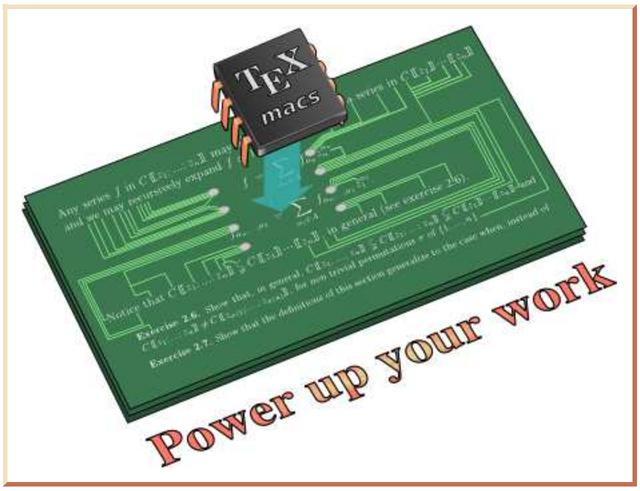
The art of guessing

Joris van der Hoeven
CNRS, École polytechnique



Fields Institute, Toronto, 2009

http://www.TEXMACS.org

Guess what

$$0.142857142857142857142857142857142857142857\cdots$$

$$0.142857142857142857142857142857142857142857\cdots$$

$$\frac{1}{7}$$

Applications:

- More comprehensive output.
- New unexpected insights.
- Guide decisions in algorithms.

Efficient algorithm

$$x_0 = a_0 + \frac{1}{x_1} \qquad a_0 = \lfloor x_0 \rfloor$$

$$x_0 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{x_2}} \qquad a_1 = \lfloor x_1 \rfloor$$

$$x_0 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_{l-2} + \cfrac{1}{a_{l-1} + x_l}}}}$$

$$x_l = \frac{p_l\, x_0 + q_l}{r_l\, x_0 + s_l} \qquad p_l, q_l, r_l, s_l \in \mathbb{Z}$$

$$x_{k+l} = \frac{p_{k,l}\, x_l + q_{k,l}}{r_{k,l}\, x_l + s_{k,l}} \qquad p_{k,l}, q_{k,l}, r_{k,l}, s_{k,l} \in \mathbb{Z}$$

$$\begin{pmatrix} p_{k,l+m} & q_{k,l+m} \\ r_{k,l+m} & s_{k,l+m} \end{pmatrix} = \begin{pmatrix} p_{k+l,m} & q_{k+l,m} \\ r_{k+l,m} & s_{k+l,m} \end{pmatrix} \begin{pmatrix} p_{k,l} & q_{k,l} \\ r_{k,l} & s_{k,l} \end{pmatrix}$$

$$\begin{aligned} \mathsf{F}(p) &= 2\,\mathsf{F}(p/2) + O(\mathsf{I}(p)) \\ &= O(\mathsf{I}(p) \log p) \\ &= O(p \log^2 p \log\log p) \end{aligned}$$

Guessing $\mathbb{Z}$-linear dependencies

Question: given $z_1, \ldots, z_n \in \mathbb{R}$, find $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}$ with

$$\lambda_1 \, z_1 + \cdots + \lambda_n \, z_n = 0$$

Example:

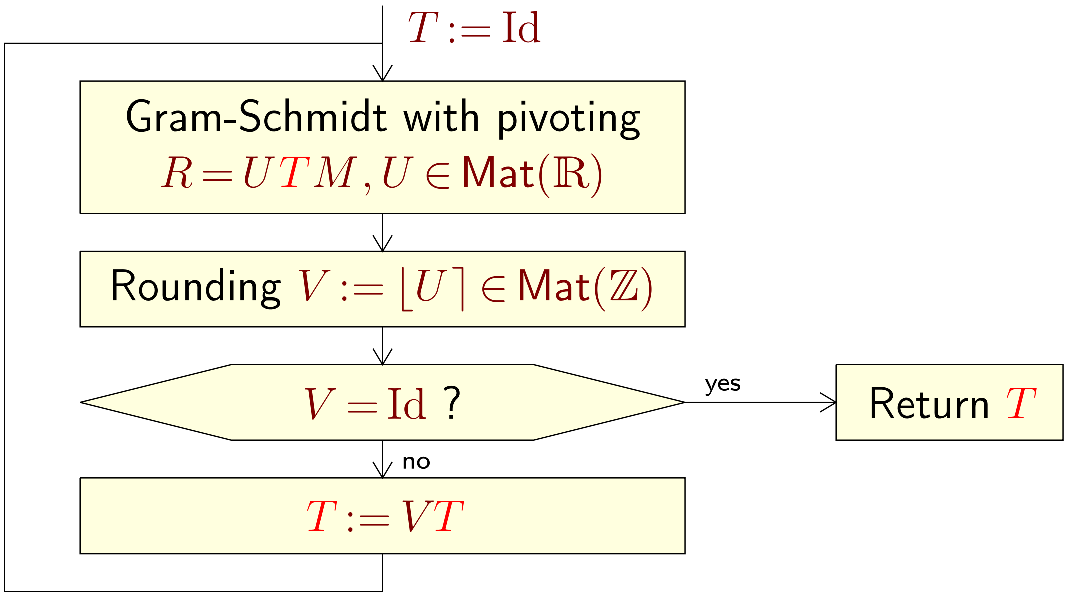
- $z_1 = 1, \ldots, z_n = \alpha^{n-1}$ for $\alpha \in \mathbb{R}$.
- $z_1 = \mathrm{Li}_{3,3}$, $z_2 = \mathrm{Li}_6$, $z_3 = \mathrm{Li}_{4,2}$, $z_4 = \mathrm{Li}_{2,2,2}$.

