A reversible framework for resource theories

Fernando G.S.L. Brandão and Martin B. Plenio

Fields Institute, Toronto 07/07/2009

Imperial College London

Institute for Mathematical Sciences

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Restricted Operations and Resources

- In Physics and information theory we commonly deal with restrictions on the physical processes/operations available.

Restricted set of operations	Free states	Resource
Local operations and Classical Communication (LOCC)	Separable states	Entanglement

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G-invariant operations, for a group G (superselection rules)	G-invariant states	reference frame

Resource Theories

- Given some set of allowed operations on a physical system, which transformations from one state of the system into another can be realized?

When and how can a resource be converted from one form into another?

- It is an extremely challenging question in general!

Resource Theories

- Usual approach: Restricted set of operations ⟹ Resource

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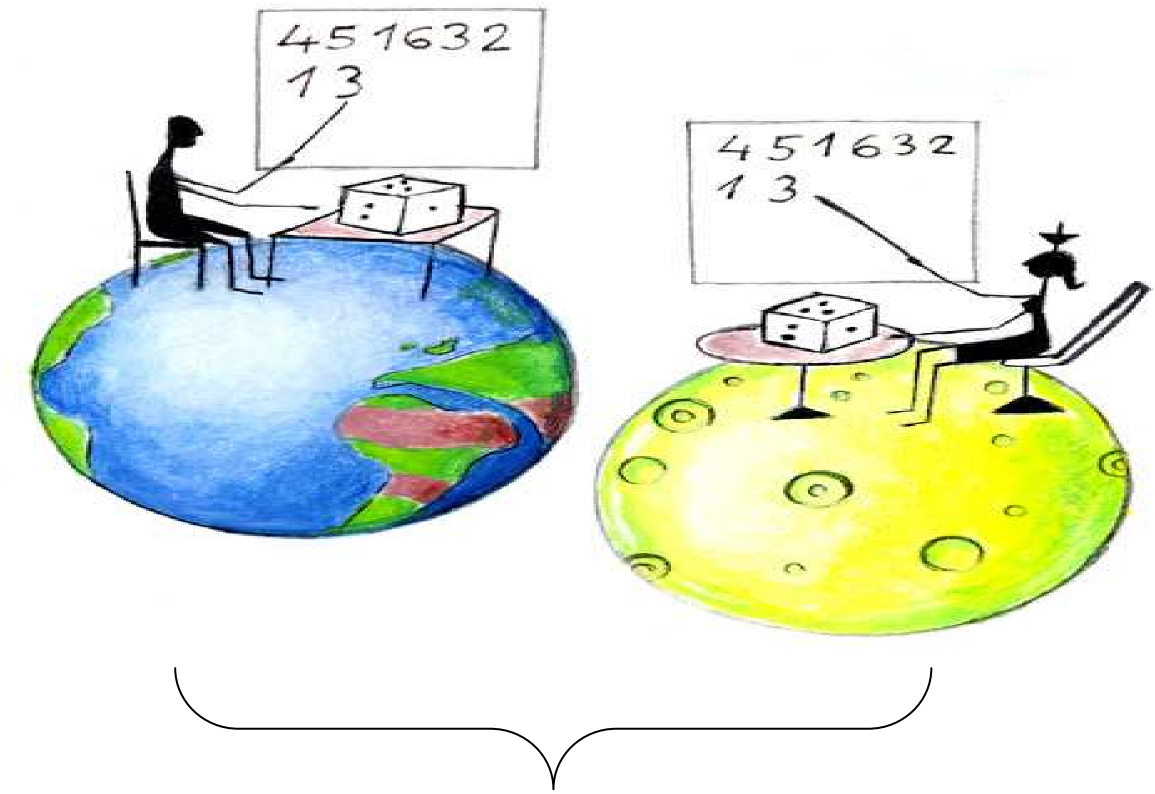
Resource Theories

- Usual approach: Restricted set of operations ⟹ Resource
- **This talk**: Resource ⟹ Restricted set of operations: *Any* operation which cannot create the resource

WHY?

