Dual Representation of Conditional Quasiconvex Maps

Marco Frittelli and Marco Maggis

Milano University

Workshop on Foundations of Mathematical Finance, Fields Institute, Toronto, January 11, 2010

1. Quasiconvex functions
2. Conditional quasiconvex maps
3. Motivations
4. Setting
5. The results
6. On the proof of the Theorem
7. Back to the Conditional Certainty Equivalent
8. On Musielak-Orlicz Spaces
9. The dual representation of the CCE

On Quasiconvexity (QCO)

- $f : E \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ is quasiconvex (QCO) if

$$f(\lambda X + (1-\lambda)Y) \leq \max\{f(X), f(Y)\},\ \lambda \in [0,1]$$

- Equivalently: f is (QCO) if all the lower level sets

$$\{X \in E \mid f(X) \leq c\} \quad \forall c \in \mathbb{R}$$

  are convex
- Findings on (QCO) real valued functions go back to De Finetti (1949), Fenchel (1949)...
- On (QCO) real valued functions and their dual representation: J-P Penot 1990 - 2007, Volle 1998, ...

Dual representation for real valued maps

As a straightforward application of the Hahn-Banach Theorem:

Proposition (Volle 98)

Let E be a locally convex topological vector space and E^* be its topological dual space. If $f : E \longrightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ is lsc and (QCO) then

$$f(x) = \sup_{x^* \in E^*} R(x^*(x), x^*),$$

where $R : \mathbb{R} \times E^* \longrightarrow \overline{\mathbb{R}}$ is defined by

$$R(t, x^*) := \inf \{ f(x) \mid x \in E \text{ such that } x^*(x) \geq t \}.$$

As a corollary of the above result:

Dual representation of static (QCO) cash-subadditive risk measures

Proposition (Cerreia-Maccheroni-Marinacci-Montrucchio, 2009)

A function $\rho : L^\infty \to \overline{\mathbb{R}}$ is quasiconvex cash-subadditive decreasing if and only if

$$\begin{aligned} \rho(X) &= \max_{Q \in M_{1,f}} R(E_Q[-X], Q), \\ R(t, Q) &= \inf \{\rho(\xi) \mid \xi \in L^\infty \text{ and } E_Q[-\xi] = t\} \end{aligned}$$

where $R : \mathbb{R} \times M_{1,f} \to \overline{\mathbb{R}}$ and $R(t, Q)$ is the reserve amount required today, under the scenario Q, to cover an expected loss t in the future.

The conditional setting: let $\mathcal{G} \subseteq \mathcal{F}$

A map

$$\pi : L(\Omega, \mathcal{F}, P) \rightarrow L(\Omega, \mathcal{G}, P)$$

is quasiconvex (QCO) if $\forall X, Y \in L(\Omega, \mathcal{F}, P)$ and for all $\mathcal{G}$-measurable r.v. Λ, $0 \leq \Lambda \leq 1$,

$$\pi(\Lambda X + (1 - \Lambda)Y) \leq \pi(X) \vee \pi(Y);$$

or equivalently if all the lower level sets

$$\mathcal{A}(Y) = \{X \in L(\Omega, \mathcal{F}, P) \mid \pi(X) \leq Y\} \quad \forall\, Y \in L(\Omega, \mathcal{G}, P)$$

are conditionally convex, i.e. for all $X_1, X_2 \in \mathcal{A}(Y)$ one has that $\Lambda X_1 + (1 - \Lambda)X_2 \in \mathcal{A}(Y)$.

The problem

Which is the dual representation of a (QCO) conditional map

$$\pi : L(\Omega, \mathcal{F}, P) \to L(\Omega, \mathcal{G}, P) \quad ?$$

The problem

Which is the dual representation of a (QCO) conditional map

$$\pi : L(\Omega, \mathcal{F}, P) \rightarrow L(\Omega, \mathcal{G}, P) \quad ?$$

As in the convex case, the dual representation of a (QCO) conditional map turns out to have the same structure of the real valued case,

The problem

Which is the dual representation of a (QCO) conditional map

$$\pi : L(\Omega, \mathcal{F}, P) \rightarrow L(\Omega, \mathcal{G}, P) \quad ?$$

As in the convex case, the dual representation of a (QCO) conditional map turns out to have the same structure of the real valued case,

...but the proof is not a straightforward application of known facts.