Lattice reduction:

$$M = \begin{pmatrix} z_1 & \varepsilon & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ z_n & 0 & 0 & \varepsilon \end{pmatrix}$$

Block lattice reduction (work in progress)

The main loop



Dichotomy over precision

1. For low precisions, compute $T \in \mathsf{Mat}(\mathbb{Z})$ as above.
2. For large precisions p, do

$$\begin{array}{lcll} T_1 & := & \mathrm{Reduce}(M \,\|\, 2^{-p/2}\, |M|\, \mathrm{Id}) & \quad \text{(at precision } p/2\text{)} \\ T_2 & := & \mathrm{Reduce}(T_1\, M) & \quad \text{(at precision } p/2\text{)} \end{array}$$

and return $T_2 T_1$.

Power series analogues

Guessing rationality

Problem: given $f \in \mathbb{K}[[z]]$, guess $P, Q \in \mathbb{K}[z]$ with $f = \frac{P}{Q}$

Solution: Padé approximation

Polynomial dependencies

Problem: given $f_1, \ldots, f_n \in \mathbb{K}[[z]]$, guess $P_1, \ldots, P_n \in \mathbb{K}[z]$ with

$$P_1 f_1 + \cdots + P_n f_n = 0.$$

Solution: Padé-Hermite (Beckermann-Labahn, Derksen, Gfun)

Applications

- $(f_n)_{n \in \mathbb{N}} = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$
- $f_1 = \varphi, f_2 = \varphi', \ldots, f_n = \varphi^{(n-1)}$, with $\varphi \in \mathbb{K}[[z]]$

Guessing singular dependencies

Example: number s_n of alcohols of the form $C_nH_{2n+1}OH$

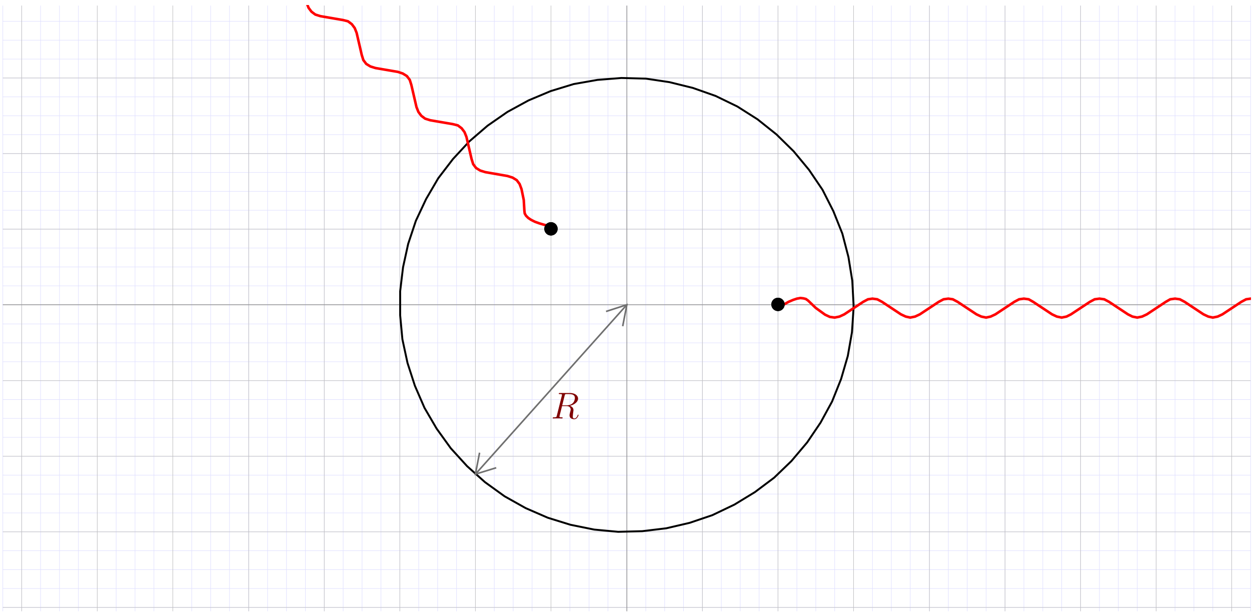
$$s(z) = 1 + z\,\frac{s(z)^3 + 2\,s(z^3)}{3}.$$

Dominant **algebraic** singularity at $r = 0.304218409\cdots$. Setting

$$\begin{aligned}\varphi(z) &= s(r+z)\\ \psi(z) &= 2\,s((r+z)^3) + 3,\end{aligned}$$

we have

$$(r+z)\,\varphi(z)^3 - 3\,\varphi(z) + \psi(z) = 0.$$



Problem: given $f_1, \ldots, f_d \in \mathbb{C}[[z]]$ and R, guess relations

$$g_1\,f_1 + \cdots + g_d\,f_d = h \qquad (g_1, \ldots, g_d, h \in \mathbb{A}_R;\ \text{i.e. analytic on } \bar{\mathcal{D}}_R)$$

