How do optimal portfolios depend on the time horizon?

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Basic question

Financial problem

- **Standard problem:** optimal portfolio choice with time horizon T, i.e.

$$E[U(x+\vartheta \cdot S_T)] = E\left[U\left(x+\int_0^T \vartheta_u\, dS_u\right)\right] = \max!$$

over admissible strategies ϑ on $[0, T]$.

- Denote solution by

$$\widehat{X}_T = x + \hat{\vartheta}^{(T)} \cdot S_T = x + \int_0^T \hat{\vartheta}(T,u)\, dS_u.$$

- **Basic question: How does all that depend on the time horizon T ?**

Mathematical problem

- **Semimartingale** S with "usual no arbitrage assumptions".
- **Utility field** $U(T, y, \omega)$ such that
  - $y \mapsto U(t, y, \omega)$ is standard utility function.
  - $(T, \omega) \mapsto U(T, y, \omega)$ is reasonable stochastic process (right-continuous, of locally integrable variation).
- For each T, we get **random variable** of the form

$$\widehat{X}_T = x + \int_0^T \hat{\vartheta}(T, u)\, dS_u.$$

- **What kind of stochastic process is $(\widehat{X}_T)_{T \geq 0}$?**
- Double dependence on parameter T makes problem difficult.

Literature ?

- ???
- **very little**
- ???

Main results

Main results in a nutshell

- **1)** The process $\widehat{X} = (\widehat{X}_T)_{T \geq 0}$ of optimal wealths has a version $\widetilde{X}$ which is a **semimartingale**.
- **2)** The optimal strategy field $\big(\hat{\vartheta}(T,u)\big)_{T \geq 0, 0 \leq u \leq T}$ has a version $\tilde{\vartheta}$ which is **of finite variation with respect to the time horizon T**.
- **3)** The semimartingale $\widetilde{X}$ admits a **unique decomposition** as

$$\widetilde{X}_T = x + \int_0^T \bar{\vartheta}_u \, dS_u + \bar{B}_T \qquad \text{for all } T \geq 0$$

for an **integrand** $\bar{\vartheta}$ which does not depend on T and a **predictable process** $\bar{B}$ which is **of finite variation**.

Potential applications

Ideas for applications

- Can replace horizon-dependent strategy $\hat{\vartheta}^{(T)}$ by "universal strategy" $\bar{\vartheta}$; then $\bar{B}_T$ quantifies "utility loss" for T-horizon problem.
- With dependence of strategy on time horizon well understood, can study random horizon models.
- Can study "random endowment" $\bar{B}_T$ as incentive to eliminate influence of time horizon on portfolio choice.
- many more possibilities ...

Outline of proof

Semimartingale version $\widetilde{X}$ for $\widehat{X}$

Getting a regular version for $\widehat{X}$

- Recall optimal wealth $\widehat{X}_T = x + \int_0^T \hat{\vartheta}(T,u)\, dS_u$.
- First prove that $T \mapsto \widehat{X}_T$ **is right-continuous in probability**.
    - Similar argument as in **Kramkov/Schachermayer**, who prove continuity with respect to x.
    - Uses right-continuity of U with respect to T (plus technical conditions).
- Next prove that $\widehat{X}$ **satisfies (NUPBR) with respect to simple integrands for** $\widehat{X}$.
    - Uses that S satisfies (NUPBR).
    - Uses translation of admissible simple integrands for $\widehat{X}$ into general admissible integrands for S.
    - Needs **technical result on good change of measure**.

Getting a semimartingale version for $\widehat{X}$

- **An adapted process satisfying (NUPBR) for simple admissible integrands admits right and left limits along the rationals.**
  - Combines stochastic analysis and mathematical finance.
  - Inspired from **Delbaen/Schachermayer**.
  - Substitute for **Bichteler–Dellacherie**.
- A process which is right-continuous in probability and has left and right limits along the rationals admits an RCLL version.
- **Indirect:** If $\widehat{X}$ is no **semimartingale**, then S admits arbitrage; uses again translation of simple admissible stochastic integrals over $\widehat{X}$ into general admissible stochastic integrals over S.

Properties of optimal strategy field $\hat{\vartheta}(T, u)$

First properties of $\hat{\vartheta}(T, u)$

- View $\hat{\vartheta}(\cdot, \cdot)$ as process on $\bar{\Omega} := \Omega \times (0, \infty)$ indexed by T.
- First prove that $T \mapsto \hat{\vartheta}(T, \cdot)$ **is right-continuous in measure** (for $P \otimes [S]$, morally).
- Then prove that $\hat{\vartheta}$ **has a product-measurable version** ϑ' on $[0, \infty) \times \bar{\Omega}$.
  - Uses that $T \mapsto \widehat{X}_T$ is right-continuous in probability.
  - Combination with good change of measure gives even continuity in L^2 (locally).
  - Use isometry of stochastic integral after change of measure.
  - $\dots$
- Prove that **diagonal** $\bar{\vartheta}$ **of** ϑ' **is predictable and** S**-integrable**.
- How to get **finite variation** property with respect to T **?**

