Equilibrium Pricing in Incomplete Markets under Translation Invariant Preferences

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Outline

1. **Convex duality in a discrete-time framework**

2. **BS$\triangle$Es with random walk noise**

3. **BSDEs in the continuous-time limit**

Ingredients

- filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$
- a group of finitely many agents $\mathbb{A}$
- agent $a \in \mathbb{A}$ is endowed with an uncertain payoff $H^a \in L^\infty(\mathcal{F}_T)$
- liquid asset $(S_t)_{t=0}^T$ satisfying (NA)
- structured product in net supply n with final payoff $R \in L^\infty(\mathcal{F}_T)$
- at time t agent a invests in such a way so as to optimize a preference functional

$$U_t^a : L^\infty(\mathcal{F}_T) \to L^\infty(\mathcal{F}_t)$$

We assume U_t^a satisfies the following properties:

(N) Normalization $U_t^a(0) = 0$

(M) Monotonicity

$U_t^a(X) \geq U_t^a(Y)$ for all $X, Y \in L^\infty(\mathcal{F}_T)$ such that $X \geq Y$

(T) Translation property

$U_t^a(X + Y) = U_t^a(X) + Y$ for all $X \in L^\infty(\mathcal{F}_T)$ and $Y \in L^\infty(\mathcal{F}_t)$

(C) $\mathcal{F}_t$-Concavity

$U_t^a(\lambda X + (1-\lambda)Y) \geq \lambda U_t^a(X) + (1-\lambda)U_t^a(Y)$

for all $X, Y \in L^\infty(\mathcal{F}_T)$ and $\lambda \in L^\infty(\mathcal{F}_t)$ such that $0 \leq \lambda \leq 1$

(TC) Time-consistency $U_{t+1}^a(X) \geq U_{t+1}^a(Y)$ implies $U_t^a(X) \geq U_t^a(Y)$

$$\Leftrightarrow \qquad U_t^a(X) = U_t^a(U_{t+1}^a(X))$$

Examples

1) $U_t^a(X) = -\frac{1}{\gamma} \log \mathsf{E}\left[e^{-\gamma X} \mid \mathcal{F}_t\right]$

2) $U_t^a(X) = (1-\lambda)\mathsf{E}\left[X \mid \mathcal{F}_t\right] - \lambda \mathsf{E}\left[(X - \mathsf{E}\left[X \mid \mathcal{F}_t\right])^2 \mid \mathcal{F}_t\right]$

3) $U_t^a(X) = (1-\lambda)\mathsf{E}\left[X \mid \mathcal{F}_t\right] - \lambda \rho_t(X)$

where ρ_t is a conditional convex risk measure

Definition of equilibrium

An equilibrium consists of an adapted process $(R_t)_{t=0}^T$ satisfying the terminal condition $R_T = R$ together with trading strategies $(\widehat{\vartheta}_t^a)_{t=1}^T$ such that the following conditions hold:

(i) individual optimality

$$U_t^a\left(H^a + \sum_{s=1}^{T} \widehat{\vartheta}_s^{a,1}\Delta S_s + \widehat{\vartheta}_s^{a,2}\Delta R_s\right)$$

$$\geq \; U_t^a\left(H^a + \sum_{s=1}^{t} \widehat{\vartheta}_s^{a,1}\Delta S_s + \widehat{\vartheta}_s^{a,2}\Delta R_s + \sum_{s=t+1}^{T} \vartheta_s^{a,2}\Delta S_s + \vartheta_s^{a,2}\Delta R_s\right)$$

for every t and all possible strategies (ϑ_s^a)

(ii) market clearing $\sum_{a\in\mathbb{A}} \widehat{\vartheta}_t^{a,2} = n$

The "representative agent"

