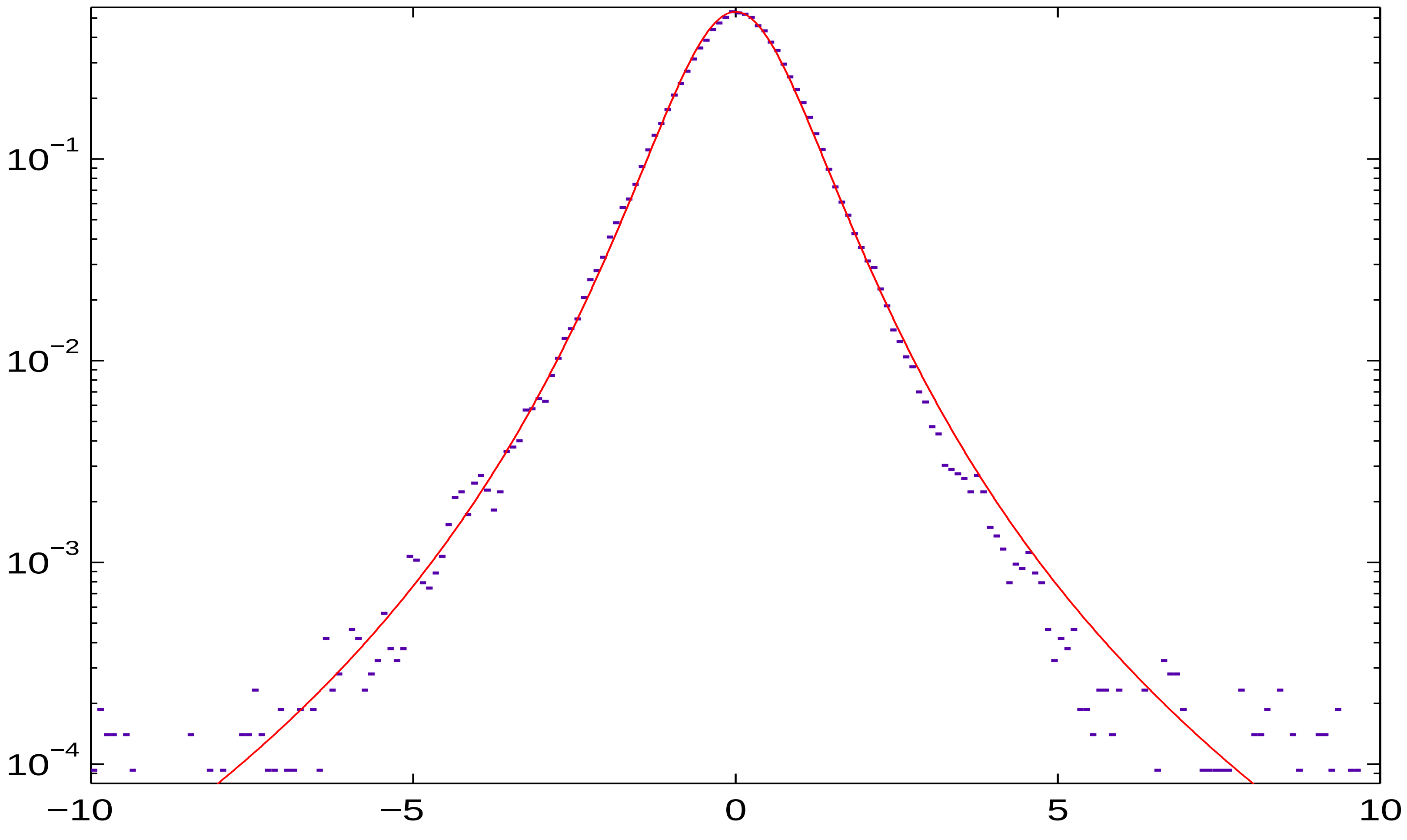

Figure 2: Log-histogram of the EWI104s log-returns and Student t density with four degrees of freedom.