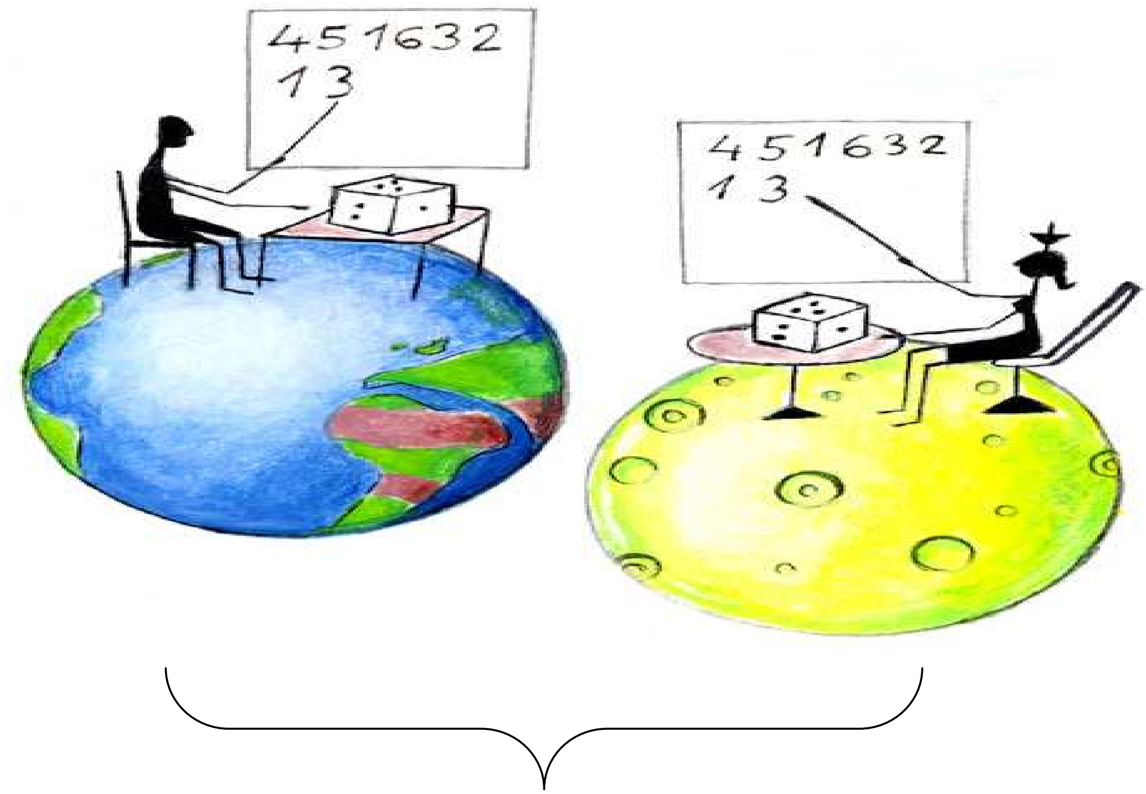
Leads to a much simpler theory, which at the same time still gives relevant information about original setting

Quantum Entanglement



$$\frac{|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle}{\sqrt{2}}$$

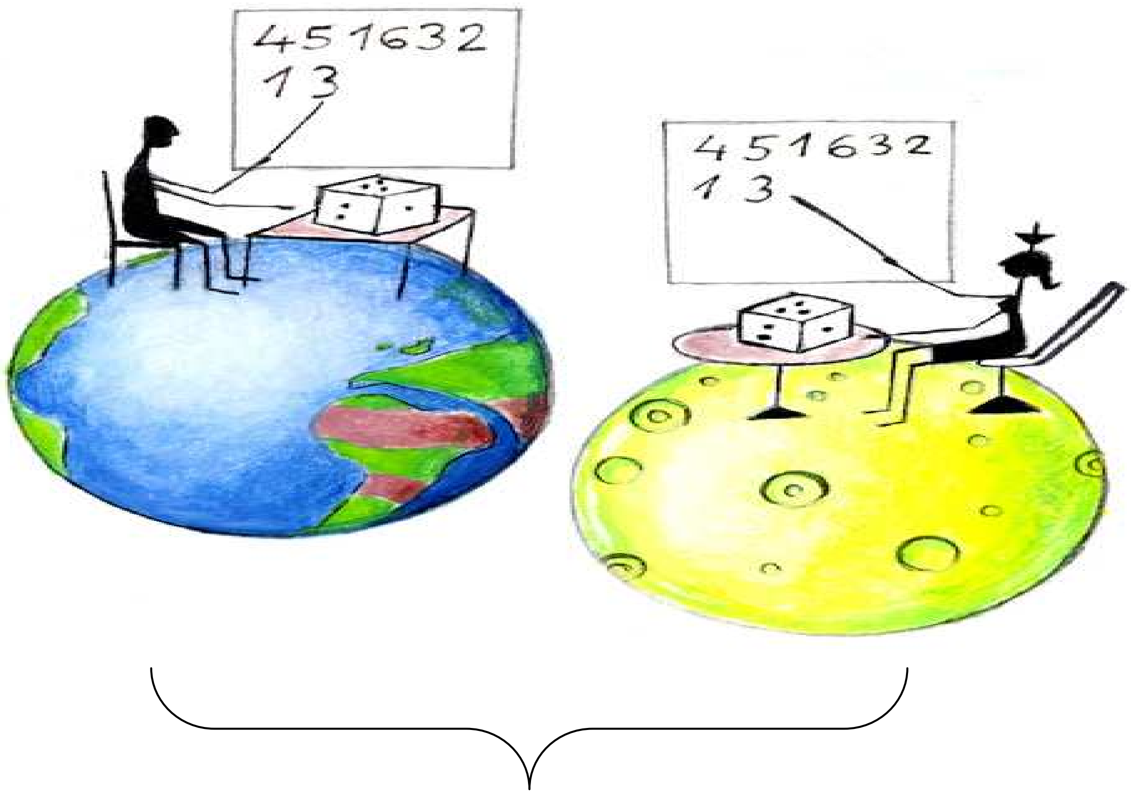
Quantum Entanglement



$$\rho \neq \sum_i p_i \rho_A^{\ i} \otimes \rho_B^i$$

(Werner 89)