Set $H^a_T = H^a$ and $H^a_{t+1} = U^a_{t+1}\left(H^a + \sum_{s=t+2}^{T} \widehat{\vartheta}^{a,1}_s \Delta S_s + \widehat{\vartheta}^{a,2}_s \Delta R_s\right)$

would like to define the representative agent by

$$\operatorname*{ess\,sup}_{\substack{\vartheta^a \in L^\infty(\mathcal{F}_t)^2 \\ \sum_{a\in\mathbb{A}} \vartheta^{a,2} = n}} \sum_{a\in\mathbb{A}} U^a_t\left(\frac{X}{|\mathbb{A}|} + H^a_{t+1} + \vartheta^{a,1}\Delta S_{t+1} + \vartheta^{a,2}\Delta R_{t+1}\right)$$

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But R_t is not known yet.

The “representative agent”

Set $H_T^a = H^a$ and $H_{t+1}^a = U_{t+1}^a \left(H^a + \sum_{s=t+2}^T \widehat{\vartheta}_s^{a,1} \Delta S_s + \widehat{\vartheta}_s^{a,2} \Delta R_s \right)$

would like to define the representative agent by

$$\underset{\substack{\vartheta^a \in L^\infty(\mathcal{F}_t)^2 \\ \sum_{a\in\mathbb{A}} \vartheta^{a,2} = n}}{\operatorname{ess\,sup}} \sum_{a\in\mathbb{A}} U_t^a \left(\frac{X}{|\mathbb{A}|} + H_{t+1}^a + \vartheta^{a,1} \Delta S_{t+1} + \vartheta^{a,2} \Delta R_{t+1} \right)$$

But R_t is not known yet. So define

$$\widehat{U}_t(X) := \underset{\substack{\vartheta^a \in L^\infty(\mathcal{F}_t)^2 \\ \sum_{a\in\mathbb{A}} \vartheta^{a,2} = 0}}{\operatorname{ess\,sup}} \sum_{a\in\mathbb{A}} U_t^a \left(\frac{X}{|\mathbb{A}|} + H_{t+1}^a + \vartheta^{a,1} \Delta S_{t+1} + \vartheta^{a,2} R_{t+1} \right)$$

and

$$\widehat{u}_t(x) := \widehat{U}_t(x R_{t+1}), \quad x \in L^\infty(\mathcal{F}_t).$$

$\widehat{U}_t$ and $\widehat{u}_t$ are $\mathcal{F}_t$-concave

For the static case, see:

Jouini, Schachermayer and Touzi (2008)

Filipovic and Kupper (2008)

...

Convex dual characterization of equilibrium

Theorem A bounded, adapted process $(R_t)_{t=0}^T$ satisfying $R_T = R$ together with trading strategies $(\widehat{\vartheta}_t^a)_{t=1}^T$, $a \in \mathbb{A}$, form an equilibrium $\iff$ for all t:

(i) $R_t \in \partial \widehat{u}_t(n)$

(ii) $\sum_{a\in\mathbb{A}} U_t^a(H_{t+1}^a + \widehat{\vartheta}_{t+1}^{a,1}\Delta S_{t+1} + \widehat{\vartheta}_{t+1}^{a,2} R_{t+1}) = \widehat{u}_t(n)$

(iii) $\sum_{a\in\mathbb{A}} \widehat{\vartheta}_{t+1}^{a,2} = n$

Existence of equilibrium

Assumption (A)

For all $t = 0, \dots, T-1$, $V^a \in L^\infty(\mathcal{F}_{t+1})$, $W \in L^\infty(\mathcal{F}_{t+1})$, there exist $\widehat{\vartheta}^a_{t+1} \in L^\infty(\mathcal{F}_t)^2$, $a \in \mathbb{A}$, such that

$$\sum_{a \in \mathbb{A}} \widehat{\vartheta}^{a,2}_{t+1} = 0$$

and

$$\sum_{a \in \mathbb{A}} U^a_t \left(V^a + \widehat{\vartheta}^{a,1}_{t+1} \Delta S_{t+1} + \widehat{\vartheta}^{a,2}_{t+1} W \right)$$
$$= \operatorname*{ess\,sup}_{\substack{\vartheta^a_{t+1} \in L^\infty(\mathcal{F}_t)^2 \\ \sum_{a \in \mathbb{A}} \vartheta^{a,2}_{t+1} = 0}} \sum_{a \in \mathbb{A}} U^a_t \left(V^a + \vartheta^{a,1}_{t+1} \Delta S_{t+1} + \vartheta^{a,2}_{t+1} W \right).$$

