Diversification Benefits: A Second-Order Approximation

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Outline

- Introduction
- Part I: Analysis of diversification benefits
- Part II: Accuracy anaylsis of the closed-form OpVaR approximation

Introduction – Research motivation

"Given the size and interconnected nature of markets, the growth in volumes, the global nature of traders and their cross-asset characteristics, managing operational risk will only become more important."

Lloyd C. Blankfein, CEO Goldman Sachs
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- One (!) important part of managing OR: Calculation of regulatory capital
- No agreed standard method for doing so

The Basel II regulatory framework for OR

- Consider a (AMA) bank's business with d sub-units of business
- Basel II requires the calculation of regulatory capital for OR through

$$\mathsf{RC}^{\mathsf{OR}} = \mathsf{VaR}_\alpha\left(\sum_{i=1}^{d} S_i\right) = (1-\delta)\sum_{i=1}^{d}\mathsf{VaR}_\alpha\left(S_i\right),$$

with $\alpha = 99.9\%$, S_i denoting the total yearly OR loss of business unit i and for some "well-reasoned" estimate diversification benefits $\delta \in \mathbb{R}$

Focus of this talk:

- Part I: Analysis of diversification benefits δ
- Part II: Calculation of $\mathsf{VaR}_\alpha\left(S_i\right)$

Part I

Analysis of diversification benefits

Based on:

Degen, M., Lambrigger D. D. and Segers, J. (2010). Risk concentration and diversification: second-order properties. *Insurance: Mathematics and Economics (to appear)*.

Practical relevance:

- So far not enough evidence to convince regulators to allow $\delta \neq 0$
- However: $\delta = 0$ only for comonotonic risks; recent empirical evidence questions this; see Cope and Antonini (2008)

Aim of our paper:

- Get a grasp on δ (analytically)
- Provide a tool that allows to assess the sensitivity of diversification benefits w.r.t. changes in the underlying input variables

Mathematical tools:

- First- and second-order asymptotic properties $(\alpha \rightarrow 1)$ for $\delta = \delta(\alpha)$

Framework

Ideal:

- Find stochastic model for $(S_1, \dots, S_d)$ that "accurately" reflects the dependence structure between business units $S_1, \dots, S_d$

- Analysis of diversification benefits δ/risk concentration C (and calculation of regulatory capital) based on this model:

$$C(\alpha) := 1 - \delta(\alpha) = \frac{\mathrm{VaR}_\alpha\left(\sum_{i=1}^d S_i\right)}{\sum_{i=1}^d \mathrm{VaR}_\alpha\left(S_i\right)}$$

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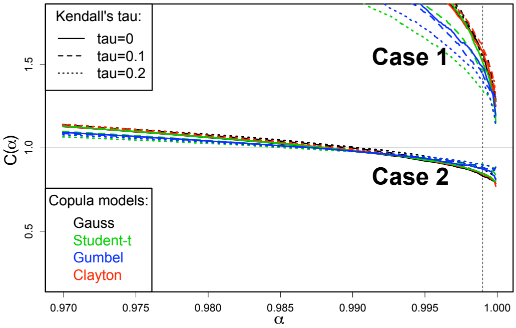
- Too ambitious (given the state of the art in dependence modeling)

Realistic: Analysis of risk concentration for $S_1, \ldots, S_d \overset{iid}{\sim} F$

- Toy model (no depdendence), but...

Diversification under dependence: Copulas vs. margins



Empirical risk concentration (10^7 simulations) under dependence with $d = 2$ identically distributed Burr margins with parameters ($\theta = 0.1, \kappa = 20$) in case 1 and ($\theta = 0.3, \kappa = 6.7$) in case 2, so that both show the same heavy-tailedness (!) (i.e. same tail index)

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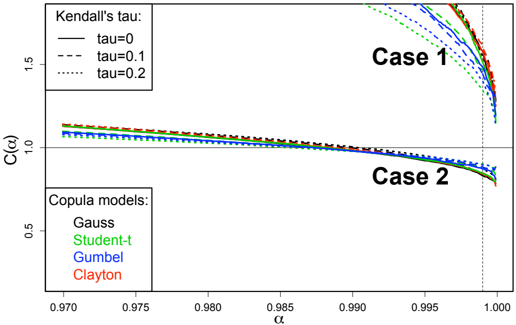

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- Fallacy: Dependence as THE main driver of diversification effects
- Instead: Tail behavior of margins matters - but in an delicate way

Back to the “toy model”...