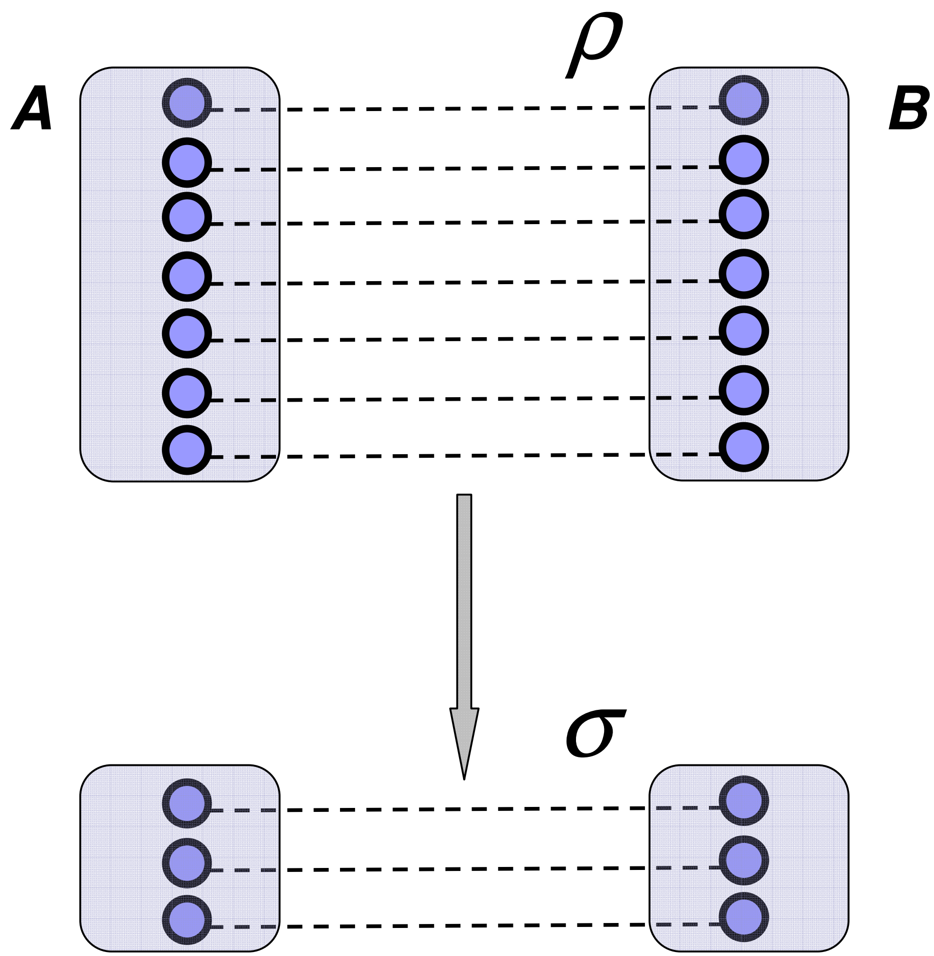
Quantum Entanglement



$$\rho \neq \sum_i p_i \rho_1^i \otimes \ldots \otimes \rho_k^i$$

Cannot be created by local operations and classical communication (LOCC)

LOCC asymptotic entanglement transformations

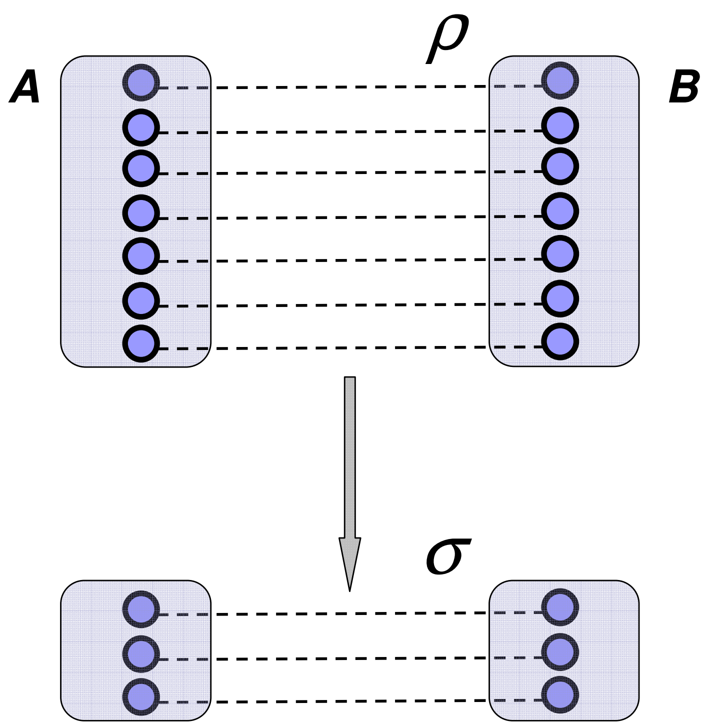


$$\rho^{\otimes n} \rightarrow_{LOCC} \sigma_n \approx \sigma^{\otimes k_n}$$

LOCC asymptotic entanglement transformations

Optimal rate of conversion

$$R(\rho \to \sigma) = \inf \left\{ \frac{n}{m} : \lim_{n\to\infty} \left(\min_{\Lambda \in LOCC} \| \Lambda(\rho^{\otimes n}) - \sigma^{\otimes m} \|_1 \right) = 0 \right\}$$