Theorem Under assumption (A) an equilibrium exists

Definition

U_0^a is sensitive to large losses if

$$\lim_{\lambda \to \infty} U_0^a(\lambda X) = -\infty$$

for all $X \in L^\infty(\mathcal{F}_T)$ such that $\mathbb{P}[X < 0] > 0$.

Theorem

If all U_0^a are sensitive to large losses,

then condition (A) is satisfied and an equilibrium exists.

Idea of the proof

Consider two agents a and b in a one time-step model

Assume $S_0 = S_1 = H_1^a = H_1^b = 0$

Consider the convolution

$$\begin{aligned} & \sup_{\vartheta \in \mathbb{R}} U^a(\vartheta R_1) + U^b(-\vartheta R_1) \\ = \; & \sup_{\vartheta \in \mathbb{R}} U^a(\vartheta (R_1 - \mathsf{E}[R_1])) + U^b(-\vartheta (R_1 - \mathsf{E}[R_1])) \end{aligned}$$

Under sensitivity to large losses it will be attained for some $\widehat{\vartheta}$.

Differentiable preferences

We say that U_t^a satisfies the differentiability condition $(\mathbf{D})$ if for all $X, Y \in L^\infty(\mathcal{F}_{t+1})$, there exists $Z \in L^1(\mathcal{F}_{t+1})$ such that

$$\lim_{k\to\infty} k\left(U_t^a\left(X+\frac{Y}{k}\right) - U_t^a(X)\right) = \mathsf{E}\left[YZ \mid \mathcal{F}_t\right].$$

If such a Z exists, it has to be unique, and we denote it by $\nabla U_t^a(X)$.

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If such a Z exists, it has to be unique, and we denote it by $\nabla U_t^a(X)$.

Theorem Assume all U_t^a satisfy (D). Then there can exist at most one equilibrium price process $(R_t)_{t=0}^T$, and if the market is in equilibrium, then $\widehat{U}_t$ satisfies (D) at $X = nR_{t+1}$ with

$$\nabla\widehat{U}_t(nR_{t+1}) = \frac{1}{|\mathbb{A}|}\sum_{a\in\mathbb{A}} \nabla U_t^a\left(H_{t+1}^a + \widehat{\vartheta}_{t+1}^{a,1}\Delta S_{t+1} + \widehat{\vartheta}_{t+1}^{a,2} R_{t+1}\right),$$

and

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \prod_{t=1}^{T} \nabla\widehat{U}_t\left(nR_{t+1}\right)$$

defines probability measure on $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$S_t = \mathsf{E}_{\mathbb{Q}}\left[S_T \mid \mathcal{F}_t\right] \quad \text{and} \quad R_t = \mathsf{E}_{\mathbb{Q}}\left[R_T \mid \mathcal{F}_t\right] \quad \text{for all } t\,.$$

Random Factors and BSΔEs

Fix $h > 0$ and $N \in \mathbb{N}$

Denote $\mathbb{T} = \{0, h, \dots, T = Nh\}$

$b_t^1, \dots, b_t^d$ $\quad d$ independent random walks with $P[\Delta b_{t+h}^i = \pm\sqrt{h}] = 1/2$

$b_t^{d+1}, \dots, b_t^D$ $\quad 2^d - (d+1)$ random walks orthogonal to $b_t^1, \dots, b_t^d$