Problem: given $f_1, \ldots, f_d \in \mathbb{C}[[z]]$ and $R = 1$, find a "minimal relation"

$$g_1\,f_1 + \cdots + g_d\,f_d = h$$

for the power series norm

$$\|\varphi\| = \sqrt{\varphi_0^2 + \varphi_1^2 + \cdots}.$$

Truncated problem at order n

$$g_1\, f_1 + \cdots + g_d\, f_d = h$$

$$G = \begin{pmatrix} g_{1,n-1} & \cdots & g_{d,n-1} & \cdots & \cdots & g_{1,0} & \cdots & g_{d,0} \end{pmatrix}$$

$$M = \begin{pmatrix}
f_{1,0} & & & & & 1 & & & & & & & & & & \\
\vdots & & & & & & \ddots & & & & & & & & & \\
f_{d,0} & & & & & & & 1 & & & & & & & & \\
f_{1,1} & f_{1,0} & & & & & & & 1 & & & & & & & \\
\vdots & \vdots & & & & & & & & \ddots & & & & & & \\
f_{d,1} & f_{d,0} & & & & & & & & & 1 & & & & & \\
\vdots & \vdots & \ddots & & & & & & & & & \ddots & & & & \\
\vdots & \vdots & & \ddots & & & & & & & & & \ddots & & & \\
f_{1,n-1} & f_{1,n-2} & \cdots & \cdots & f_{1,0} & & & & & & & & & 1 & & \\
\vdots & \vdots & & & \vdots & & & & & & & & & & \ddots & \\
f_{d,n-1} & f_{d,n-2} & \cdots & \cdots & f_{d,0} & & & & & & & & & & & 1
\end{pmatrix}.$$

$$G\,M = \begin{pmatrix} h_{n-1} & h_{n-2} & \cdots & \cdots & h_0 & g_{1,n-1} & \cdots & g_{d,n-1} & \cdots & g_{1,0} & \cdots & g_{d,0} \end{pmatrix}$$

Ultimate correctness

Definition. *Given $n \in \mathbb{N} \cup \{+\infty\}$, let $\mathbf{\Phi}_{;n}$ be the set of vectors*

$$\begin{aligned} \varphi &= (h_{n-1}, \ldots, h_0, g_{1,n-1}, \ldots, g_{d,n-1}, \ldots\ldots, g_{1,0}, \ldots, g_{d,0}), \\ \|\varphi\| &< +\infty \end{aligned}$$

We say that φ is ***i*-normal** *if $g_{i,0} = 1$ and $g_{j,0} = 0$ for all $j > i$.*

Theorem.

1) If $\mathbf{\Phi}_{;\infty} \neq 0$, then $\mathbf{\Phi}_{;\infty}$ contains a minimal normal relation.

2) Assume that $\varphi \in \mathbf{\Phi}_{;\infty}$ is a minimal i-normal relation. For each $n \in \mathbb{N}$, let $f_{;n}$ be the truncation of f at order n and consider the corresponding minimal i-normal relation $\varphi_{;n} \in \mathbf{\Phi}_{;n}$. Then the relations $\varphi_{;n}$ converge to φ.

3) If $\mathbf{\Phi}_{;\infty} = 0$, then the $\|\varphi_{;n}\|$ are unbounded for $n \to \infty$.

Example I

$$f_1 \;=\; \frac{1}{1-\lambda\, z} \qquad (\lambda > 1)$$

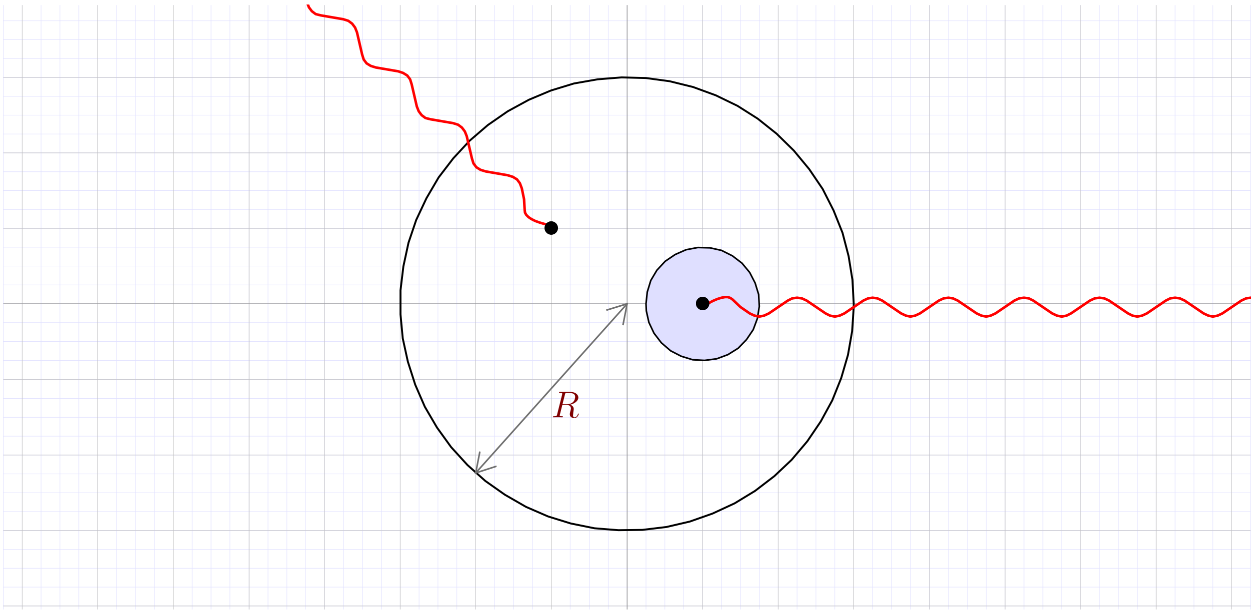
$$\begin{aligned}
g_{;4,1} &= 1.0000 - 1.6000\, z - 0.60000\, z^2 \\
&\quad - 0.20000\, z^3 \\
g_{;16,1} &= 1.0000 - 1.6180\, z - 0.61803\, z^2 - \cdots \\
&\quad - 5.8340\, 10^{-6}\, z^{14} - 1.9447\, 10^{-6}\, z^{15} \\
g_{;64,1} &= 1.0000 - 1.6180\, z - 0.61803\, z^2 - \cdots \\
&\quad - 5.0484\, 10^{-26}\, z^{62} - 1.6828\, 10^{-26}\, z^{63} \\
g_{;256,1} &= 1.0000 - 1.6180\, z - 0.61803\, z^2 - \cdots \\
&\quad - 2.8307\, 10^{-106}\, z^{254} - 9.4357\, 10^{-107}\, z^{255}
\end{aligned}$$