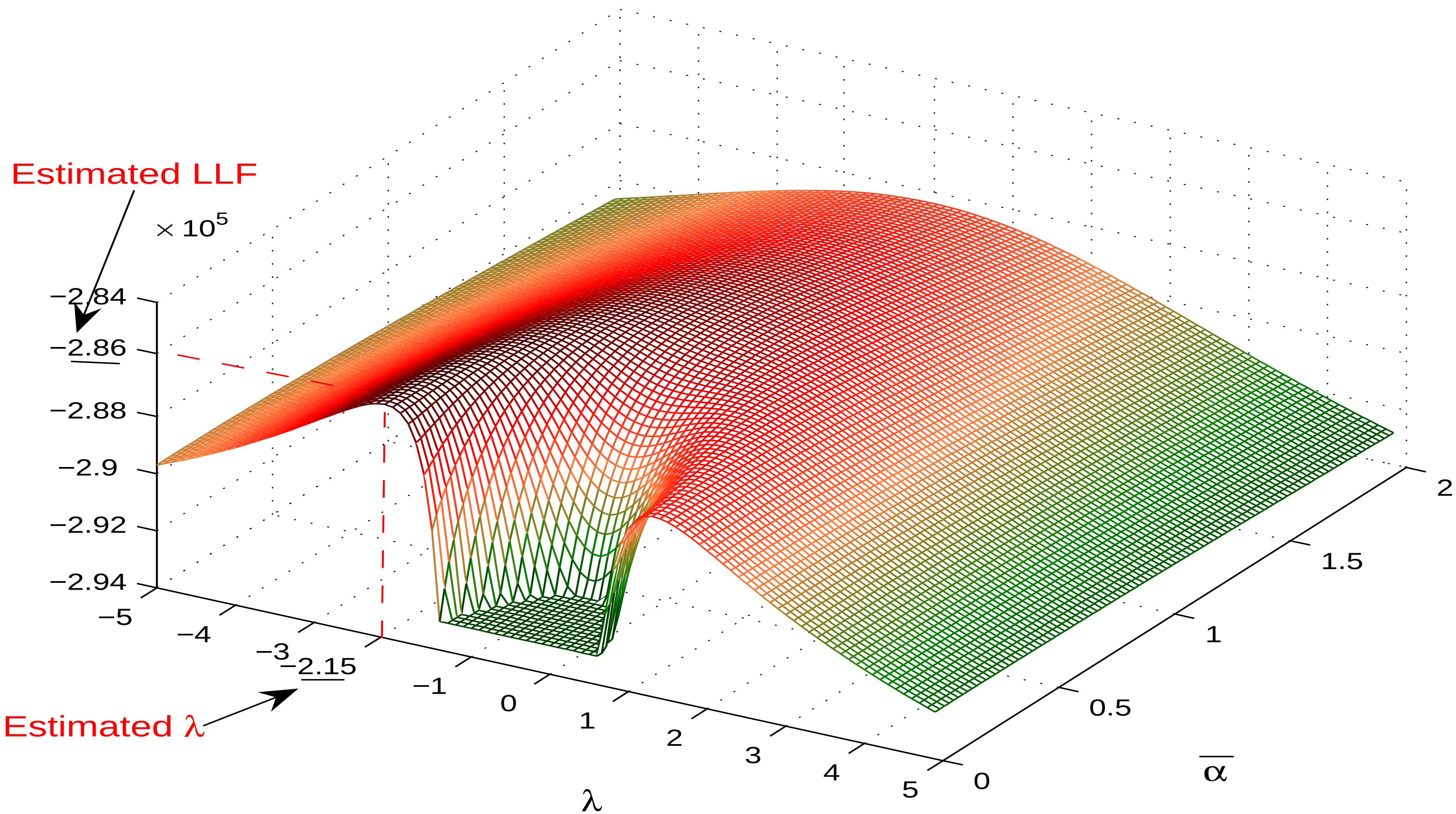

Figure 3: Log-likelihood function based on the EWI104s.