First motivation: Dynamic (QCO) Risk Measures

- Let Λ, $0 \leq \Lambda \leq 1.$, be $\mathcal{G}$-measurable random variables
- The convexity of $\pi : L(\Omega, \mathcal{F}, P) \to L(\Omega, \mathcal{G}, P)$

$$\pi(\Lambda X + (1-\Lambda)Y) \leq \Lambda\pi(X) + (1-\Lambda)\pi(Y)$$

implies:

$$\pi(\Lambda X + (1-\Lambda)Y) \leq \Lambda\pi(X) + (1-\Lambda)\pi(Y) \leq \pi(X) \vee \pi(Y).$$

- Quasiconvexity alone:

$$\pi(\Lambda X + (1-\Lambda)Y) \leq \pi(X) \vee \pi(Y)$$

allows to control the risk of a diversified position.

A second Motivation: the Conditional Certainty Equivalent

A stochastic dynamic utility (SDU)

$$u : \mathbb{R}\times[0,\infty)\times\Omega \to \mathbb{R}\cup\{-\infty\}$$

satisfies the following conditions

(a) The effective domain, $\mathcal{D} := \{x \in \mathbb{R} : u(x,t,\omega) > -\infty\}$, and the range, $\mathcal{R} := \{u(x,t,\omega) \mid x \in \mathcal{D}\}$ do not depend on $(t,\omega) \in [0,\infty)\times\Omega$ and are not empty.

(b) For almost any $\omega \in \Omega$ and for any $t \in [0,t)$ the function $x \to u(x,t,\omega)$ is strictly increasing on $\mathcal{D}$ and is concave, increasing and upper semicontinuos on $\mathbb{R}$.

(c) $\omega \to u(x,t,\omega)$ is $\mathcal{F}_t-$measurable for all $(x,t) \in \mathcal{D}\times[0,\infty)$

An additional possible assumption

(d) $t \to u(x,t,\omega)$ is decreasing for all $(x,\omega) \in \mathcal{D}\times\Omega$

Valuation mechanism

- Such SDU have recently been used to formulate the forward utility Theory (Musiela Zariphopoulou 06-08, Berrier Rogers Theranchi 08)
- Here we use the SDU to define a valuation mechanism, the backward conditional certainty equivalent - denoted by $C_{s,t}(X)$ - that represents the time-s-value of the time t claim X, for

$$0 \leq s \leq t < \infty.$$

- Set

$$\mathcal{U}(t) = \{X \in L^0(\Omega, \mathcal{F}_t, P) \,|\, u(X, t, \omega) \in L^1(\Omega, \mathcal{F}, P)\}.$$

Conditional Certainty Equivalent: CCE

Definition

Let u be a SDU and X be a random variable in $\mathcal{U}(t)$. For each $s \in [0, t]$, the backward Conditional Certainty Equivalent $C_{s,t}(X)$ of X is the random variable in $\mathcal{U}(s)$ solution of the equation:

$$u(C_{s,t}(X), s) = E\left[u(X, t)|\mathcal{F}_s\right].$$

Thus the CCE defines the valuation operator

$$C_{s,t} : \mathcal{U}(t) \to \mathcal{U}(s), \;\; C_{s,t}(X) = u^{-1}\left(E\left[u(X, t)|\mathcal{F}_s\right]\right), s).$$

This definition is the natural generalization to the dynamic and stochastic environment of the classical definition of the certainty equivalent, as given in Pratt 1964.

Equivalent definition of the CCE

Definition (Conditional Certainty Equivalent process)

Let u be a SDU and X be a random variable in $\mathcal{U}(t)$. The backward conditional certainty equivalent of X is the only process $\{Y_s\}_{0\leq s\leq t}$ such that $Y_t \equiv X$ and the process $\{u(Y_s, s)\}_{0\leq s\leq t}$ is a martingale.

Equivalent definition of the CCE

Definition (Conditional Certainty Equivalent process)

Let u be a SDU and X be a random variable in $\mathcal{U}(t)$. The backward conditional certainty equivalent of X is the only process $\{Y_s\}_{0\leq s\leq t}$ such that $Y_t \equiv X$ and the process $\{u(Y_s,s)\}_{0\leq s\leq t}$ is a martingale.