- Non-negative $S_1, \dots, S_d \overset{iid}{\sim} F$ with $\overline{F} \in \mathrm{RV}_{-1/\xi}$ for some $\xi > 0$
- Let $G = F^{*d}$, $U_F = (1/\overline{F})^{\leftarrow}$ $(\in RV_\xi)$
- We show that, as $\alpha \to 1$,

$$C(\alpha) = \frac{1}{d} \frac{G^{\leftarrow}(\alpha)}{F^{\leftarrow}(\alpha)} \to d^{\xi - 1}$$

- First-order approximation: $C_1(\alpha) = d^{\xi-1}$ for large values of α

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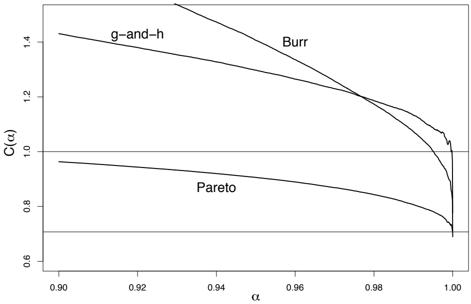
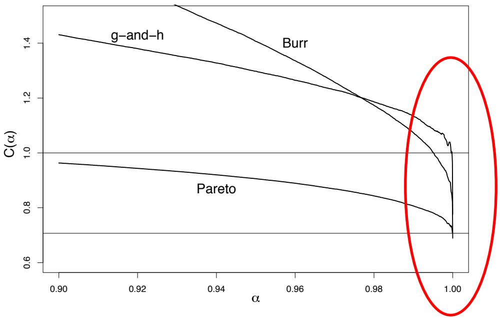
known unknown



Empirical risk concentration (based on 10^7 simulations) together with first-order approximation $C_1 \equiv \sqrt{2}/2 \approx 0.71$ for two iid rvs from a Burr ($\tau = 0.25, \kappa = 8$), a Pareto ($\xi = 0.5$) and a g-and-h ($g = 2, h = 0.5$) distribution - same tail index!



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- In relevant regions $C(\alpha)$ very sensitive to small changes of α
- Driving factors? ($\rightarrow$ second-order properties)

Towards a second-order result

- Find non-degenerate K and A, with $\lim_{\alpha\to 1} \frac{C(\alpha) - d^{\xi-1}}{A(\alpha)} = K(d,\xi)$

- Hard part is finding convergence rate $A(\cdot)$ & it turns out that two different asymptotic regimes matter:

$\frac{\overline{G}(x)}{\overline{F}(x)} \to d$	vs.	$\frac{U_F(td)}{U_F(t)} \to d^\xi$
second-order subexponentiality		second-order regular variation
$\rightsquigarrow$ rate $b(\cdot)$		$\rightsquigarrow$ rate $a(\cdot)$

- Which one dominates in the limit?

- Then, "putting together all the epsilons"...

Main result

Second-order risk concentration

For $S_1, \ldots, S_d \overset{iid}{\sim} F$ positive random variables and under some mild conditions on $U = (1/\overline{F})^{\leftarrow}$ (see D., Lambrigger, Segers (2010) for details), one has for fixed $d \geq 2$ and as $\alpha \to 1$,

$$\boxed{C(\alpha) = d^{\xi-1} + K_{\xi,\rho}(d)A(\alpha) + o\big(A(\alpha)\big),}$$

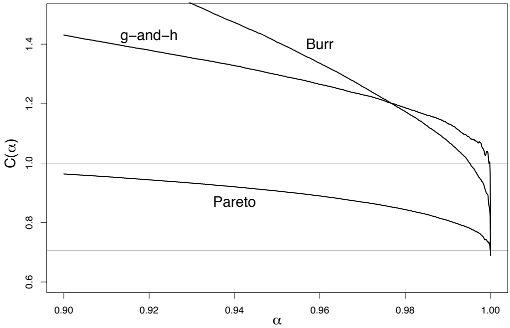
for some constant $K_{\xi,\rho}(d) \in \mathbb{R}$ and with

$$A(\alpha) = \begin{cases} b(F^{\leftarrow}(\alpha)), & \rho < -(1 \wedge \xi), \\ a(1/(1-\alpha)), & \rho > -(1 \wedge \xi). \end{cases}$$