Fast convergence, but **not** $g_1 = 1 - \lambda\, z$. In fact

$$\begin{aligned}
g_1 &= \frac{1-\lambda\, z}{1-\alpha\, z} \\
\alpha + \frac{1}{\alpha} &= \lambda + \frac{2}{\lambda} \qquad (\alpha < 1)
\end{aligned}$$

Example II

$$f_1 = s(0.15\,(z+0.25))$$
$$f_2 = f_1'$$



$$\begin{aligned}
g_{;32,1} &= 1.6836 - 0.29080\,z + 0.0013079\,z^2 + \cdots \\
&\quad + 0.022645\,z^{30} - 0.0029284\,z^{31} \\
g_{;32,2} &= 1.0000 - 2.4006\,z - 0.88670\,z^2 - \cdots \\
&\quad + 0.0061880\,z^{30} - 9.9191 \cdot 10^{-4}\,z^{31} \\
g_{;64,1} &= 1.7104 - 0.36445\,z + 0.23778\,z^2 - \cdots \\
&\quad + 1.6968 \cdot 10^{-4}\,z^{62} - 1.0104 \cdot 10^{-5}\,z^{63} \\
g_{;64,2} &= 1.0000 - 2.3257\,z - 1.2027\,z^2 - \cdots \\
&\quad + 5.2359 \cdot 10^{-5}\,z^{62} - 3.4225 \cdot 10^{-6}\,z^{63} \\
g_{;128,1} &= 1.7105 - 0.36215\,z + 0.23696\,z^2 - 0.052535\,z^3 + 0.033518\,z^4 - \cdots \\
&\quad + 4.6860 \cdot 10^{-9}\,z^{126} - 1.3437 \cdot 10^{-10}\,z^{127} \\
g_{;128,2} &= 1.0000 - 2.3235\,z - 1.2138\,z^2 - 0.0067226\,z^3 - 0.080434\,z^4 + \cdots \\
&\quad + 1.5192 \cdot 10^{-9}\,z^{126} - 4.5516 \cdot 10^{-11}\,z^{127}
\end{aligned}$$

Slow convergence: $\|\varphi_{;32}\| = 4.3276$, $\|\varphi_{;64}\| = 4.3845$, $\|\varphi_{;128}\| = 4.3863$.

Counter-Example III

$$\begin{aligned} f_1 &= \psi_{i,\lambda} \\ \psi_{1,\lambda} &= \log(1-\lambda\, z) \\ \psi_{2,\lambda} &= \sqrt{1-\lambda\, z} \\ \psi_{3,\lambda} &= \mathrm{e}^{\frac{\lambda z}{1-\lambda z}} \\ \psi_{4,\lambda} &= \mathrm{random}(\lambda\, z) \end{aligned}$$

	32	64	128	256
$\psi_{1,\sqrt{2}}$	2.5897	4.2958	$1.1107 \cdot 10^1$	$7.5308 \cdot 10^1$
$\psi_{1,2}$	$1.2324 \cdot 10^1$	$7.7306 \cdot 10^1$	$3.3839 \cdot 10^3$	$6.4461 \cdot 10^6$
$\psi_{1,4}$	$2.7074 \cdot 10^3$	$3.1503 \cdot 10^6$	$5.0806 \cdot 10^{12}$	$1.3215 \cdot 10^{25}$
$\psi_{1,8}$	$5.7101 \cdot 10^6$	$1.6078 \cdot 10^{13}$	$1.2829 \cdot 10^{26}$	$8.6064 \cdot 10^{51}$
$\psi_{1,16}$	$6.0964 \cdot 10^{10}$	$1.7814 \cdot 10^{21}$	$1.5612 \cdot 10^{42}$	$1.2279 \cdot 10^{84}$
$\psi_{1,32}$	$1.1152 \cdot 10^{15}$	$9.1674 \cdot 10^{29}$	$4.9020 \cdot 10^{59}$	$1.6662 \cdot 10^{119}$
$\psi_{2,\sqrt{2}}$	2.2960	3.7208	9.7394	$6.5890 \cdot 10^1$
$\psi_{2,2}$	9.3539	$6.1551 \cdot 10^1$	$2.6678 \cdot 10^3$	$5.0111 \cdot 10^6$
$\psi_{2,4}$	$1.7232 \cdot 10^3$	$2.1500 \cdot 10^6$	$3.4149 \cdot 10^{12}$	$8.7128 \cdot 10^{24}$
$\psi_{2,8}$	$3.5980 \cdot 10^6$	$1.0031 \cdot 10^{13}$	$7.9396 \cdot 10^{25}$	$5.2284 \cdot 10^{51}$
$\psi_{3,\sqrt{2}}$	6.2155	$1.3295 \cdot 10^1$	$4.1722 \cdot 10^1$	$4.2679 \cdot 10^2$
$\psi_{3,2}$	$4.0164 \cdot 10^1$	$3.9310 \cdot 10^2$	$1.4047 \cdot 10^4$	$1.1091 \cdot 10^7$
$\psi_{3,4}$	$6.9272 \cdot 10^3$	$6.8304 \cdot 10^6$	$4.7564 \cdot 10^{11}$	$5.1308 \cdot 10^{20}$
$\psi_{3,8}$	$3.6660 \cdot 10^6$	$9.4896 \cdot 10^{11}$	$4.2745 \cdot 10^{21}$	$1.1809 \cdot 10^{39}$
$\psi_{4,2}$	$8.0487 \cdot 10^4$	$3.5565 \cdot 10^9$	$2.1354 \cdot 10^{19}$	$4.8792 \cdot 10^{38}$
$\psi_{4,4}$	$5.0774 \cdot 10^9$	$1.1548 \cdot 10^{19}$	$2.9916 \cdot 10^{38}$	$1.2335 \cdot 10^{77}$
$\psi_{4,8}$	$3.2564 \cdot 10^{14}$	$4.3151 \cdot 10^{28}$	$5.1348 \cdot 10^{57}$	$2.9654 \cdot 10^{115}$