Bipartite pure state entanglement transformations

- Transformations are reversible in the asymptotic limit

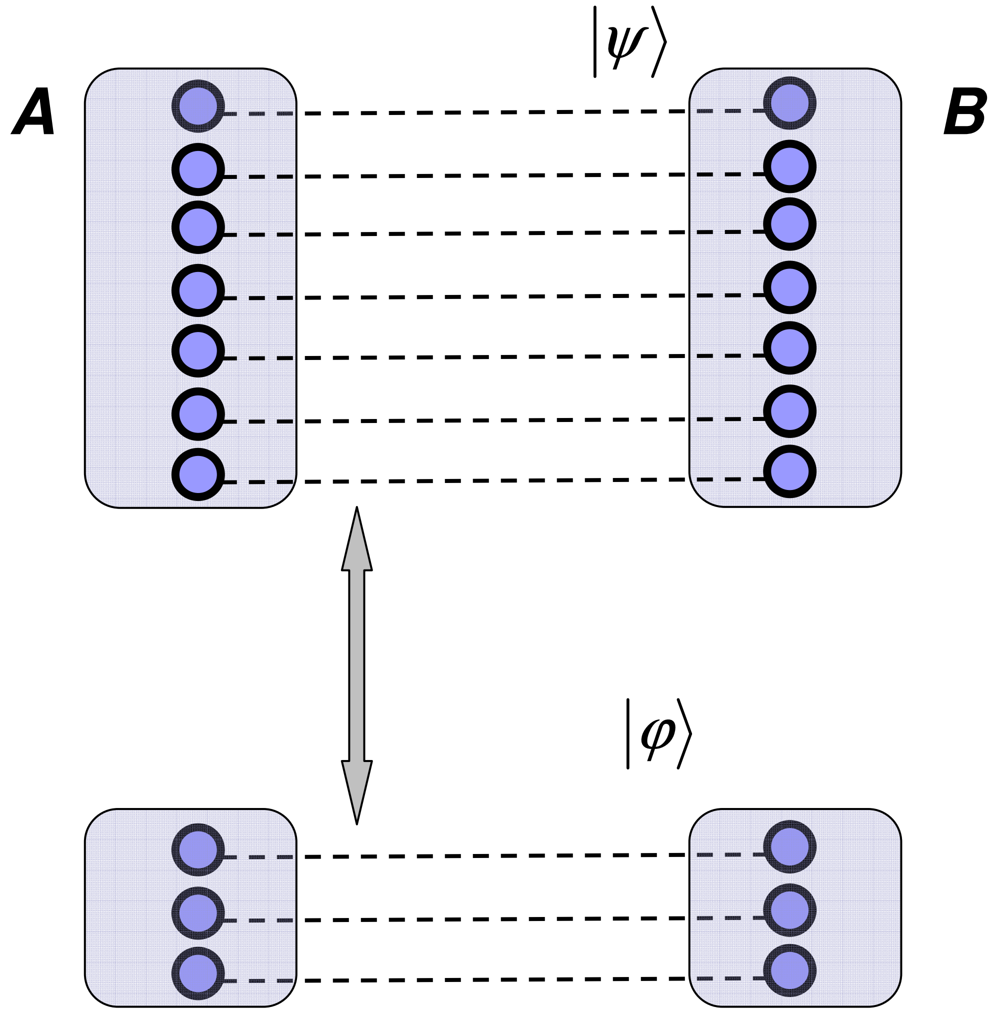
$$|\psi\rangle^{\otimes nE(\psi)} \to_{LOCC} |\varphi\rangle^{\otimes nE(\varphi)}$$

$$|\varphi\rangle^{\otimes nE(\varphi)} \to_{LOCC} |\psi\rangle^{\otimes nE(\psi)}$$

$$E(\rho) = S(\rho_A)$$

- $$|\psi\rangle^{\otimes n} \to_{LOCC} |\varphi\rangle^{\otimes n}$$

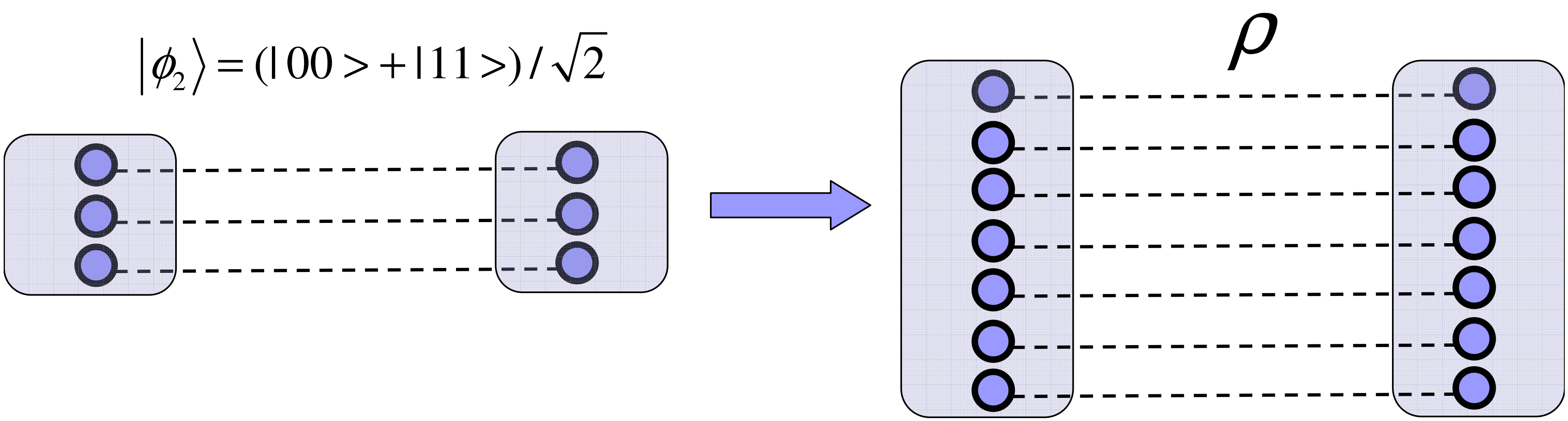
iff $$E(\psi) \geq E(\varphi)$$



(Bennett, Bernstein, Popescu, Schumacher 96)