	SGH	Student t	NIG	Hyperbolic	VG
σ	0.9807068	0.7191163	0.9697258	0.9584118	0.9593693
$\bar{\alpha}$	0.0000000		0.9694605	0.7171357	
λ	-2.1629649				1.4912414
ν		4.3259646			
$\ln(\mathcal{L}^*)$	-285796.3865295	-285796.3865297	-286448.9371892	-287152.0787956	-287499.8259143
L_n		**0.0000004**	1305.1013194	2711.3845322	3406.8787696

Table 1: Results for log-returns of the EWI104s

Country	Student-t	NIG	Hyperbolic	VG	ν
Australia	0.000000	76.770817	150.202282	181.632971	4.281222
Austria	0.000000	39.289103	77.505683	102.979330	4.725907
Belgium	0.000000	31.581622	60.867570	83.648470	4.989912
Brazil	2.617693	5.687078	63.800349	60.078395	2.713036
Canada	0.000000	47.506215	79.917741	104.297607	5.316154
Denmark	0.000000	41.509921	87.199686	114.853658	4.512101
Finland	0.000000	28.852844	68.677271	88.553080	4.305638
France	0.000000	26.303544	57.639325	80.567283	4.722787
Germany	0.000000	27.290205	52.667918	71.120798	5.005856
Greece	0.000000	60.432172	104.789463	125.601499	4.674626
Hong.Kong	0.000000	42.066531	100.834255	122.965326	3.930473
India	0.000000	74.773701	163.594078	198.002956	3.998713
Ireland	0.000000	77.727856	136.505582	170.013644	4.761519
Italy	0.000000	25.196598	55.185625	75.481897	4.668983
Japan	0.000000	37.630363	77.163656	102.967380	4.649745
Korea.S.	0.000000	120.904983	304.829431	329.854620	3.289204

Malaysia	0.000000	79.714054	186.013963	221.061290	3.785195
Netherlands	0.000000	26.832761	51.625813	71.541627	5.084056
Norway	0.000000	42.243851	89.012090	115.059003	4.472349
Portugal	0.000000	61.177624	137.681039	165.689683	3.984860
Singapore	0.000000	36.379685	77.600590	98.124375	4.251472
Spain	0.000000	56.694545	109.533768	138.259224	4.517153
Sweden	0.000000	77.618384	143.420049	178.983373	4.546640
Taiwan	0.000000	41.162560	96.283628	115.186585	3.914719
Thailand	0.000000	78.250621	254.590254	267.508143	3.032038
UK	0.000000	26.693076	55.937248	80.678494	4.952843
USA	0.000000	40.678242	79.617362	100.901197	4.636661