$$\log \|\varphi_{;n}\| \approx \frac{n}{2} \log \lambda.$$

$$\log \|\varphi_{;n}\| \approx \frac{n}{d+1} \log \lambda.$$

Only poles

Problem: given $f \in \mathbb{C}[[z]]$, guess $P \in \mathbb{C}[z]$ with

$$P f \in \mathbb{A}_1.$$

Algorithm `denom`(f, N, D)

INPUT: the first $N > 2D$ coefficients of f and a degree bound D

OUTPUT: $\approx$ minimal monic polynomial P with $\deg P \leqslant D$, $P f \in \mathbb{A}_1$, or **failed**

Step 1. [Initialize]

$d := 0$

Step 2. [Determine P]

Solve $$\begin{pmatrix} f_{N-2d+1} & \cdots & f_{N-d} \\ \vdots & & \vdots \\ f_{N-d} & \cdots & f_{N-1} \end{pmatrix} \begin{pmatrix} P_{d-1} \\ \vdots \\ P_0 \end{pmatrix} + \begin{pmatrix} f_{N-2d} \\ \vdots \\ f_{N-d-1} \end{pmatrix} = 0.$$

Set $P := z^d + P_{d-1} z^{d-1} + \cdots + P_0$

Step 3. [Terminate or loop]

If $P f \in \mathbb{A}_1$ then return P

If $d = N$ then return **failed**

Set $d := d + 1$ and go to step 2

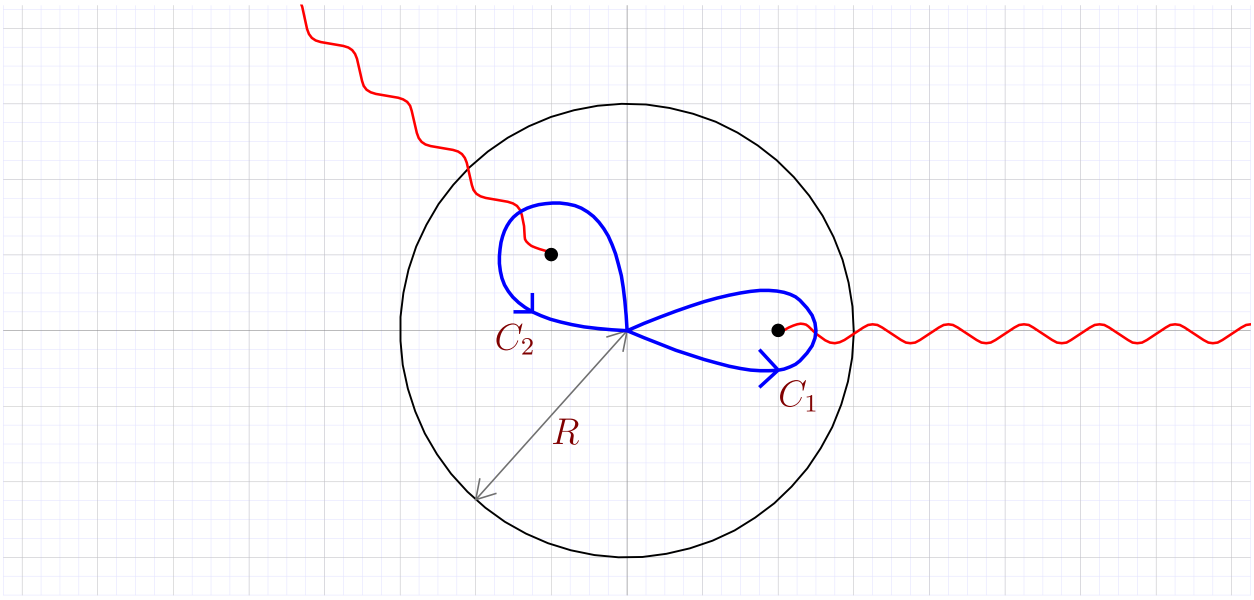
Theorem 1. *Exponential convergence in N.*

Only algebraic singularities

Problem: given $f \in \mathbb{C}[[z]]$, guess $P_d, \ldots, P_0 \in \mathbb{C}[z]$ with

$$P_d f^d + \cdots + P_0 \in \mathbb{A}_1. \tag{1}$$

Assume: algorithm for analytic continuation of f.