- As for the g-expectation - which is a local mechanism - this definition provides a non linear evaluation.
- Even if u is concave the CCE is not a concave functional, but it is conditionally quasiconcave

Setting for the dual representation

$$\pi : L_{\mathcal{F}} \longrightarrow L_{\mathcal{G}}$$

We now state the assumptions on the spaces $L_{\mathcal{F}}$ and $L_{\mathcal{G}}$ and on the quasiconvex conditional map π in order to obtain the dual representation.

Notations

- $L^p_{\mathcal{F}} := L^p(\Omega, \mathcal{F}, P)$, $p \in [0, \infty]$.
- $L_{\mathcal{F}} := L(\Omega, \mathcal{F}, P) \subseteq L^0(\Omega, \mathcal{F}, P)$ is a locally convex lattice
- $L_{\mathcal{G}} := L(\Omega, \mathcal{F}, P) \subseteq L^0(\Omega, \mathcal{G}, P)$ is a lattice of $\mathcal{G}$ measurable random variables.
- $L^*_{\mathcal{F}} = (L_{\mathcal{F}}, \geq)^*$ is the order continuous dual of $(L_{\mathcal{F}}, \geq)$, which is also a lattice.

Standing assumptions on the spaces

1. $L_{\mathcal{F}}$ (resp. $L_{\mathcal{G}}$) satisfies the property 1_F (resp 1_G):

$$X \in L_{\mathcal{F}} \text{ and } A \in \mathcal{F} \Longrightarrow (X\mathbf{1}_A) \in L_{\mathcal{F}}. \qquad (1_F)$$

2. $L^*_{\mathcal{F}} = (L_{\mathcal{F}}, \geq)^*$ is a Banach Lattice and $L^*_{\mathcal{F}} \hookrightarrow L^1(\Omega, \mathcal{F}, P)$.
3. $L_{\mathcal{F}} \hookrightarrow L^1(\Omega, \mathcal{F}, P)$.

Examples of spaces satisfying the assumptions

- The L^p spaces: $L_{\mathcal{F}} := L^p_{\mathcal{F}}$, with $p \in [1, \infty]$.
  Then: $L^*_{\mathcal{F}} = L^q_{\mathcal{F}} \hookrightarrow L^1_{\mathcal{F}}$ (with $q = 1$ when $p = \infty$).
- The Orlicz spaces $L_{\mathcal{F}} := L^{\Psi}_{\mathcal{F}}$, for any Young function Ψ.
  Then $L^*_{\mathcal{F}} = L^{\Psi^*} \hookrightarrow L^1_{\mathcal{F}}$, where Ψ^* is the conjugate of Ψ.
- The Morse subspace $L_{\mathcal{F}} := M^{\Psi}$ for any continuous Young function Ψ.
  Then $L^*_{\mathcal{F}} = L^{\Psi^*} \hookrightarrow L^1_{\mathcal{F}}$.

Conditions on π

Let $X_1, X_2 \in L^\infty(\Omega, \mathcal{F}, P)$ and consider

$$\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$$

(MON) $X_1 \leq X_2 \Longrightarrow \pi(X_1) \leq \pi(X_2)$

Conditions on π

Let $X_1, X_2 \in L^\infty(\Omega, \mathcal{F}, P)$ and consider

$$\pi : L_{\mathcal{F}} \rightarrow L_{\mathcal{G}}$$

(MON) $X_1 \leq X_2 \Longrightarrow \pi(X_1) \leq \pi(X_2)$

(τ-LSC) the lower level set

$$\mathcal{A}_Y = \{X \in L_{\mathcal{F}} \mid \pi(X) \leq Y\}$$

is τ closed for each $\mathcal{G}$-measurable Y

Conditions on π

Let $X_1, X_2 \in L^\infty(\Omega, \mathcal{F}, P)$ and consider

$$\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$$

(MON) $X_1 \leq X_2 \Longrightarrow \pi(X_1) \leq \pi(X_2)$

(τ-LSC) the lower level set

$$\mathcal{A}_Y = \{X \in L_{\mathcal{F}} \mid \pi(X) \leq Y\}$$

is τ closed for each $\mathcal{G}$-measurable Y

(REG) $\forall A \in \mathcal{G}$, $\pi(X_1 \mathbf{1}_A + X_2 \mathbf{1}_A^C) = \pi(X_1)\mathbf{1}_A + \pi(X_2)\mathbf{1}_A^C$

On continuity from below (CFB)

(CFB) $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ is *continuous from below if*

$$X_n \uparrow X \quad P \text{ a.s.} \quad \Rightarrow \quad \pi(X_n) \uparrow \pi(X) \quad P \text{ a.s.}$$

Proposition

If $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ satisfies (MON) and (QCO), then are equivalent:
(i) π is $\sigma(L_{\mathcal{F}}, L^*_{\mathcal{F}})$-(LSC)
(ii) π is (CFB)
(iii) π is order-(LSC) (i.e. the Fatou property)

Conclusion: in the following results, we may replace $\sigma(L_{\mathcal{F}}, L^*_{\mathcal{F}})$-(LSC) with (CFB).