Getting finite variation for $T \mapsto \hat{\vartheta}(T, \cdot)$

- Again view $\hat{\vartheta}(\cdot, \cdot)$ as process indexed by T and defined on $\bar{\Omega} := \Omega \times (0, \infty)$. Use constant filtration $\mathcal{P}_t \equiv \mathcal{P}$ on $\bar{\Omega}$.
- **Idea:** Prove that ϑ' has a version $\tilde{\vartheta}$ which is a **semimartingale** (indexed by T) on $\bar{\Omega}$.
  - First show that **mapping** $f \mapsto \int_0^\infty f(T)\,\vartheta'(dT, \cdot)$ **is continuous** on very simple predictable processes f on $\bar{\Omega}$.
  - Exploits structure of ϑ' as family of optimal strategies.
  - Prove that ϑ' is a **quasimartingale** over $\bar{\Omega}$.
  - Deduce existence of version $\tilde{\vartheta}$ which is **right-continuous in** T.
  - Now use again continuity and Bichteler–Dellacherie to get **semimartingale** property for $\tilde{\vartheta}$.

Decomposition of $\widetilde{X}$ and stochastic Fubini

Idea for decomposition

- $T \mapsto \tilde{\vartheta}(T, \cdot)$ is of finite variation; so

$$\begin{aligned}\widetilde{X}_T &= x + \int_0^T \tilde{\vartheta}(T, u)\, dS_u \\ &= x + \int_0^T \tilde{\vartheta}(u, u)\, dS_u + \int_0^T \int_u^T \tilde{\vartheta}(dt, u)\, dS_u.\end{aligned}$$

- So $\bar{\vartheta}$ must be **diagonal** of $\tilde{\vartheta}(\cdot, \cdot)$ — and we know that **diagonal $\bar{\vartheta}$ of $\tilde{\vartheta}(\cdot, \cdot)$ is good integrand for S.**

Decomposition of $\widetilde{X}$: main problem

- How about second term in decomposition?
- Consider the process

$$\bar{B}_T := \int_0^T \int_u^T \tilde{\vartheta}(dt, u)\, dS_u \qquad \text{for } T \geq 0 :$$

- Is it of **finite variation ?**
- Is it **predictable ?**
- Can we get rid of double dependence on T somehow?
  "Obvious" idea: Use Fubini somehow.

Towards a stochastic Fubini result

- **Idea:** Since $\tilde{\vartheta}(\cdot, u)$ charges only $(u, T]$, write

$$\bar{B}_T = \int_0^T \int_u^T \tilde{\vartheta}(dt, u)\, dS_u \qquad = \qquad \int_0^T \Big(\tilde{\vartheta}(\cdot, u)\,\big(I_{(0,T]}\big)\Big)\, dS_u$$

$$\overset{(???)}{=} \qquad \left(\int_0^T \tilde{\vartheta}(\cdot, u)\, dS_u\right)\big(I_{(0,T]}\big)\,.$$

- **(???) :** How and why **?**
- View $\tilde{\vartheta}$ as process indexed by u and taking values in the space of finite signed measures (with respect to T).
- So we need . . .

Fubini ingredients and aspects . . .

- . . . a **measure-valued stochastic integral** $\phi \cdot M$, with respect to a martingale M, of a measure-valued integrand ϕ.
- . . . a version of **Fubini's theorem** for such stochastic integrals.
- . . . a result on **predictability**.
- . . . why martingale? Again, use **good change of measure** . . .
- . . . ideas inspired by **Björk/Di Masi/Kabanov/Runggaldier**, but technically different.
- . . . generalises several existing results on "stochastic Fubini theorems".

Assumptions

Assumptions on S

- S is **semimartingale**.
- S satisfies a **variant of "existence of an equivalent σ-martingale measure"**, namely $\mathcal{Z}_{e,\sigma} \neq \emptyset$**:** There exists local P-martingale $Z > 0$ such that ZS is P-σ-martingale and such that $Z \log Z$ **is locally P-integrable**.

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  - **Globally weaker** than (NFLVR), since Z need not be a true UI martingale (so need no existence of equivalent measure).
  - **Locally stronger** than (NFLVR), since we want a little bit more than local integrability for Z.
  - Very remarkably, this condition is **invariant under change to $Q \approx P$ if dQ/dP is bounded**.

Assumptions on U

- $y \mapsto U(T, y, \omega)$ is **standard utility function on** $[0, \infty)$.
- $(T, \omega) \mapsto U(T, y, \omega)$ is **adapted, right-continuous and of locally integrable variation**.
- For each finite time horizon T, problem of maximising $E[U(T, x + \vartheta \cdot S_T)]$ over all x-admissible ϑ for S up to time T has **unique solution** $\widehat{X}_T$ and **finite value**.

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- For any uniformly bounded sequence of stopping times (τ_n) decreasing to τ and any $0 < a \leq b < \infty$, we have P-a.s.

$$\lim_{n \to \infty} \sup_{a \leq y \leq b} |U(\tau_n, y, \omega) - U(\tau, y, \omega)| = 0.$$

References, etc.

Last words

This is work still in progress (but hopefully finished very soon).

Look for it on my homepage

http://www.math.ethz.ch/∼mschweiz

Thank you for your attention !