Every $X \in L^\infty(\mathcal{F}_{t+h})$ can be represented as

$$X = \mathsf{E}\left[X \mid \mathcal{F}_t\right] + \pi_t(X) \cdot \Delta b_{t+h}$$

for

$$\pi_t(X) \cdot \Delta b_{t+h} = \sum_{i=1}^{D} \pi_t^i(X) \Delta b_{t+h}^i \quad \text{and} \quad \pi_t^i(X) = \frac{1}{h} \mathsf{E}\left[X \Delta b_{t+h}^i \mid \mathcal{F}_t\right].$$

$$U_t^a(X) = U_t^a\left(\mathsf{E}\left[X|\mathcal{F}_t\right] + \pi_t(X)\cdot\Delta b_{t+h}\right) = \mathsf{E}\left[X \mid \mathcal{F}_t\right] - f_t^a(\pi_t(X))h$$

for the $\mathcal{F}_t$-convex function $f_t^a : L^\infty(\mathcal{F}_t)^D \to L^\infty(\mathcal{F}_t)$ given by

$$f_t^a(z) := -\frac{1}{h}U_t^a(z\cdot\Delta b_{t+h})\,.$$

Assume condition (A) is satisfied and all U_t^a satisfy the differentiability condition (D).

Then there exists $\nabla f_t^a(z) \in L^\infty(\mathcal{F}_t)^D$ such that

$$\lim_{k\to\infty} k\left(f_t^a(z+z'/k) - f_t^a(z)\right) = z'\cdot\nabla f_t^a(z)$$

For given S_{t+h}, R_{t+h}, H^a_{t+h} denote

$$\begin{aligned}
Z^S_{t+h} &:= \pi_t(S_{t+h}) \\
Z^R_{t+h} &:= \pi_t(R_{t+h}) \\
Z^a_{t+h} &:= \pi_t(H^a_{t+h}) \\
Z_{t+h} &= (Z^S_{t+h}, Z^R_{t+h}, Z^a_{t+h}, a \in \mathbb{A})\,.
\end{aligned}$$

and define the function $f_t : L^\infty(\mathcal{F}_t)^{(3+|\mathbb{A}|)D} \longrightarrow L^\infty(\mathcal{F}_t)$ by

$$\begin{aligned}
f_t(v, Z_{t+h}) \;=\; & \operatorname*{ess\,inf}_{\substack{\vartheta^a \in L(\mathcal{F}_t)^2 \\ \sum_{a\in\mathbb{A}} \vartheta^{a,2} = 0}} \; \sum_{a\in\mathbb{A}} f^a_t \left(\frac{v}{|\mathbb{A}|} + Z^a_{t+h} + \vartheta^{a,1}_{t+h} Z^S_{t+h} + \vartheta^{a,2}_{t+h} Z^R_{t+h} \right) \\
& - \frac{\vartheta^{a,2}_{t+h}}{h} \mathsf{E}\left[\Delta S_{t+h} \mid \mathcal{F}_t \right].
\end{aligned}$$

Set

$$
\begin{aligned}
g_t^R(Z_{t+h}) &:= Z_{t+h}^R \cdot \nabla^v f_t(n Z_{t+h}^R, Z_{t+h}) \\
g_t^a\left(Z_{t+h}\right) &:= f_t^a\left(Z_{t+h}^a + \widehat{\vartheta}_{t+h}^{a,1} Z_{t+h}^S + \widehat{\vartheta}_{t+h}^{a,2} Z_{t+h}^R\right) \\
&\quad -\widehat{\vartheta}_{t+h}^{a,1} \frac{1}{h} \mathsf{E}\left[\Delta S_{t+h} \mid \mathcal{F}_t\right] - \widehat{\vartheta}_{t+h}^{a,2} g_t^R(Z_{t+h}).
\end{aligned}
$$