Implications

- Two different regimes of diversification effects (depending on first- and second-order tail behavior of F)
- Second-order approximation $C_2(\alpha) = d^{\xi-1} + K_{\xi,\rho}(d)A(\alpha)$
- (Recall: first-order approximation $C_1(\alpha) \equiv d^{\xi-1}$)

First-order approximation for C:



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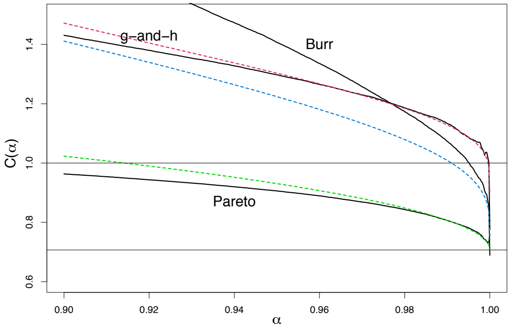
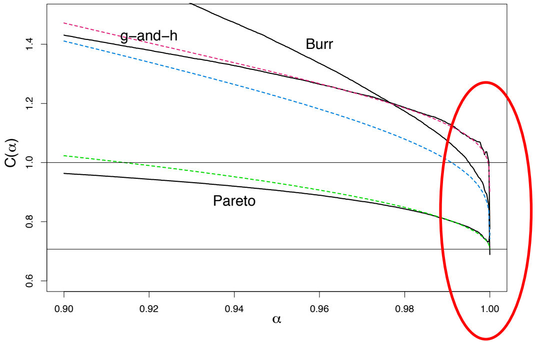
Second-order approximation for C:



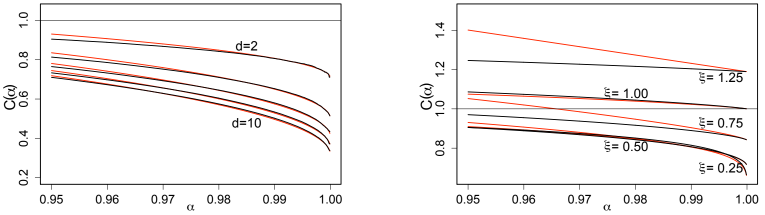
Empirical risk concentration (full, based on 10^7 simulations) together with first-order approximation $C_1 \equiv \sqrt{2}/2 \approx 0.71$ (full) and second-order approximation C_2 (dashed) for $d = 2$ iid rvs from a Burr ($\tau = 0.25, \kappa = 8$), a Pareto ($\xi = 0.5$) and a g-and-h ($g = 2, h = 0.5$) distribution - same tail index!

Second-order approximation for C:



Empirical risk concentration (full, based on 10^7 simulations) together with first-order approximation $C_1 \equiv \sqrt{2}/2 \approx 0.71$ (full) and second-order approximation C_2 (dashed) for $d = 2$ iid rvs from a Burr ($\tau = 0.25, \kappa = 8$), a Pareto ($\xi = 0.5$) and a g-and-h ($g = 2, h = 0.5$) distribution - same tail index!

Sensitivity analysis of diversification benefits



Behavior of diversification benefits for d iid Pareto(ξ) rvs together with respective second-order approximations (red lines). In the right panel $d = 2$ is fixed with varying ξ (theoretical $C(\cdot)$, $G^{\leftarrow}$ numerically inverted). In the left panel $\xi = 0.5$ is fixed and $d = 2, 4, 6, 8, 10$ (simulated $C(\cdot)$, based on $n = 10^7$ simulations).

- Theoretical/empirical ($n = 10^7$, took > 30 minutes) vs. approximation
- Error negligible in area where we need it ($\alpha = 99.9\%$)
- Hence, no need to simulate tons of (very) heavy-tailed data

Conclusion (1/3) – Implications for practice

- Fallacy: Diversification effects occur mainly/only due to dependence in the data

- At least as important driver is the tail behavior (second-order properties !) of underlying loss model F

- Diversification benefits are highly sensitive to VaR-level α

- Negative diversification (at 99.9%) occurs more often than is commonly believed – in finite mean models (!)