Algorithm `alg_dep`$(f, (\sigma_1, \ldots, \sigma_s), N, D, B)$
INPUT: analytic function f above $\bar{\mathcal{D}}_1 \setminus \{\sigma_1, \ldots, \sigma_s\}$ and bounds N, D, B
OUTPUT: normalized dependency (1) with $d \leqslant B$, $\deg P_d \leqslant D$, or **failed**

Step 1. [Initialize]

Set $\boxed{\Phi := \{f\}}$

Step 2. [Saturate]

If $C_i \Phi \subseteq \Phi$ for all i, then go to step 3

If $\operatorname{card}(\Phi) > N$ then return **failed**

$\boxed{\Phi := C_1 \Phi \cup \cdots \cup C_s \Phi}$

Repeat step 2

Step 3. [Terminate]

Denote $\Phi := \{\varphi_1, \ldots, \varphi_k\}$

Compute $\boxed{Q := (F - \varphi_1) \cdots (F - \varphi_k) = Q_k F^k + \cdots + Q_0}$

For each $i \in \{0, \ldots, k\}$, compute $\boxed{D_i := \operatorname{denom}(Q_i, r, D)}$

If D_i = **failed** for some i, then return **failed**

Return $\boxed{\operatorname{lcm}(D_0, \ldots, D_k)\, Q}$

Only Fuchsian singularities

Problem: given $f \in \mathbb{C}[[z]]$, guess $L_r, \ldots, L_0 \in \mathbb{C}[z]$ with

$$L_r f^{(r)} + \cdots + L_0 \in \mathbb{A}_1. \tag{2}$$

Assume: algorithm for analytic continuation of f.

Algorithm `fuch_dep`$(f, (\sigma_1, \ldots, \sigma_s), N, D, B)$

INPUT: analytic function f above $\bar{\mathcal{D}}_1 \setminus \{\sigma_1, \ldots, \sigma_s\}$ and bounds N, D, B

OUTPUT: normalized dependency (2) with $r \leqslant B$, $\deg L_r \leqslant D$, or **failed**

Step 1. [Initialize]

Set $\boxed{\Phi := \{f\}}$

Step 2. [Saturate]

If $\mathrm{Vect}(C_i \Phi) \subseteq \mathrm{Vect}(\Phi)$ for all i, then go to step 3

If $\mathrm{card}(\Phi) > N$ then return **failed**

$\boxed{\Phi := \Phi \cup \{C_i \varphi\}}$ for i and $\varphi \in \Phi$ with $\boxed{C_i \varphi \notin \mathrm{Vect}(\Phi)}$

Repeat step 2

Step 3. [Terminate]

Denote $\Phi := \{\varphi_1, \ldots, \varphi_k\}$

Compute $\boxed{K := \mathrm{lcm}(\partial - \varphi_1^\dagger, \ldots, \partial - \varphi_k^\dagger) = K_k \partial^k + \cdots + K_0}$

in the skew polynomial ring $\mathbb{C}((z))[\partial]$, where $\varphi^\dagger$ denotes φ' / φ

For each $i \in \{0, \ldots, k\}$, compute $D_i := \mathtt{denom}(K_i, r, D)$

If $D_i =$ **failed** for some i, then return **failed**

Return $\boxed{\mathrm{lcm}(D_0, \ldots, D_k)\, K}$

Remark. Only works for Fuchsian singularities, because of

$$f = \mathrm{e}^{\frac{1}{z - \sigma}} + \mathrm{e}^{\frac{-1}{z - \sigma}}.$$

Guessing asymptotic behaviour

Guessing Stirling's formula

```
Mmx] use "symbolix"; use "multimix"; use "jorix"
```

```
Mmx] include "jorix/extrapolate_extra.mmx"
```

```
Mmx] bit_precision := 256; significant_digits := 12;
```

```
Mmx] v == [ 1.0 * n! | n in 0..1000 ];
```

```
Mmx] guess_asymptotics (v, 7)
```

$$2.50662828099\,\mathrm{e}^{1.000000000000 n \log(n) - 0.999999999999 n + 0.499999999637 \log(n)} + \frac{0.208884786159\,\mathrm{e}^{1.000000000000 n \log(n) - 0.999999999999 n + 0.499999999637 \log(n)}}{n} + O\left(\frac{\mathrm{e}^{1.000000000000 n \log(n) - 0.999999999999 n + 0.499999999637 \log(n)}}{n^{2.00000000000}}\right)$$

```
Mmx]
```

Number of alcohols: $s(z) = \frac{s(z)^3 + 2\, s(z^3)}{2}$

```
Mmx] use "symbolix"; use "multimix"; use "jorix"
```

```
Mmx] include "jorix/extrapolate_extra.mmx"
```

```
Mmx] bit_precision := 256; significant_digits := 15;
```

```
Mmx] s == fixed_point_series (f :->
         ((f*f*f + 2*dilate(f,3)) / 3) << 1, 1)
```

$$1 + z + z^2 + 2\, z^3 + 5\, z^4 + 11\, z^5 + 28\, z^6 + 74\, z^7 + 199\, z^8 + 551\, z^9 + O\left(z^{10}\right)$$

```
Mmx] s[1000]
```

