FRAGILITY OF ARBITRAGE AND BUBBLES IN DIFFUSION MODELS

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Fields Institute, Toronto, 15th January 2010

Trading model

– Probability space with a filtration $\mathcal{F}_t, t \in [0, T]$. $\mathcal{F}_0$ is assumed trivial.

– $S_t, t \in [0, T]$ is a d-dimensional continuous semimartingale with strictly positive components, representing the price evolution of d risky assets.

– A trading strategy H is an S-integrable process. The corresponding wealth process is $V_t^H := V_0 + (H \cdot S)_t$ where V_0 is the initial capital.

– H is admissible if, for some $c > 0$, $V_t^H \geq -c$ a.s. for all $t \in [0, T]$. The family of such strategies is denoted by $\mathcal{A}$.

Arbitrage

– We say that the model admits arbitrage if there is $H \in \mathcal{A}$ such that starting from $V_0 = 0$ we get to a position $V_T^H \geq 0$ a.s. and $P(V_T^H > 0) > 0$.

Example. Take the (unique strong) solution of

$$dS(t) = \frac{1}{S(t)}dt + dW(t), \; S(0) := 1,$$

where $W(t)$ is Brownian motion. This is called Bessel process and is notorious for providing arbitrage opportunities.

Bubbles

– No standard definition, we informally mean that an asset is overpriced.

– For today's talk, a bubble is some admissible strategy H satisfying

$$x + \int_0^T H(u)dS(u) \geq S_i(T)$$

for some $i \in \{1, \ldots, d\}$ and $x < S_i(0)$.

– If there were no admissibility constraints, a bubble would also form an arbitrage.

Strict local martingales

Definition. $S_t, t \in [0, T]$ is a local martingale if there is a nondecreasing sequence of stopping times τ_n tending to ∞ almost surely such that $S_{t \wedge \tau_n}, t \in [0, T]$ is a martingale for each n. A strict local martingale is a local martingale which is not a martingale.

– Strict local martingales are suggested for modelling bubbles in a large number of recent papers: Loewenstein, Willard, Jarrow, Protter, Shimbo, Pal, Cox, Hobson, Ekström, Tysk.

Example 1: inverse Bessel process

Take $U(t) = 1/S(t)$, where $S(t)$ is the Bessel process mentioned above. This satisfies the equation

$$dU(t) = -U^2(t)dW(t), \qquad U(0) = 1.$$

U_t is a supermartingale, but not a martingale and P is the unique local martingale measure for U_t. Hence, by the characterization of superhedging, there is admissible H with

$$ES(T) + \int_0^T H(t)dS(t) \geq S(T),$$

and, since $ES(T) < S(0)$, a bubble phenomenon arises.

Example 2: stochastic volatility model

Consider

$$dP(t) = P(t)V(t)dB(t), \quad P(0) := 1,$$

$$dV(t) = V(t)\rho dB(t) + V(t)\sqrt{1-\rho^2}dW(t) + V(t)dt, \quad V(0) := 1$$

where $(B(t), W(t))$ is a 2-dimensional Brownian motion, $0 < \rho < 1$.

– $P(t)$ is thought to represent the price of an asset (under a risk-neutral measure) while $V(t)$ its volatility. $P(t)$ is a strict local martingale and admits a bubble.

Absence of arbitrage and bubbles

– If there is $Q \sim P$ such that S is a Q-local martingale then there is no arbitrage.

– If S is a true Q-martingale then there are no bubbles either.

Local martingales in discrete time

Theorem. If $S_t,\ t \in \mathbb{N}$ is a local martingale then there exists $Q \sim P$ with arbitrarily small $\|Q - P\|_{tv}$ such that S_t is a Q-martingale (with prescribed integrablity conditions).

Dalang-Morton-Willinger, Schachermayer, Kabanov, Prokaj, Rásonyi.