Conclusion (2/3) – Summary of Part I

1) Second-order approximation C_2 as tool to assess the sensitivity of diversification benefits w.r.t. changes in the

- **i)** underlying loss model F,
- **ii)** number of risks d,
- **iii)** level α

2) The iid case is the Fréchet-lower bound case and hence the "best case" scenario with regards to diversification

3) Validation/consistency check of models (e.g. for given model, is diversification benefit of, say, 20% justified—at 99.9% level)

Conclusion (3/3) – Future Work

4) Impose dependence structure (ambitious), start with

- $S_1, \ldots, S_d \overset{iid}{\sim} F \quad \rightsquigarrow \quad S_i$'s independent, df F_i
- $\boldsymbol{S} = (S_1, \ldots, S_d)$ with (Archimedean) Copula

5) Estimation of δ (idea: penultimate approximations)

Part II

Accuracy anaylsis of the closed-form OpVaR approximation

(Application of Part I; work in progress)

Recall regulatory capital charge for operational risk:

$$\mathsf{RC}^{\mathsf{OR}} = \mathsf{VaR}_\alpha \left(\sum_{i=1}^{d} S_i \right) = (1-\delta) \sum_{i=1}^{d} \mathsf{VaR}_\alpha (S_i),$$

with S_i denoting the total yearly OR loss of business unit i

- Part I: Analysis of diversification benefit δ ✓
- Part II: Calculation of $\mathsf{VaR}_\alpha (S_i)$

Framework:

Classical actuarial model for non-life insurance, with total OR-loss process for business unit S given by

$$S(t) = \sum_{k=1}^{N(t)} X_k,$$

with $(X_i)_{i \geq 1} \overset{iid}{\sim} F$ as the single OR-losses, independent of the claim arrival process $(N(t))_{t \geq 0}$.

- In OR context: t is fixed to be 1 year (and henceforth suppressed)
- Then: $G(x) := \mathbb{P}[S \leq x] = \sum_{n=0}^{\infty} \mathbb{P}[N = n] F^{n*}(x)$

- Under some mild conditions on N (Embrechts et al. 1979):

$$\overline{G}(x) \sim E(N)\overline{F}(x), \quad x \to \infty$$

- Relates single-loss model F to total-loss model G

- Böcker and Klüppelberg (2005) show

$$VaR_\alpha(S) := G^{-1}(\alpha) = F^{-1}\left(1 - \frac{1-\alpha}{E(N)}(1+o(1))\right), \quad \alpha \to 1$$

The closed-form OpVaR approximation

1) For $\overline{F} \in RV$ and with $\tilde{\alpha} := 1 - (1-\alpha)/\mathbb{E}[N]$, one has for large α-values:

$$VaR_{\alpha}(S) \approx VaR_{\tilde{\alpha}}(X)$$

2) Approximation **1)** is used by at least one "systemically important" bank to calculate regulatory capital

3) Goodness of approximation: not considered in detail so far (!)

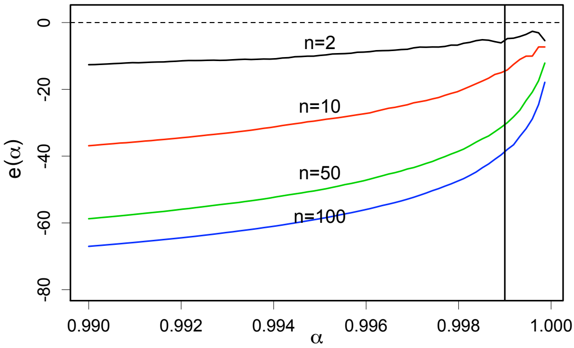
4) Does it matter? - Yes!

Motivating example

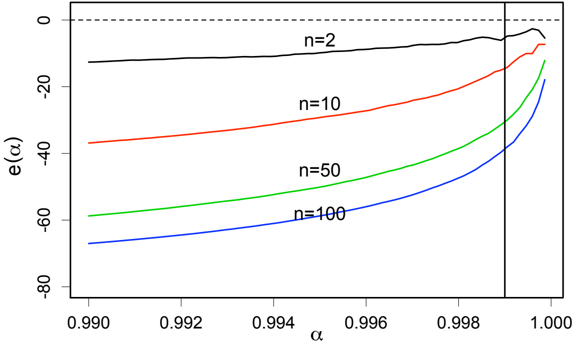
- Relative approximation error: $e(\alpha) = \dfrac{F^{-1}(\tilde{\alpha})}{G^{-1}(\alpha)} - 1$

Example: For simplicity consider the case

- Pareto distribution as single loss model F
- Non-random number of losses $N = n$ a.s.