7141405838260856302799626971573503958693316602056092784244684\
1983772993528261108462272215625226898378369376033103164938017\
3669561825664464622626151233542455259174511379297519997180160\
2579846795790849542198688525904188085884723919363179856107974\
0289288926951041200409300695154710153948689237950994002858702\
9019470508730834892794468407130382550381419051983920864847676\
5824090791612448742977833366797686342252975727420405871855752\
2846876249646987023836756747448731476010055025269524756569973\
4188694861730348

```
Mmx] w == [ 1.0 * s[n] | n in 0..1000 ];
```

```
Mmx] guess_asymptotics (w, 9)
```

$$0.346304267394184\, \mathrm{e}^{1.19000938463478 n - 1.50000000000000 \log(n)} - \frac{0.146261408814197\, \mathrm{e}^{1.19000938463478 n - 1.50000000000000 \log(n)}}{n} + \frac{0.138424784853633\, \mathrm{e}^{1.19000938463478 n - 1.50000000000000 \log(n)}}{n^{2.00000000000000}} + \frac{0.109383962532057\, \mathrm{e}^{1.19000938463478 n - 1.50000000000000 \log(n)}}{n^{3.00000000000000}} + O\left(\frac{\mathrm{e}^{1.19000938463478 n - 1.50000000000000 \log(n)}}{n^{4.00000000000000}}\right)$$

```
Mmx] exp (-1.19000938463478)
```

0.304218409074645

```
Mmx]
```

Transseries

```
Mmx] use "symbolix"; use "multimix"
```

```
Mmx] x == infinity ('x);
```

```
Mmx] lengthen (1 / (x - 1), 2)
```

$$\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\frac{1}{x^4}+\frac{1}{x^5}+\frac{1}{x^6}+O\left(\frac{1}{x^7}\right)$$

```
Mmx] 1 / (1 - 1/x - exp(-x))
```

$$1+\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+O\left(\frac{1}{x^4}\right)+\frac{1}{e^x}+\frac{2}{x\,e^x}+\frac{3}{x^2\,e^x}+O\left(\frac{1}{x^3\,e^x}\right)+\frac{1}{e^{2x}}+\frac{3}{x\,e^{2x}}+O\left(\frac{1}{x^2\,e^{2x}}\right)+\frac{1}{e^{3x}}+O\left(\frac{1}{x\,e^{3x}}\right)$$

```
Mmx] 1 / ((1 - 1/x) * (1 - exp(-x))) - 1 / (1 - 1/x)
```

$$\frac{1}{e^x}+\frac{1}{x\,e^x}+\frac{1}{x^2\,e^x}+O\left(\frac{1}{x^3\,e^x}\right)+\frac{1}{e^{2x}}+\frac{1}{x\,e^{2x}}+O\left(\frac{1}{x^2\,e^{2x}}\right)+\frac{1}{e^{3x}}+O\left(\frac{1}{x\,e^{3x}}\right)$$

```
Mmx] integrate (exp (x^2), x)
```

$$\frac{e^{x^2}}{2\,x}+\frac{e^{x^2}}{4\,x^3}+\frac{3\,e^{x^2}}{8\,x^5}+\frac{15\,e^{x^2}}{16\,x^7}+O\left(\frac{e^{x^2}}{x^9}\right)$$

```
Mmx] integrate (x^x, x)
```

$$\frac{e^{x\log(x)}}{\log(x)}-\frac{e^{x\log(x)}}{\log(x)^2}+\frac{e^{x\log(x)}}{\log(x)^3}-\frac{e^{x\log(x)}}{\log(x)^4}+O\left(\frac{e^{x\log(x)}}{\log(x)^5}\right)+\frac{e^{x\log(x)}}{x\log(x)^3}-\frac{3\,e^{x\log(x)}}{x\log(x)^4}+\frac{6\,e^{x\log(x)}}{x\log(x)^5}+O\left(\frac{e^{x\log(x)}}{x\log(x)^6}\right)+\frac{e^{x\log(x)}}{x^2\log(x)^4}-\frac{e^{x\log(x)}}{x^2\log(x)^5}+O\left(\frac{e^{x\log(x)}}{x^2\log(x)^6}\right)+\frac{2\,e^{x\log(x)}}{x^3\log(x)^5}+O\left(\frac{e^{x\log(x)}}{x^4\log(x)^6}\right)$$

```
Mmx] lengthen (sum (log x, x), 7)
```

$$x\log(x)-x-\frac{\log(x)}{2}+\frac{1}{12\,x}-\frac{1}{360\,x^3}+\frac{1}{1260\,x^5}-\frac{1}{1680\,x^7}+\frac{1}{1188\,x^9}+O\left(\frac{1}{x^{11}}\right)$$

```
Mmx] euler == exp (-x) * integrate (exp (x) / x, x);
```

```
Mmx] lengthen (euler, 5)
```

$$\frac{1}{x}+\frac{1}{x^2}+\frac{2}{x^3}+\frac{6}{x^4}+\frac{24}{x^5}+\frac{120}{x^6}+\frac{720}{x^7}+\frac{5040}{x^8}+\frac{40320}{x^9}+O\left(\frac{1}{x^{10}}\right)$$

```
Mmx] eval (euler, x, 1000.0)
```

0.0010010020060241207

```
Mmx]
```

Given: $f_0, \ldots, f_N$

Assume: $f_n = a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots$

Phase 0

$$f_n = a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots$$

Phase 1

$$\begin{aligned} f_n &= a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots \\ (\Delta f)_n = f_{n+1} - f_n &= \frac{-a_1}{n^2} + \frac{a_1 - 2a_2}{n^3} + \frac{3a_2 - a_1}{n^4} + \cdots \end{aligned}$$

Phase 2

$$\begin{aligned} f_n &= a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots \\ (\Delta f)_n &= \frac{-a_1}{n^2} + \frac{a_1 - 2a_2}{n^3} + \frac{3a_2 - a_1}{n^4} + \cdots \\ (\Delta(n^2 \Delta f))_n &= \frac{2a_2 - a_1}{n^2} + \frac{3a_1 - 8a_2}{n^3} + \frac{23a_2 - 7a_1}{n^4} + \cdots \end{aligned}$$