The dual representation of conditional quasiconvex maps

Theorem

If $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ *is (MON), (QCO), (REG) and* $\sigma(L_{\mathcal{F}}, L^*_{\mathcal{F}})$*-LSC then*

$$\pi(X) = \sup_{Q \in L^*_{\mathcal{F}} \cap \mathcal{P}} K(X, Q)$$

where

$$K(X, Q) := \inf_{\xi \in L_{\mathcal{F}}} \{\pi(\xi) \mid E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}]\}$$

$$\mathcal{P} =: \left\{ \frac{dQ}{dP} \mid Q << P \text{ and } Q \text{ probability} \right\}$$

Exactly the same representation of the real valued case, but with conditional expectations!

$Q = P$ on $\mathcal{G}$

Corollary

Suppose that the assumptions of the Theorem hold true.
If for $X \in L_{\mathcal{F}}$ *there exists* $\eta \in L_{\mathcal{F}}$ *and* $\varepsilon > 0$ *such that* $\pi(\eta) + \varepsilon < \pi(X)$, *then*

$$\pi(X) = \sup_{Q \in L^*_{\mathcal{F}} \cap \mathcal{P}_{\mathcal{G}}} K(X, Q),$$

where

$$\mathcal{P}_{\mathcal{G}} =: \left\{ \frac{dQ}{dP} \mid Q \in \mathcal{P} \text{ and } Q = P \text{ on } \mathcal{G} \right\}.$$

NOTE: The (weak) additional assumption allows us to replace $\mathcal{P} =: \left\{ \frac{dQ}{dP} \mid Q << P \text{ and } Q \text{ probability} \right\}$ with the same set $\mathcal{P}_{\mathcal{G}}$ that is used in the convex conditional case.

The role of Monotonicity

- The assumption (MON) is only used to obtain the dual representation over the set of positive elements of the dual space, i.e. on probability measures.
- Only in the next two results the (MON) plays a role.
- Obviously, (MON) allows to replace "$\geq$" with "$=$":

Lemma

Let $Q \in L^*_{\mathcal{F}} \cap \mathcal{P}_{\mathcal{G}}$. If $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ is (MON) and (REG) then

$$K(X, Q) = \inf_{\xi \in L_{\mathcal{F}}} \{\pi(\xi) \mid E_Q[\xi|\mathcal{G}] =_Q E_Q[X|\mathcal{G}]\}$$

Cash additivity

- A map $\pi : L_{\mathcal{F}} \rightarrow L_{\mathcal{G}}$ is said to be

(CAS) *cash additivity if for all* $X \in L_{\mathcal{F}}$ *and* $\Lambda \in L_{\mathcal{G}}$

$$\pi(X + \Lambda) = \pi(X) + \Lambda.$$

Cash additivity

- A map $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ is said to be

  (CAS) *cash additivity if for all* $X \in L_{\mathcal{F}}$ *and* $\Lambda \in L_{\mathcal{G}}$

$$\pi(X + \Lambda) = \pi(X) + \Lambda.$$

- Note: (CAS) and (QCO) implies Convexity.

Cash additivity

- A map $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ is said to be

  (CAS) *cash additivity if for all* $X \in L_{\mathcal{F}}$ *and* $\Lambda \in L_{\mathcal{G}}$

$$\pi(X + \Lambda) = \pi(X) + \Lambda.$$

- Note: (CAS) and (QCO) implies Convexity.
- Next, we show that we recover the result of Detlefsen Scandolo 05 for convex conditional maps.