Table 2: L_n test statistic of the EWI104s for different currency denominations

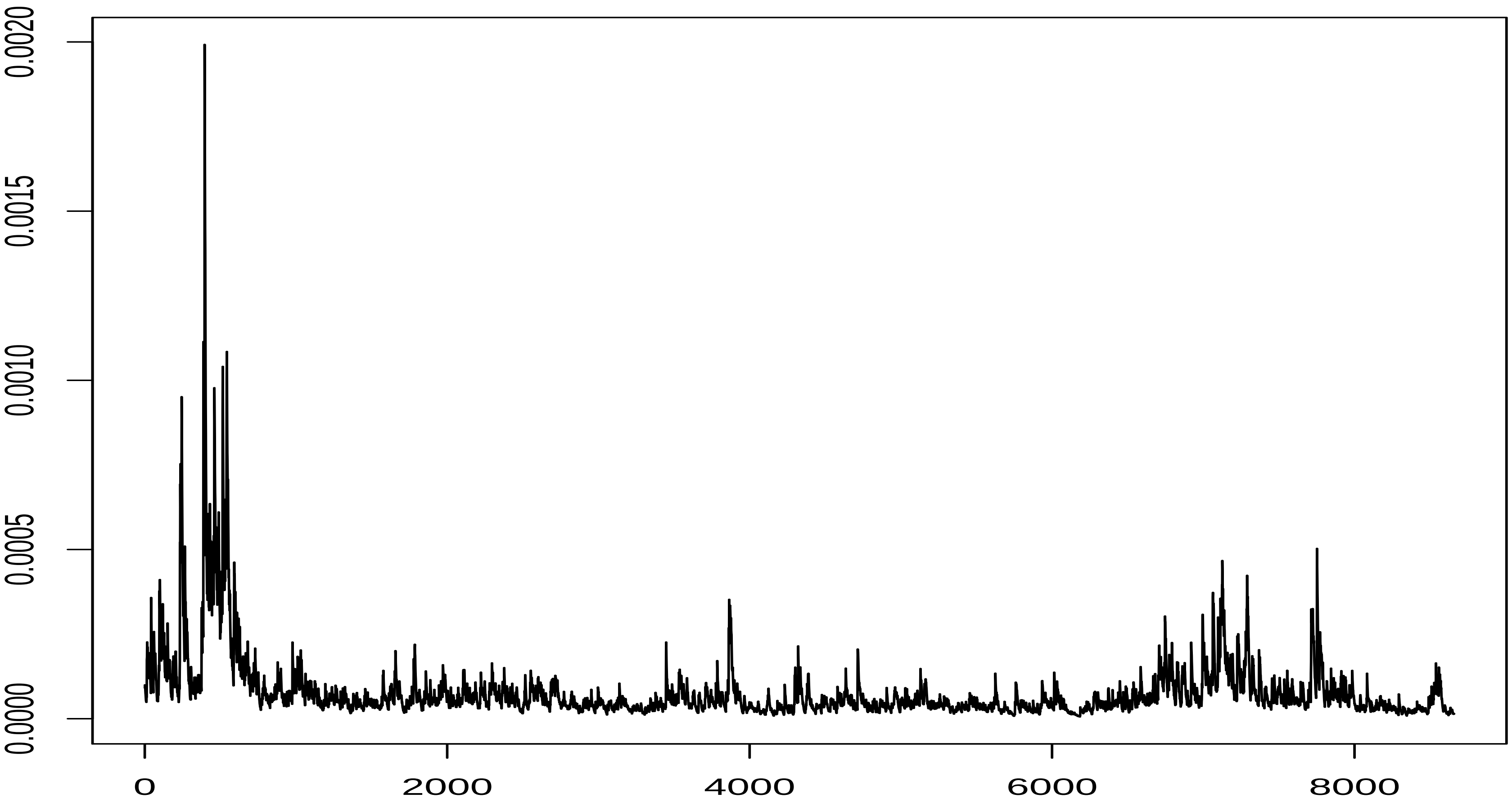

Figure 4: Squared volatility.