Theorem The processes (R_t) and (H_t^a) satisfy the following coupled system of BSΔEs

$$
\begin{aligned}
\Delta R_{t+h} &= g_t^R(Z_{t+h})h + Z_{t+h}^R \cdot \Delta b_{t+h}\,, \quad R_T = R \\
\Delta H_{t+h}^a &= g_t^a\left(Z_{t+h}\right)h + Z_{t+h}^a \cdot \Delta b_{t+h}\,, \quad H_T^a = H.
\end{aligned}
$$

Example (discrete time)

Let $(b_t^S)_{t\in\mathbb{T}}, (b_t^R)_{t\in\mathbb{T}}, (b_t^a)_{t\in\mathbb{T}}$, $a \in \mathbb{A}$, be independent random walks with

$$P[\Delta b_t^S = \pm\sqrt{h}] = P[\Delta b_t^R = \pm\sqrt{h}] = P[\Delta b_t^a = \pm\sqrt{h}] = 1/2\,.$$

Assume that the price of the standard asset is given by

$$S_{t+h} = S_t + \mu S_t h + \sigma S_t \Delta b_{t+h}^S\,, \qquad S_0 = s$$

and

$$R_T = r(b_T^R)\,, \qquad H^a = h^a(b_T^S, b_T^R + b_T^a)$$

for bounded Lipschitz functions r, h^a.

Suppose that agent a's preference functional is

$$U_t^a(X) = -\frac{1}{\gamma^a}\log \mathrm{E}\left[\exp(-\gamma^a X) \mid \mathcal{F}_t\right] \quad \text{for some} \quad \gamma^a > 0\,.$$

Then

$$U_t^a(X) = \mathrm{E}\left[X \mid \mathcal{F}_t\right] - f_t^a(\pi_t(X))h$$

for

$$f_t^a(z) = \frac{1}{h\gamma^a}\log \mathrm{E}\left[\exp(-\gamma^a z \cdot \Delta b_{t+h})\right]\,.$$

Neglect the random walks $b^{d+1}, \ldots, b^D$

and use the approximation

$$\frac{1}{h\gamma^a} \sum_{i=1}^{d} \log\cosh\left(\sqrt{h}\gamma^a z^i\right) \approx \frac{\gamma^a}{2} \sum_{i=1}^{d} (z^i)^2$$

Then the BSΔE of the last theorem yields ...

... the recursive algorithm

$$
\begin{aligned}
R_t &= \mathsf{E}\left[R_{t+1} \mid \mathcal{F}_t\right] - g_t^R h\,, & R_T &= R \\
H_t^a &= \mathsf{E}\left[H_{t+1}^a \mid \mathcal{F}_t\right] - g_t^a h\,, & H_T^a &= H^a\,,
\end{aligned}
$$

where

$$
\begin{aligned}
g_t^R &= \frac{1}{c^{SS}}\left[c^{RS}\mu S_t + \gamma\left(n\left\{c^{SS}c^{RR} - c^{SR}c^{SR}\right\} + c^{RA}c^{SS} - c^{SR}c^{SA}\right)\right] \\
g_t^a &= \frac{\gamma^a}{2}\left\|Z_{t+h}^a + \widehat{\vartheta}_{t+h}^{a,1} Z_{t+h}^S + \widehat{\vartheta}_{t+h}^{a,2} Z_{t+h}^R\right\|_2^2 - \widehat{\vartheta}_{t+h}^{a,1}\mu S_t - \widehat{\vartheta}_{t+h}^{a,2} g_t^R \\
\widehat{\vartheta}_{t+h}^{a,1} &= \frac{\mu S_t}{\gamma^a c^{SS}} + \frac{c^{SR}c^{Ra} - c^{Sa}c^{RR}}{c^{SS}c^{RR} - c^{SR}c^{SR}} - \frac{c^{SR}}{c^{SS}}\frac{\gamma}{\gamma^a}\left(n + \frac{c^{SS}c^{RA} - c^{SR}c^{SA}}{c^{SS}c^{RR} - c^{SR}c^{SR}}\right) \\
\widehat{\vartheta}_{t+h}^{a,2} &= n\frac{\gamma}{\gamma^a} + \frac{c^{SR}c^{Sa} - c^{Ra}c^{SS} - \frac{\gamma}{\gamma^a}\left(c^{SR}c^{SA} - c^{SS}c^{RA}\right)}{c^{SS}c^{RR} - c^{SR}c^{SR}}
\end{aligned}
$$