The Fenchel convex conjugate

Definition

Suppose that $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ is convex and $Q \in L^*_{\mathcal{F}} \cap \mathcal{P}_{\mathcal{G}}$. The conditional Fenchel convex conjugate π^* is the extended valued $\mathcal{G}-$measurable random variable:

$$\pi^*(Q) = \sup_{\xi \in L_{\mathcal{F}}} \{E_Q[\xi|\mathcal{G}] - \pi(\xi)\}.$$

The conditional convex case

Corollary

Suppose that the assumptions of the Theorem hold true.
Suppose that for every $Q \in L^*_{\mathcal{F}} \cap \mathcal{P}_{\mathcal{G}}$ *and* $\xi \in L_{\mathcal{F}}$ *we have* $E_Q[\xi|\mathcal{G}] \in L_{\mathcal{F}}$.
If $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ *satisfies in addition (CAS) then*

$$K(X, Q) = E_Q[X|\mathcal{G}] - \pi^*(Q).$$

and

$$\pi(X) = \sup_{Q \in L^*_{\mathcal{F}} \cap \mathcal{P}_{\mathcal{G}}} \left\{ E_Q[X|\mathcal{G}] - \pi^*(Q) \right\}.$$

Why the proofs of the real valued case and convex case do not work

- We cannot directly apply Hahn-Banach to $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$, as it happened in the real case, since

$$\{\xi \in L_{\mathcal{F}} \mid \pi(\xi) \leq \pi(X) - \varepsilon\}^C$$

is not any more convex!

- Convexity is preserved by the map:

$$\pi_0 : L_{\mathcal{F}} \to \mathbb{R} \quad \pi_0(X) := E[\pi(X)]$$

but not quasiconvexity!

Approximation argument

The idea is to approximate π with combinations of quasiconvex real valued functions π_A

$$\pi_A(X) := ess \sup_{\omega \in A} \pi(X), \; A \in \mathcal{G}.$$

We consider finite partitions $\Gamma = \{A^\Gamma\}$ of $\mathcal{G}$ measurable sets A^Γ and

$$\pi^\Gamma(X) := \sum_{A^\Gamma \in \Gamma} \pi_{A^\Gamma}(X) \mathbf{1}_{A^\Gamma},$$

$$H^\Gamma(X) := \sup_{Q \in L^*_{\mathcal{F}} \cap \mathcal{P}} \inf_{\xi \in L_{\mathcal{F}}} \left\{ \pi^\Gamma(\xi) \mid E_Q[\xi|\mathcal{G}] \geq E_Q[X|\mathcal{G}] \right\}$$

Steps of the proof

I First we show $H^\Gamma(X) = \pi^\Gamma(X)$.

II Then it is a simple matter to deduce

$$\pi(X) = \inf_\Gamma \pi^\Gamma(X) = \inf_\Gamma H^\Gamma(X)$$

III Finally we prove that

$$\begin{aligned} \inf_\Gamma H^\Gamma(X) &= \inf_\Gamma \sup_{Q \in L_t^* \cap \mathcal{P}} \inf_{\xi \in L_t} \left\{ \pi^\Gamma(\xi) | E_Q[\xi|\mathcal{F}_s] \geq E_Q[X|\mathcal{F}_s] \right\} \\ &= \sup_{Q \in L_t^* \cap \mathcal{P}} \inf_{\xi \in L_t} \{ \pi(\xi) | E_Q[\xi|\mathcal{F}_s] \geq E_Q[X|\mathcal{F}_s] \} \end{aligned}$$

that is based on a uniform approximation result.

Recall...the Conditional Certainty Equivalent (CCE)

$$\mathcal{U}(t) = \{X \in L^0(\Omega, \mathcal{F}_t, P) \,|\, u(X, t, \omega) \in L^1(\Omega, \mathcal{F}, P)\}.$$

Definition

Let u be a SDU and X be a random variable in $\mathcal{U}(t)$. For each $s \in [0, t]$, the backward Conditional Certainty Equivalent $C_{s,t}(X)$ of X is the random variable in $\mathcal{U}(s)$ solution of the equation:

$$u(C_{s,t}(X), s) = E\left[u(X, t)|\mathcal{F}_s\right].$$

Thus the CCE defines the valuation operator

$$C_{s,t} : \mathcal{U}(t) \to \mathcal{U}(s), \;\; C_{s,t}(X) = u^{-1}\left(E\left[u(X, t)|\mathcal{F}_s\right]\right), s).$$

Conditional risk premium

Definition

Let u be a SDU and X be a random variable in $\mathcal{U}(t)$. For each $s \in [0, t]$, the *conditional risk premium* of X is the random variable $r_{s,t}(X) \in L^0(\Omega, \mathcal{F}_s, P; \mathcal{D})$ defined by:

$$r_{s,t}(X) := E[X|\mathcal{F}_s] - C_{s,t}(X).$$

- One then looks for local expressions of $r_{s,t}$ in terms of the risk aversion coefficient and impatience factor that can be written in terms of the partial derivatives of u.
- But we now present the dual representation of the CCE.