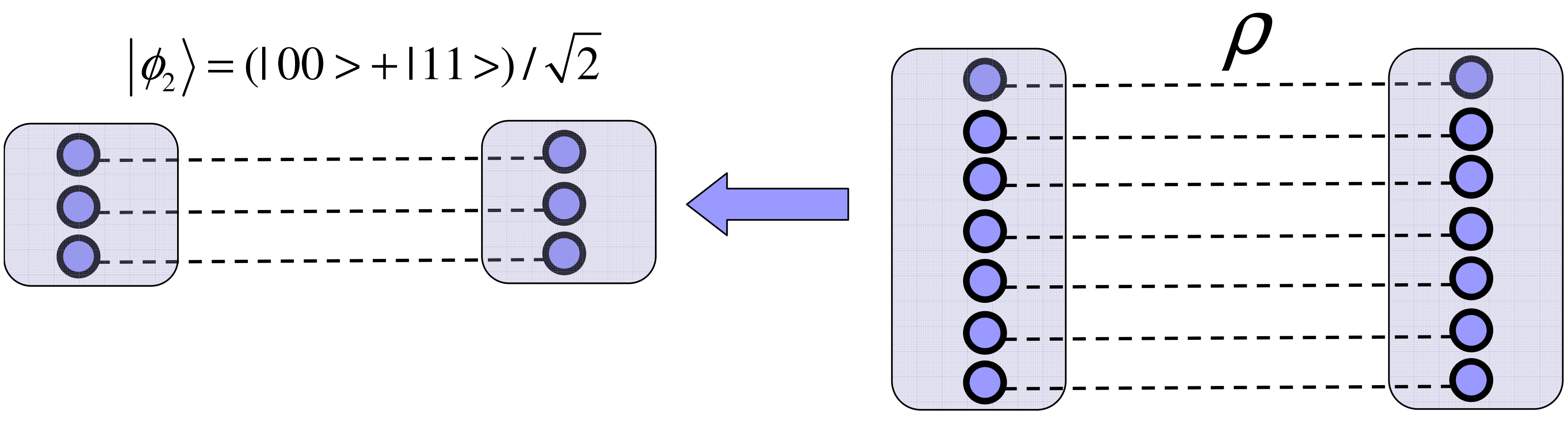
Mixed state entanglement

- Entanglement cost: $$E_C(\rho) = R(\phi_2 \rightarrow \rho)$$



Mixed state entanglement

- Distillable entanglement: $E_D(\rho) = R(\rho \rightarrow \phi_2)^{-1}$



Mixed state entanglement

Manipulation of mixed state entanglement under LOCC is irreversible

- In general: $E_C(\rho) > E_D(\rho)$

- Extreme case, *bound* entanglement: $E_C(\rho) > 0, E_D(\rho) = 0$

(Horodecki^3 98, Vidal&Cirac 01)

- No unique measure for entanglement manipulation under LOCC

Entanglement beyond LOCC

- In many cases it's helpful to consider the manipulation of entanglement under larger classes of quantum operations than LOCC, e.g.

1. Separable Operations
2. PPT Operations
3. LOCC + bound entanglement

Rains 97, 99, 00, Bennett et al 98 Eggeling,Vollbrecht, Werner, Wolf 01

Audenaert, Plenio, Eisert 03, Horodecki, Oppenheim, Horodecki 03

Ishizaka 04, Ishizaka&Plenio 04,05, Matthews&Winter 08

Is there a non-trivial class of operations for which we recover the total order found in pure bipartite states for *all* entangled states?

What’s the largest class of operations which does not create entanglement?

Non-entangling operations

- Definition: A quantum operation (trace preserving completely positive map) $\Lambda : D(H_1 \otimes ... \otimes H_k) \rightarrow D(H_{1'} \otimes ... \otimes H_{k'})$ is non-entangling if $\Lambda(\sigma)$ is separable for every separable state $\sigma \in D(H_1 \otimes ... \otimes H_k)$

More generally,

What's the largest class of operations which does not create a resource?