for

$$
\begin{aligned}
\gamma &:= \left(\sum_a (\gamma^a)^{-1}\right)^{-1} \\
c^{SS} &:= Z_{t+h}^S \cdot Z_{t+h}^S, \quad c^{SR} := Z_{t+h}^S \cdot Z_{t+h}^R, \quad c^{SA} := Z_{t+h}^S \cdot \sum_a Z_{t+h}^a, \quad \ldots
\end{aligned}
$$

Example (continuous time)

Let B_t^S, B_t^R, B_t^a, $a \in \mathbb{A}$, be independent Brownian motions

$$dS_t = \mu S_t dt + \sigma S_t dB_t^S\,, \quad S_0 = s$$

and

$$R_T = r(B_T^R)\,, \quad H^a = h^a(B_T^S, B_T^R + B_T^a)$$

for bounded Lipschitz functions r, h^a.

Suppose that agent a's preference functional is

$$U_t^a(X) = -\frac{1}{\gamma^a} \log \mathsf{E}\left[\exp(-\gamma^a X) \mid \mathcal{F}_t\right] \quad \text{for some} \quad \gamma^a > 0\,.$$

The BSDE corresponding to the above BSΔE is

$$
\begin{aligned}
dR_t &= g_t^R dt + Z_t^R \cdot dB_t\,, \qquad R_T = R \\
dH_t^a &= g_t^a dt + Z_t^a \cdot dB_t\,, \qquad H_T^a = H^a\,,
\end{aligned}
$$

where

$$
\begin{aligned}
g_t^R &= \frac{1}{c^{SS}}\left[c^{RS}\mu S_t + \gamma\left(n\left\{c^{SS}c^{RR} - c^{SR}c^{SR}\right\} + c^{RA}c^{SS} - c^{SR}c^{SA}\right)\right] \\
g_t^a &= \frac{\gamma^a}{2}\left\|Z_t^a + \hat{\vartheta}_t^{a,1} Z_t^S + \hat{\vartheta}_t^{a,2} Z_t^R\right\|_2^2 - \hat{\vartheta}_t^{a,1}\mu S_t - \hat{\vartheta}_t^{a,2} g_t^R \\
\hat{\vartheta}_t^{a,1} &= \frac{\mu S_t}{\gamma^a c^{SS}} + \frac{c^{SR}c^{Ra} - c^{Sa}c^{RR}}{c^{SS}c^{RR} - c^{SR}c^{SR}} - \frac{c^{SR}}{c^{SS}}\frac{\gamma}{\gamma^a}\left(n + \frac{c^{SS}c^{RA} - c^{SR}c^{SA}}{c^{SS}c^{RR} - c^{SR}c^{SR}}\right) \\
\hat{\vartheta}_t^{a,2} &= n\frac{\gamma}{\gamma^a} + \frac{c^{SR}c^{Sa} - c^{Ra}c^{SS} - \frac{\gamma}{\gamma^a}\left(c^{SR}c^{SA} - c^{SS}c^{RA}\right)}{c^{SS}c^{RR} - c^{SR}c^{SR}}
\end{aligned}
$$

for

$$
c^{SS} := Z_t^S \cdot Z_t^S, \quad c^{SR} := Z_t^S \cdot Z_t^R, \quad c^{SA} := Z_t^S \cdot \sum_a Z_t^a, \quad \ldots
$$