In continuous time

Theorem. $b : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ measurable, $\sigma : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}$ continuous, $\sigma(t,x)\sigma^T(t,x)$ positive definite for all t, x.

For each $0 \le t < T, x \in \mathbb{R}^d$ the SDE

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t), \quad X(0) = x,$$

has a unique solution measure $P^{t,x}$ on $C[t,T]$.

Fix $u \in \mathbb{R}^d$. Take $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ with $\mathcal{F}_0$ trivial and an $\mathcal{F}_t$-Brownian motion $W(t)$ such that $X^{0,u}(t)$ is a solution of the above equation with $X^{0,u}(0) = u$.

Theorem continued. Define

$$S_i(t) := \exp\{X_i^{0,u}(t)\}, \quad i = 1, \ldots, d, \quad t \in [0, T].$$

Then for each $\varepsilon > 0$ there exists a d-dimensional process $\tilde{S}(t),\ t \in [0, T]$ and a probability $Q \sim P$ such that

$$\frac{1}{1+\varepsilon} \leq \frac{\tilde{S}_i(t)}{S_i(t)} \leq (1+\varepsilon)$$

almost surely for all $i = 1, \ldots, d$, $t \in [0, T]$ and $\tilde{S}(t)$ is a Q-martingale (with respect to $\mathcal{F}_t$).

Consequences

Take $S(t)$ the Bessel process. Our main result applies and, for any $\varepsilon, T > 0$, there is an arbitrage-free price process $\tilde{S}(t), t \in [0, T]$ which is ε-close to $S(t)$ on the logarithmic scale.

Consequences

Let $U(t)$ be the inverse Bessel process. The same $\tilde{S}(t)$ as in the previous example provides a uniformly ε-close model, i.e. for all $t \in [0, T]$,

$$\frac{1}{1+\varepsilon} \leq \frac{\tilde{S}(t)}{U(t)} \leq (1+\varepsilon)$$

and $\tilde{S}(t)$ models prices without bubbles.

Consequences

The stochastic volatility model $S(t) = (P(t), V(t)),\ t \in [0,T]$ satisfies the conditions of the main theorem, so there is $Q \sim P$ and a Q-martingale $M(t) := \tilde{S}_1(t),\ t \in [0,T]$ such that

$$\frac{1}{1+\varepsilon} \leq \frac{M(t)}{P(t)} \leq (1+\varepsilon)$$

holds a.s. for $t \in [0,T]$. Thus our results apply to incomplete market models, too.

Markets with transaction costs

– If there are proportional transaction costs present in the market then the objects corresponding to equivalent martingale measures in the frictionless case are exactly processes like $\tilde{S}_t$.

– Our results could be reformulated and proved in the following manner: as soon as there are transaction costs (arbitrarily small), there is no arbitrage and there are no bubbles in diffusion models satisfying the hypotheses of our main theorem.

Counterexample

This example is due to Christophe Stricker. Let

$$X_t := \exp(W_t - t/2),\ t \geq 0,$$

where W is a Brownian motion and $(\mathcal{F}_t)_{t\geq 0}$ is its natural filtration. Define

$$\tau := \inf\{t : X_t = 1/2\},$$

and set

$$S_t := X_{\tau \wedge \mathrm{tg}\, t},\ 0 \leq t < \pi/2, \quad S_t = 1/2,\ t \geq \pi/2.$$

Counterexample

– Set $\mathcal{G}_t := \mathcal{F}_{\mathrm{tg}\, t}$. We find that S_t is a $(\mathcal{G}_t)_{t\geq 0}$-local martingale.

– As $S_{\pi/2}$ is constant, we cannot get uniformly ε-close to S_t with any martingale on $[0, \pi/2]$.

– This is one more (drastic) example for a bubble.

Conclusions

– Arbitrage is fragile in diffusion models: a small mis-specification or the introduction of transaction costs destroys them.

– Strict local martingales are fragile models for bubbles.