Let $M_n \subseteq D(H^{\otimes n})$ denote the non-resource states,
and with some abuse of notation, denote by *M* this family of sets

Any state *not* in *M* is a resource

We assume *M* is closed

Resource non-Generating Operations

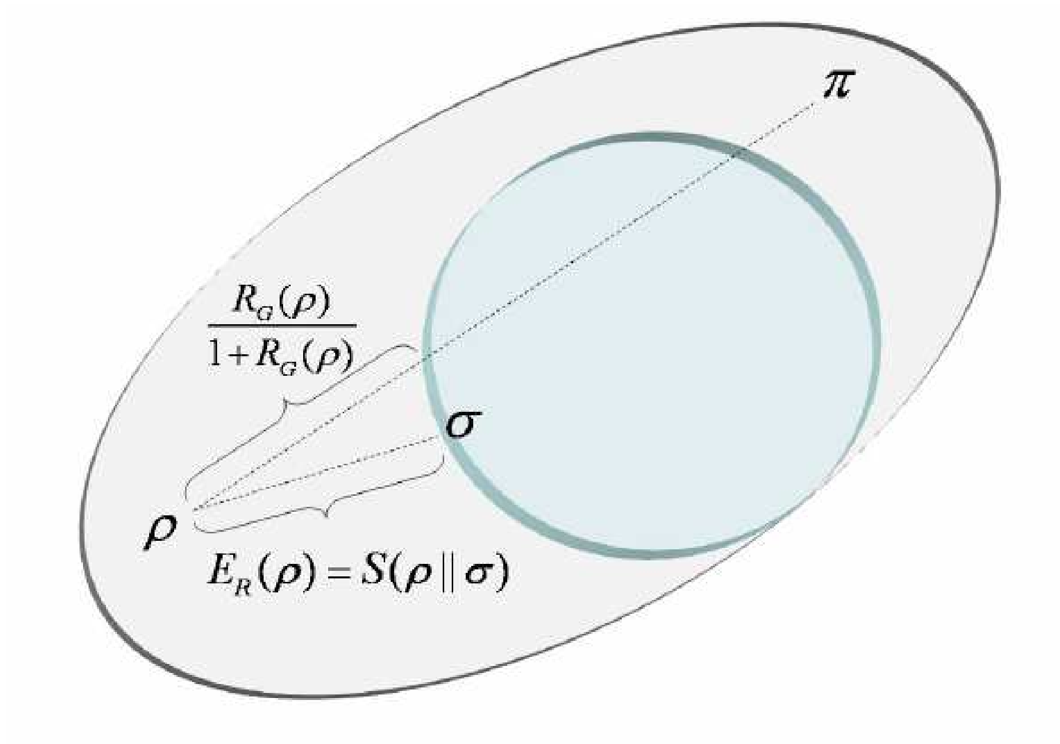
- Definition: A quantum operation (trace preserving completely positive map) $\Lambda : D(H) \to D(H)$ is resource non-generating if $\Lambda(\sigma)$ is a non-resource state for every non-resource state σ

Two measures of resource

- The relative entropy of *M*-resource is given by

$$E_M(\rho) = \min_{\sigma \in M} S(\rho \| \sigma) \qquad S(\rho\|\sigma) = tr(\rho(\log\rho - \log\sigma))$$

Vedral&Plenio 97



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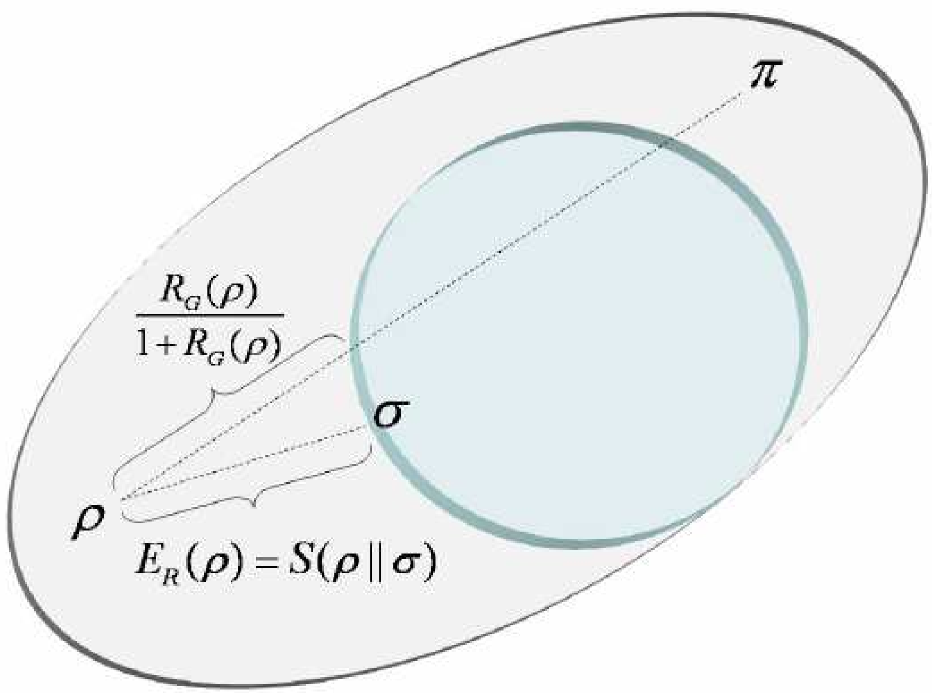
Vedral&Plenio 97