Phase 2-bis

$$\begin{aligned} f_n &= a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots \\ (\Delta f)_n &= \frac{-a_1}{n^2} + \frac{a_1 - 2a_2}{n^3} + \frac{3a_2 - a_1}{n^4} + \cdots \\ (\Delta(n^2 \Delta f))_n &= \frac{2a_2 - a_1}{n^2} + \frac{3a_1 - 8a_2}{n^3} + \frac{23a_2 - 7a_1}{n^4} + \cdots \\ n^2 \Delta(n^2 \Delta f) &\approx (n^2 \Delta(n^2 \Delta f))_{N-2} = c_2 \end{aligned}$$

Phase 1-bis

$$
\begin{aligned}
f_n &= a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots \\
(\Delta f)_n &= \frac{-a_1}{n^2} + \frac{a_1 - 2a_2}{n^3} + \frac{3a_2 - a_1}{n^4} + \cdots \\
(\Delta(n^2 \Delta f))_n &= \frac{2a_2 - a_1}{n^2} + \frac{3a_1 - 8a_2}{n^3} + \frac{23a_2 - 7a_1}{n^4} + \cdots \\
n^2 \Delta(n^2 \Delta f) &\approx c_2 \\
n^2 \Delta f &\approx \left[n^2 \Delta f\right]_{N-1} + \Delta_{N-1}^{-1}\left(\frac{c_2}{n^2}\right) = c_1 - \frac{c_2}{n} - \frac{c_2}{2n^2} - \\
&\quad \frac{c_2}{6n^3} + \cdots
\end{aligned}
$$

Phase 0-bis

$$
\begin{aligned}
f_n &= a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots \\
(\Delta f)_n &= \frac{-a_1}{n^2} + \frac{a_1 - 2a_2}{n^3} + \frac{3a_2 - a_1}{n^4} + \cdots \\
(\Delta(n^2 \Delta f))_n &= \frac{2a_2 - a_1}{n^2} + \frac{3a_1 - 8a_2}{n^3} + \frac{23a_2 - 7a_1}{n^4} + \cdots \\
n^2 \Delta(n^2 \Delta f) &\approx c_2 \\
n^2 \Delta f &\approx c_1 - \frac{c_2}{n} - \frac{c_2}{2n^2} - \frac{c_2}{6n^3} + \cdots \\
f &\approx f_N + \Delta_N^{-1}\left(\frac{c_1}{n^2} - \frac{c_2}{n^3} - \cdots\right) = c_0 - \frac{c_1}{n} + \frac{c_2 - c_1}{2n^2} + \\
&\quad \frac{4c_2 - c_1}{6n^3} + \cdots
\end{aligned}
$$

See also: E-algorithm

Constant tactic

Assumption:

$$f_n = c + R_n, \qquad R_n \prec 1$$

Criterion:

$$\varepsilon_{f, f_N} < \delta$$

$$\varepsilon_{f,\varphi} = \max_{k \in \{L, \ldots, N\}} \frac{|f_n - \varphi_n|}{|f_n| + |\varphi_n|}$$

Transformation:

$$f \longmapsto g = \Delta f$$

Inverse transformation:

$$\begin{aligned} g_n &\approx \tilde{g}_n \\ f_n &\approx \tilde{f}_n = \left[f_N - \sum_{0 \leqslant i < N} \tilde{g}_i \right] + \sum_{0 \leqslant i < n} \tilde{g}_i \end{aligned}$$

Explicit tactic

Assumption:

$$f_n = \psi_n (c + R_n), \qquad R_n \prec 1$$

Criterion:

$$\varepsilon_{f, \psi f_N / \psi_N} < \delta_\psi$$

$$\varepsilon_{f, \varphi} = \max_{k \in \{L, \ldots, N\}} \frac{|f_n - \varphi_n|}{|f_n| + |\varphi_n|}$$

Transformation:

$$f \longmapsto g = \Delta(f/\psi)$$

Inverse transformation:

$$\begin{aligned} g_n &\approx \tilde{g}_n \\ f_n &\approx \tilde{f}_n = \psi_n \left(\left[f_N / \psi_N - \sum_{0 \leqslant i < N} \tilde{g}_i \right] + \sum_{0 \leqslant i < n} \tilde{g}_i \right) \end{aligned}$$

Exponential tactic

Assumption:

$$f_n = \pm\, \mathrm{e}^{R_n} \qquad (R_n \succ 1).$$

Criterion:

$$\frac{f_n}{f_N} > 0 \qquad (n = L, \ldots, N),$$

Transformation:

$$f \longmapsto g = \log \frac{f}{\operatorname{sign} f_N}$$

Inverse transformation:

$$\begin{aligned} g_n &\approx \tilde{g}_n \\ f_n &\approx \tilde{f}_n = (\operatorname{sign} f_N) \exp \tilde{g}_N \end{aligned}$$

Asymptotic extrapolation

Combining tactics

Several tactics with transformations $\Phi_1, \ldots, \Phi_p$ and inverses $\tilde{\Phi}_1, \ldots, \tilde{\Phi}_p$

Compute all valid $\tilde{f}_{i_1,\ldots,i_l} = (\tilde{\Phi}_{i_1} \circ \cdots \circ \tilde{\Phi}_{i_l} \circ \Phi_{i_l} \circ \cdots \circ \Phi_{i_1})(f)$ until specified l

Return best $\tilde{f}_{i_1,\ldots,i_l}$

Inverse transformations using transseries

Inverse transformations Δ^{-1} and $\exp$ done on "transseries"

Numerical evaluations done using "summing to the least term"