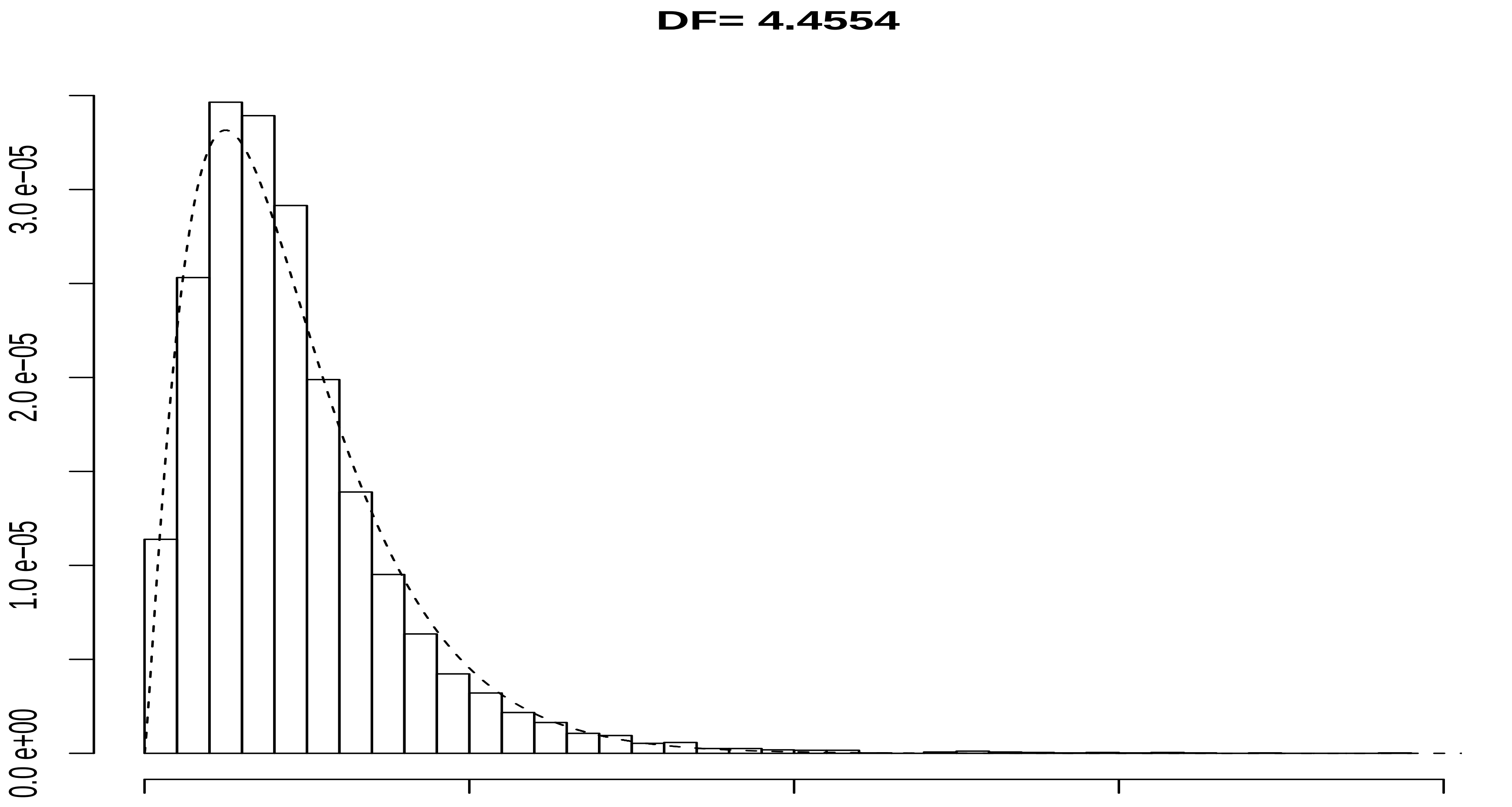

Figure 5: Histogram of inverse squared volatility.