•The robustness of *M*-resource is given by

$$R_M(\rho) = \min \; s : \frac{\rho + s\pi}{1+s} \in M$$

Vidal&Tarrach 99 and Harrow&Nielsen 03



Asymptotically resource non-generating operations

- Definition: A quantum operation Λ is ε-resource non-generating if

  $R_M(\Lambda(\sigma)) \leq \varepsilon$ for every non-resource state σ

- We say that a sequence of maps $\{\Lambda_n\}$ is asymptotically resource non-generating if each Λ_n is ε_n-resource non-generating and

$$\lim_{n \to \infty} \varepsilon_n = 0$$

Asymptotically resource non-generating operations

- Optimal rate of conversion under asymptotically resource non-generating operations:

$$R(\rho \to \sigma) = \inf \left\{ \frac{n}{m} : \lim_{n\to\infty} \left(\min_{\Lambda \in RNG(\varepsilon_n)} \| \Lambda(\rho^{\otimes n}) - \sigma^{\otimes m} \|_1 \right) = 0, \ \lim_{n\to\infty} \varepsilon_n = 0 \right\}$$

- $RNG(\varepsilon)$ denotes the class of ε - resource non-generating operations

The main result

Under a few assumptions on *M*, for every quantum states ρ, σ

$$R(\rho \to \sigma) = E_M^{\infty}(\sigma) / E_M^{\infty}(\rho)$$

$$E_M^{\infty}(\rho) = \lim_{n \to \infty} \frac{E_M(\rho^{\otimes n})}{n}$$

Implies: $\rho^{\otimes n} \to \sigma^{\otimes n} \quad iff \quad E_M^{\infty}(\rho) \geq E_M^{\infty}(\sigma)$

The main idea

The main idea is to connect the *convertibility* of resource states to the *distinguishability* of resource states from non-resource ones

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- **Quantum Hypothesis Testing**: given several i.i.d. copies of a quantum state and the promise that you are given either ρ (null hypothesis) or σ (alternative hypothesis), decide which you have

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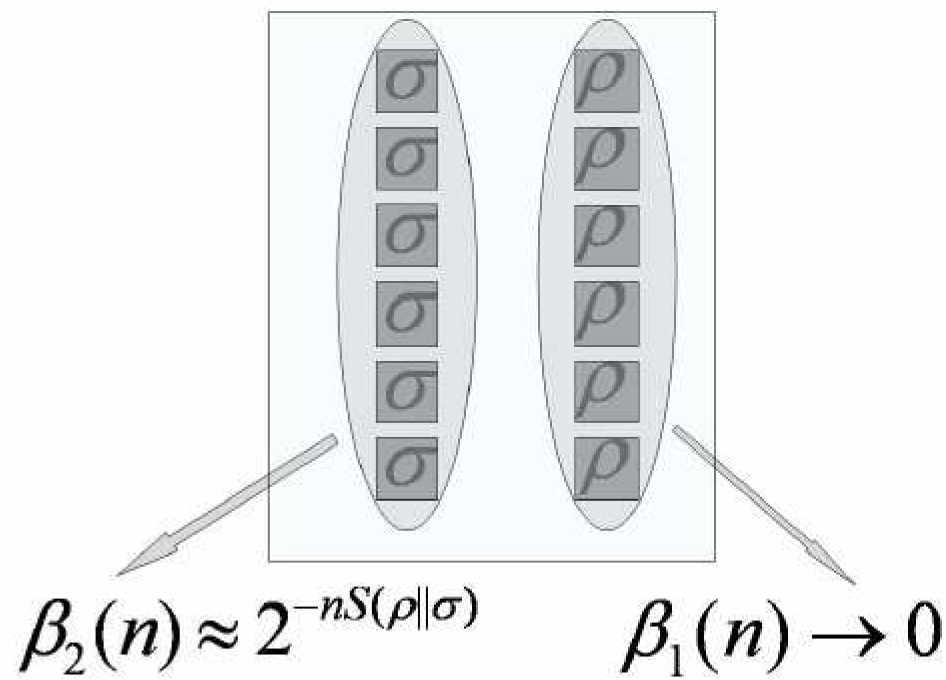
- **Quantum Stein's Lemma**:

$$\beta_1(A_n) := tr(\rho^{\otimes n}(I - A_n))$$

$$\beta_2(A_n) := tr(\sigma^{\otimes n} A_n)$$



Hiai&Petz 91 and Ogawa&Nagaoka 99

The main idea

- ***Resource* Hypothesis Testing**: given a sequence of quantum states ω_n acting on $H^{\otimes n}$, with the promise that either $\{\omega_n\}_{n\in \mathrm{N}}$ is a sequence of unknown non-resource states or $\omega_n = \rho^{\otimes n}$, for some resource state ρ, decide which is the case.

Probabilities of Error:

$$\beta_1(A_n) := tr(\rho^{\otimes n}(I - A_n))$$

$$\beta_2(A_n) := \max_{\omega_n \in M} tr(\omega_n A_n)$$

The main idea

- We say a resource theory defined by the non-resource states *M* has the *exponential distinguishing* (**ED**) property if for every resource state ρ

$$\beta_n(\rho,\varepsilon) := \min_{0 \le A_n \le I} \left(\beta_2(A_n) : \beta_1(A_n) \le \varepsilon\right) \approx 2^{-nE(\rho)}$$

For a non-identically zero function *E*

$$\beta_2(A_n) := \max_{\omega_n \in M} tr(\omega_n A_n) \qquad \beta_1(A_n) := tr(\rho^{\otimes n}(I - A_n))$$

Theorem I

- **Theorem:** If M satisfies **ED**, then

$$E(\rho) = \min_{\{\rho_n\}} \lim_{n\to\infty} \frac{\log(1+R_M(\rho_n))}{n} \quad : \quad \| \rho_n - \rho^{\otimes n} \|_1 \to 0$$

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and for every σ such that $E(\sigma) > 0$

$$R(\rho \to \sigma) = E(\rho) / E(\sigma)$$

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Proof main idea: Take maps of the form

$$\Lambda_n(.) = tr(A_n .)\sigma^{\otimes nE(\sigma)/E(\rho)} + tr((I - A_n).)\pi_n$$

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The theorem is completely general.