Properties of the CCE

Proposition

Let u be a SDU, $0 \leq s \leq v \leq t < \infty$ and $X, Y \in \mathcal{U}(t)$.

(*) $C_{s,t}(X) = C_{s,v}(C_{v,t}(X))$ *and* $C_{t,t}(X) = X$.

MON $X \leq Y \Rightarrow C_{s,t}(X) \leq C_{s,t}(Y)$.

REG $C_{s,t}(X\mathbf{1}_A + Y\mathbf{1}_{A^C}) = C_{s,t}(X)\mathbf{1}_A + C_{s,t}(Y)\mathbf{1}_{A^C}$, $\forall A \in F_s$ *and* $C_{s,t}(X)\mathbf{1}_A = C_{s,t}(X\mathbf{1}_A)\mathbf{1}_A$.

QCO *Quasiconcavity: the upper level set* $\{X \in \mathcal{U}_t \mid C_{s,t}(X) \geq Y\}$ *is conditionally convex for every* $Y \in L^0_{\mathcal{F}_s}$.

CFA *Continuity From Above:* $X_n \downarrow X \quad \Rightarrow \quad \pi(X_n) \downarrow \pi(X)$.

(**) *If in addition u is decreasing with time then:*

- $C_{s,t}(X) \leq E[C_{v,t}(X)|\mathcal{F}_s]$ *and* $E[C_{s,t}(X)] \leq E[C_{v,t}(X)]$.
- $C_{s,t}(X) \leq E[X|\mathcal{F}_s]$ *and* $E[C_{s,t}(X)] \leq E[X]$.

Example

Consider $u : \mathbb{R} \times [0, \infty) \times \Omega \to \mathbb{R}$ defined by

$$u(x, t, \omega) = 1 - e^{-\alpha_t(\omega)x + A_t(\omega)}$$

where $\alpha_t > 0$ and A_t are adapted stochastic processes.

$$C_{s,t}(X) = -\frac{1}{\alpha_s} \ln \left\{ \mathbb{E}[e^{-\alpha_t X + A_t} | \mathcal{F}_s] \right\} + \frac{A_s}{\alpha_s}.$$

If $\alpha_t(\omega) \equiv \alpha \in \mathbb{R}$ and $A_t \equiv 0$ then

$$C_{0,t}(X) = -\frac{1}{\alpha} \ln \left\{ \mathbb{E}[e^{-\alpha X}] \right\}$$

$$C_{s,t}(X) = -\frac{1}{\alpha} \ln \left\{ \mathbb{E}[e^{-\alpha X} | \mathcal{F}_s] \right\}$$

$\rho_u(X) := -C_{0,t}(X)$ is the risk measure induced by the exponential utility. By introducing a time dependence in the risk aversion coefficient one looses the monetary property.

Cash super-additive property

$$C_{s,t}(X+Y) \geq C_{s,t}(X) + Y,\ Y \in \mathcal{F}_s,\ Y \geq 0.$$

When the risk aversion coefficient is purely stochastic we have no chance that $C_{s,t}$ has any monetary or cash super-additive property.

Proposition

If the process $\{\alpha_t\}_{t\geq 0}$ is almost surely increasing then the (CCE) $C_{s,t}(X) = -\frac{1}{\alpha_s}\ln\left\{\mathbb{E}[e^{-\alpha_t X + A_t}|\mathcal{F}_s]\right\} + \frac{A_s}{\alpha_s}$ is cash super-additive

Cash super-additive property

$$C_{s,t}(X+Y) \geq C_{s,t}(X) + Y,\ Y \in \mathcal{F}_s,\ Y \geq 0.$$

When the risk aversion coefficient is purely stochastic we have no chance that $C_{s,t}$ has any monetary or cash super-additive property.