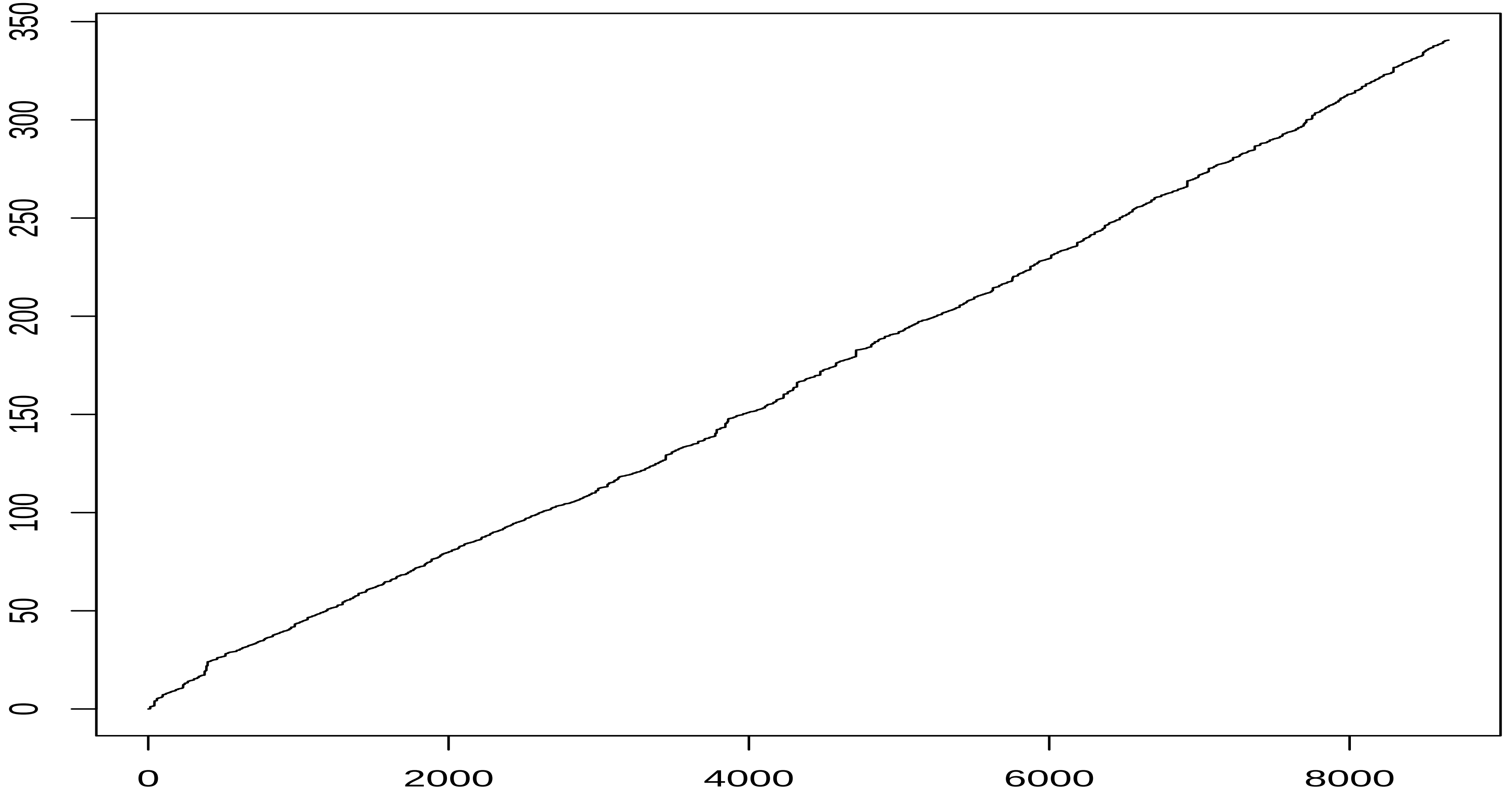

Figure 6: Quadratic variation of log-squared volatility.

Empirical Behavior of a Diversified WSI

Breymann, Lüthi, Pl. (2009)

- WSI intraday 1 April 1996 - 30 June 2001

- WSI daily 1 Jan 1973 - 10 March 2006

- S&P500 monthly 1920 - 2008

Scaling Behavior

qth absolute moment

$$E\left(\left|\frac{S^{\delta_*}_{t_n+\Delta} - S^{\delta_*}_{t_n}}{S^{\delta_*}_{t_n}}\right|^q\right) \sim \Delta^{f(q)}$$

$$f(q) = q\,H(q)$$

$H(q)$ - Hurst exponent

estimate

$$H(q) = c_1 + c_2\,q + O(q^2)$$

There is scaling ! $H(q) > 0.5$

But not much Multi-Scaling !