The trouble is of course how to prove that the set of non-resource of interest satisfies **ED**... Difficult in general!

Theorem II

- **Theorem:** If *M* satisfies

1.Closed and convex and contain the max. mixed state

2.If $\sigma \in M_n,\ \pi \in M_m \Rightarrow \sigma \otimes \pi \in M_{n+m}$

3.If $\sigma \in M_{n+1} \Rightarrow tr_{n+1}(\sigma) \in M_n$

4.If $\sigma \in M_n \Rightarrow P_\pi \sigma P_\pi \in M_n, \forall \pi \in S_n$

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$$P_\pi |\psi_1\rangle|\psi_2\rangle \ldots |\psi_n\rangle = |\psi_{\pi^{-1}(1)}\rangle|\psi_{\pi^{-1}(2)}\rangle \ldots |\psi_{\pi^{-1}(n)}\rangle$$

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Then **ED** holds true, $E(\rho) = E_M^\infty(\rho)$, and by the Theorem I

$$R(\rho \rightarrow \sigma) = E_M^\infty(\sigma) / E_M^\infty(\rho)$$

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Proof: Original quantum Stein's Lemma + exponential de Finetti theorem + Lagrange duality

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$$E'(\rho) = \min_{\{\rho_n\}} \lim_{n\to\infty} \frac{\log(1 + R'_M(\rho_n))}{n} \quad : \ \| \rho_n - \rho^{\otimes n} \|_1 \to 0$$

- **Equivalent** to $E(\rho) = E'(\rho)$

The choice of R_M

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- For the case of entanglement, we can show that if we use the minimum trace distance to the set of separable states, or *any asymptotically continuous entanglement measure*, then the theory is trivial:

$$0 = E_C(\rho) < E_D(\rho) = \infty$$

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$$0 = E_C(\rho) < E_D(\rho) = \infty$$

- Whether the result holds true for more stringent measures than R_M is an open question. Example:

$$\| \quad \|_\infty \; / \dim$$

Application 1

We can show that for every entangled state ρ, $E_R^{\infty}(\rho) > 0$, by constructing a distillation protocol under asymptotically non-entangling maps with a *non-zero* rate.

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$$R_{LOCC}(\rho \to \sigma) \geq R(\rho \to \sigma) = E_M^\infty(\sigma) / E_M^\infty(\rho) > 0$$

The mathematical definition of multipartite entanglement is equivalent to its operational definition

Bipartite case solved by Yang, Horodecki, Horodecki, Synak-Rydtke in 2005
Independent proof by Marco Piani (see arXiv:

Application 2

Recently Beige and Shor found the following application of our main result:

Given a entanglement criterion which

- is necessary for separability, but not sufficient
- if ρ is not detected, then $\rho \otimes \rho$ is not detected either

(e.g. PPT test, realignment test, Doherty et al hierarchy of tests, etc)

Then for every $\varepsilon > 0$ there is a state π not detected by the test such that

$$\min_{\omega \in S} || \pi - \sigma ||_1 > 2 - \varepsilon$$

Application 2

Proof Sketch: Take any entangled state π not detected. We show

$$\min_{\omega\in S} \| \pi^{\otimes n} - \sigma \|_1 \to 2$$

Consider the optimal sequence of asymptotically non-entangling maps for distilling π:

$$\min_{\omega\in S} \| \pi^{\otimes n} - \sigma \|_1 \geq \min_{\omega\in S} \| \Lambda_n(\pi^{\otimes n}) - \Lambda_n(\sigma) \|_1 \approx 2 - 2^{-nE_R^{\infty}(\rho)}$$

Open Problem:

$$E_R^{\infty}(\rho \otimes \sigma) \stackrel{?}{=} E_R^{\infty}(\rho) + E_R^{\infty}(\sigma)$$

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Define $$CE_R(\rho) = \inf E_R(\pi_{12}) - E_R(\pi_2) : tr_2(\pi_{12}) = \rho$$

Yang, Horodeki 07

Assuming a certain conjecture: $E_R^{\infty}(\rho) = CR(\rho)$

Joint work with Michal Horodecki

Open Problem:

$$E_R^\infty(\rho \otimes \sigma) \overset{?}{=} E_R^\infty(\rho) + E_R^\infty(\sigma)$$

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Yang, Horodeki 07

Assuming a certain conjecture: $E_R^\infty(\rho) = CR(\rho)$

Would imply E_R^∞ is strongly super-additive

Would also imply $QMA(k) = QMA(2) \ \forall k \geq 2$

and error amplification for QMA(2) Aaronson et al 09

Joint work with Michal Horodecki

The conjecture

For every projector *P* acting on $C^d \otimes C^d$

$$tr(P) \overset{?}{\leq} d^{2+\varepsilon} \left(\max_{|a\rangle,|b\rangle} \langle a,b|P|a,b\rangle \right)^2$$

For $\varepsilon < 1$

Thank you!