Proposition

If the process $\{\alpha_t\}_{t\geq 0}$ is almost surely increasing then the (CCE) $C_{s,t}(X) = -\frac{1}{\alpha_s} \ln \left\{ \mathbb{E}[e^{-\alpha_t X + A_t} | \mathcal{F}_s] \right\} + \frac{A_s}{\alpha_s}$ is cash super-additive

In order to apply our dual representation theorem to the (CCE), we still need to define it on an appropriate lattice.

Selection of the right spaces

In the literature the generalization of Orlicz spaces to the case of stochastic (not time dependent) functions are known as *Musielak – Orlicz* spaces (Musielak, 83). In our framework:

$$L^{\hat{u}} = \left\{ X \in L^0 \mid \int_\Omega \hat{u}(\lambda X(\omega), \omega) P(d\omega) < \infty \text{ for some } \lambda > 0 \right\}$$

$$M^{\hat{u}} = \left\{ X \in L^0 \mid \int_\Omega \hat{u}(\lambda X(\omega), \omega) P(d\omega) < \infty \, \forall \, \lambda > 0 \right\}$$

where

$$\widehat{u}(x,t) := u(0,t) - u(-|x|,t).$$

is the Young function associated to u, introduced by Biagini-F. 08.

These spaces are well defined under an additional integrability assumption, that we now show for the time dependent utility.

The dynamic version of Musielak-Orlicz space

Consider a SDU $u = u(x, t, \omega)$ and set

$$\widehat{u}(x, t, \omega) := u(0, t, \omega) - u(-|x|, t, \omega).$$

We suppose that $\widehat{u}$ satisfies the integrability condition:

(int) $\forall t \geq 0$, $\forall x \in \mathcal{D}_{\widehat{u}} = \{y \in \mathbb{R} \mid \widehat{u}(y, t, \omega) < +\infty\}$ we have:

$$E[\widehat{u}(x, t, \omega)] < \infty.$$

...Musielak Orlicz spaces

and consider the Banach spaces

$$L^{\hat{u}_t}(\Omega, \mathcal{F}_t, P) = \left\{ X \in L^0(\Omega, \mathcal{F}_t, P) \,\middle|\, \int_\Omega \hat{u}(\lambda X(\omega), t, \omega) P(d\omega) < \infty \text{ some } \lambda >\right.$$

$$M^{\hat{u}_t}(\Omega, \mathcal{F}_t, P) = \left\{ X \in L^0(\Omega, \mathcal{F}_t, P) \,\middle|\, \int_\Omega \hat{u}(\lambda X(\omega), t, \omega) P(d\omega) < \infty \, \forall \, \lambda > 0 \right\}$$

endowed with the Luxemburg norm

$$N_{\hat{u}}(X) = \inf \left\{ c > 0 \,\middle|\, \int_\Omega \hat{u} \left(\frac{X(\omega)}{c}, t, \omega \right) P(d\omega) \leq 1 \right\}.$$

The CCE is consistent on $M^{\widehat{u}_t}$

Example: we apply this setting to the SDU:

$$u(x, t, \omega) = 1 - e^{-\alpha_t(\omega)x + A_t(\omega)}$$

The additional integrability assumption on $\widehat{u}$ is satisfied if, for example:

$E[e^{\alpha_t|x|+A_t}] < \infty$, $\forall\, x \in \mathbb{R}$ and $A_t \in L^\infty(\mathcal{F}_t)$.

Proposition

$$C_{s,t} : M^{\widehat{u}_t} \longrightarrow M^{\widehat{u}_s}$$

We apply our dual representation theorem to the CCE

Theorem

For every $X \in L^{\widehat{u}_t}$

$$C_{s,t}(X) = \inf_{Q \in L^{(\widehat{u}_t)^*} \cap \mathcal{P}} G(E_Q[X|\mathcal{F}_s], Q),$$

where

$$G(Y,Q) = \sup_{\xi \in L^{\widehat{u}_t}} \left\{ C_{s,t}(\xi) \mid E_Q[\xi|\mathcal{F}_s] = Y \right\}, Y \in L^0_{\mathcal{F}_s}.$$

Moreover if $X \in M^{\widehat{u}_t}$ *then the essential infimum is actually a minimum.*

The theorem holds for all SDU u satisfying our assumptions.

Conclusion

Conclusion

- THANK YOU ... and

Conclusion

- THANK YOU ... and

- WELCOME TO THE RECEPTION !!!