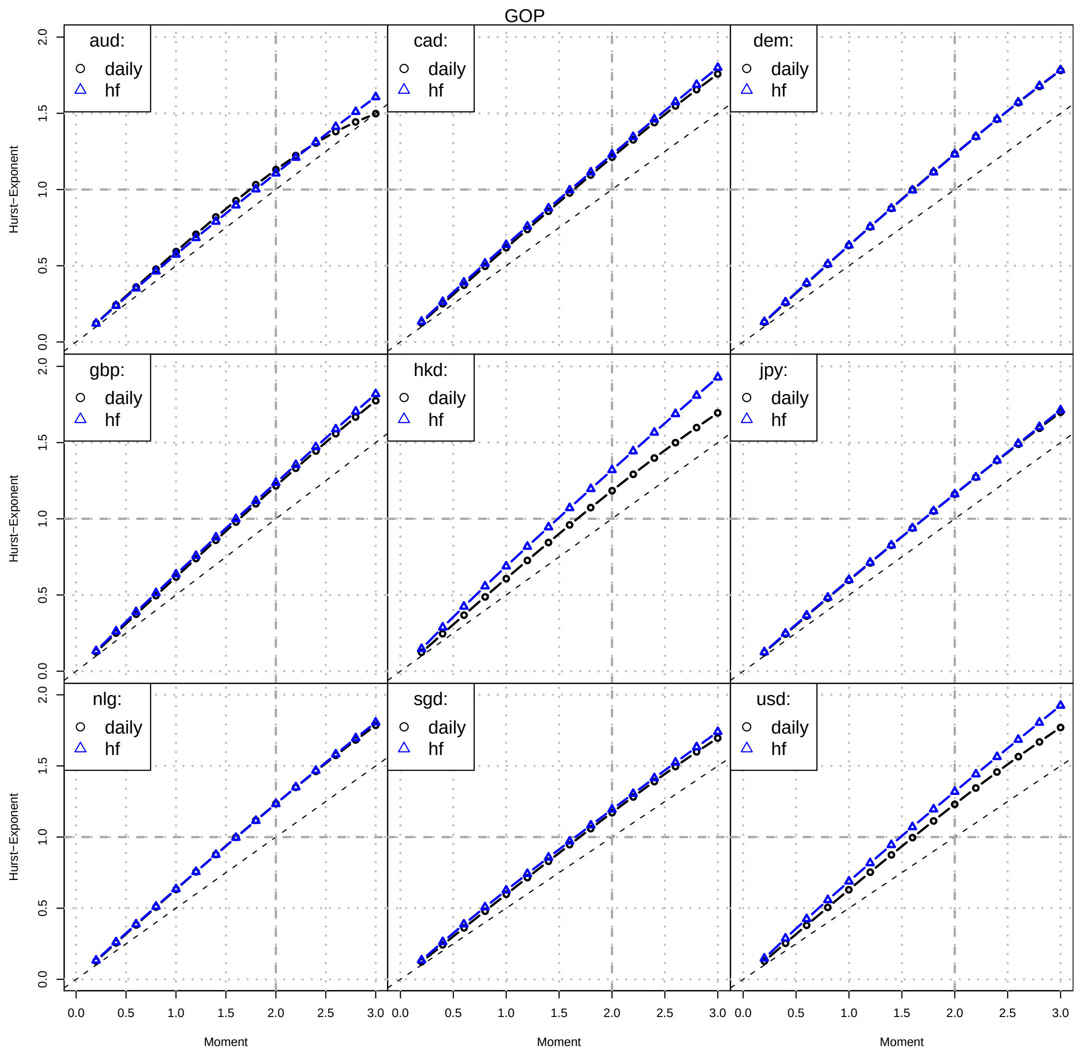

Figure 7: Scaling exponents $f(q)$ as function of the order q of the absolute moment of WSI log returns.