Relative approximation error (in %) of the closed-form OpVaR approximation (based on 10^6 simulations) for sums of $n = 2, 10, 50$ and 100 iid Pareto ($\xi = 0.5$) losses



Relative approximation error (in %) of the closed-form OpVaR approximation (based on 10^6 simulations) for sums of $n = 2, 10, 50$ and 100 iid Pareto ($\xi = 0.5$) losses

- OpVaR approximation vastly underestimates (!) regulatory capital
- Driving factors?

- Using Part I with "deterministic" replaced by "stochastic" sums:

Accuracy of the closed-form OpVaR approximation

Under some mild conditions on N and $U = (1/\overline{F})^{-1}$ one has,

$$\boxed{e(\alpha) = \frac{F^{-1}\left(1 - \frac{1-\alpha}{E[N]}\right)}{G^{-1}(\alpha)} - 1 = KA(\alpha) + o\big(A(\alpha)\big), \quad \alpha \to 1,}$$

for some $K = K_{\xi,\rho}(N) \in \mathbb{R}$ and with $A(\alpha)$ as in Theorem 1.

- The relative error $e(\alpha)$ grows like $K_{\xi,\rho}(N)A(\alpha)$

Implications

Example: $N \sim pois(\lambda)$, $F \sim Pareto(\xi)$, $\xi < 1$, then:

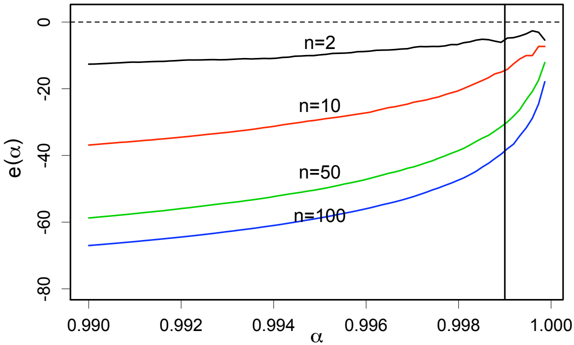
$$K_{\xi,\rho}(N) = -\lambda^{1-\xi} \quad \text{and} \quad A(\alpha) = \frac{(1-\alpha)^{\xi}}{1-\xi}$$

So, for given level $\alpha = 99.9\%$ and

- for fixed ξ, error increases with increasing λ
- for fixed λ, error decreases with increasing ξ

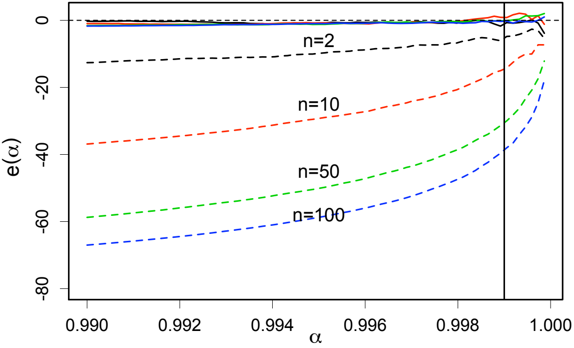
- Second-order approximation: $G^{-1}(\alpha) \approx F^{-1}(\tilde{\alpha})\,(1 - KA(\alpha))$
- As opposed to first-order: $G^{-1}(\alpha) \approx F^{-1}(\tilde{\alpha})$

Simulation result



Relative approximation error (in %) of the closed-form OpVaR approximation (based on 10^6 simulations) for sums of $n = 2, 10, 50$ and 100 iid Pareto ($\xi = 0.5$) losses

...together with a second-order refinement



Relative approximation error (in %) of the closed-form OpVaR approximation (dashed) compared with second-order refinement (full) (based on 10^6 simulations) for sums of $n = 2, 10, 50$ and 100 iid Pareto ($\xi = 0.5$) losses

▶ Second-order term seems to be able to explain the discrepancy between true $G^{-1}(\alpha)$ and closed-forem approximation $F^{-1}(\tilde{\alpha})$

Summary of Part II: It seems as...

- for some loss models the closed-form OpVaR approximation highly underestimates regulatory capital

- a second-order refinement might be helpful in understanding why this is the case (and to act correspondingly)

- the practical usefulness of the closed-form OpVaR approximation needs to be rejudged carefully

Thank you!

References

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