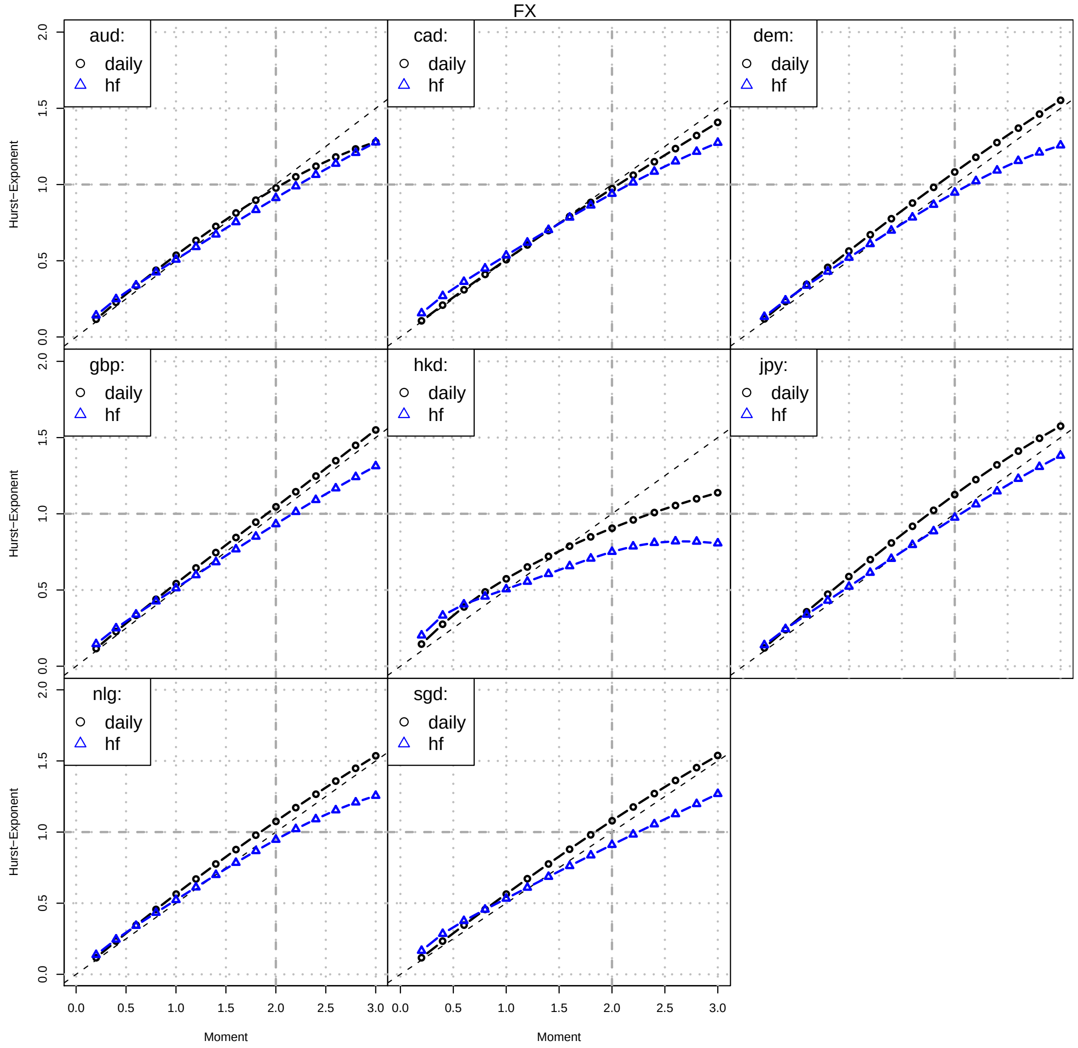

Figure 8: Same as Fig. 7 for FX log returns.

Fractional Market Model

- **discounted numeraire portfolio**

$$\bar{S}^{\delta_*}_{t_{n+1}} - \bar{S}^{\delta_*}_{t_n} \approx \bar{S}^{\delta_*}_{t_n}\, \theta_{t_n} \left(\theta_{t_n}\, \Delta + \Delta\, B^H_{t_{n+1}} \right)$$

- **discounted numeraire portfolio drift:**

$$\alpha_{t_n} = \bar{S}^{\delta_*}_{t_n} \, \theta^2_{t_n}$$

$\Longrightarrow$

volatility

$$\theta_{t_n} = \sqrt{\frac{\alpha_{t_n}}{\bar{S}^{\delta_*}_{t_n}}}$$

$\Longrightarrow$

- time transformed fractional squared Bessel process

$$\bar{S}^{\delta_*}_{t_{n+1}} - \bar{S}^{\delta_*}_{t_n} \approx \alpha_{t_n}\,\Delta + \sqrt{\alpha_{t_n}\,\bar{S}^{\delta_*}_{t_n}}\,\Delta\,B^H_{t_{n+1}}$$

long-range dependence

Squared Stochastic Volatility

$$\theta^2_{t_n} \approx \left(\frac{\bar{S}^{\delta_*}_{t_n}}{\alpha_{t_n}} \right)^{-1}$$

$$\theta^2_{t_{n+1}} - \theta^2_{t_n} \approx \theta^2_{t_n} \left(\eta_{t_n} \Delta + \Delta^{2H} \right) - \left(\theta^2_{t_n} \right)^{\frac{3}{2}} \Delta B^H_{t_{n+1}}$$

fractional $\frac{3}{2}$-model, Pl. (1997), Lewis (2000)

stationary process

inverse gamma distribution with 4 degrees of freedom

long-range dependence

mono scaling

volatility clustering

implied volatility surface negatively skewed

return distribution

$$\frac{\bar{S}^{\delta_*}_{t_{n+1}} - \bar{S}^{\delta_*}_{t_n}}{\bar{S}^{\delta_*}_{t_n}} \approx \frac{1}{\sqrt{Y^{\Delta}_{t_n}}} \, \Delta \, B^H_{t_{n+1}} + \frac{1}{Y^{\Delta}_{t_n}} \, \Delta$$

Gaussian normal mixture

fractional MMM $\quad \Longrightarrow \quad$ Student $t(4)$

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