On the existence of subgaussian directions for log-concave measures

joint work with A. Giannopoulos and P. Valettas.

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Notation

$\mathcal{P}_{[n]}$: the class of all probability measures in $\mathbb{R}^n$ which are absolutely continuous with respect to the Lebesgue measure.

μ in $\mathcal{P}_{[n]}$ is called log-concave if for any Borel sets A, B and any $\lambda \in (0,1)$, $\mu(\lambda A + (1-\lambda)B) \geq \mu(A)^{\lambda}\mu(B)^{1-\lambda}$.

$\mu \in \mathcal{P}_{[n]}$ is called centered if for all $\theta \in S^{n-1}$, $\int_{\mathbb{R}^n} \langle x, \theta \rangle d\mu(x) = 0$.

Let K a convex body in $\mathbb{R}^n$. Then $d\mu := \mathbf{1}_K dx$ is log-concave.

Notation

A direction $\theta \in S^{n-1}$ is subgaussian for μ with constant $r > 0$ if

$$\|\langle \cdot, \theta \rangle\|_{\psi_2} \leq r m_\theta,$$

where m_θ is the median of $|\langle \cdot, \theta \rangle|$ with respect to μ, and

$$\|f\|_{\psi_2} = \inf \left\{ t > 0 : \int_{\mathbb{R}^n} \exp\left((|f(x)|/t)^2 \right) \, d\mu(x) \leq 2 \right\}.$$

Let K centered convex body in $\mathbb{R}^n$, $|K| = 1$,

$$h_{Z_2(K)}(\theta) = \left(\int_K |\langle x, \theta \rangle|^2 dx \right)^{\frac{1}{2}}, \ \theta \in S^{n-1}$$

We say that K is isotropic if $Z_2(K) = L_K B_2^n$.

The Question

[Bourgain] If $r_K(\theta) = O(1)$ for all θ , then $L_K = O(1)$.

Question: [V. Milman] is it true that every convex body K has at least one "subgaussian" direction (with constant $r = O(1)$)?
[Bobkov- Nazarov] Yes, if K is 1-uncoditional.
[Klartag] Yes, with $r = \log^2 n$.
[G-P-P] Yes, with $r = \log n$.
[G-P-V] Yes, with $r = \sqrt{\log n}$.

The result

Theorem (i) If K is a centered convex body of volume 1 in $\mathbb{R}^n$, then there exists $\theta \in S^{n-1}$ such that

$$|\{x \in K : |\langle x, \theta\rangle| \geq cth_{Z_2(K)}(\theta)\}| \leq e^{-\frac{t^2}{\log(t+1)}}$$

for all $t \geq 1$, where $c > 0$ is an absolute constant.

(ii) If μ is a centered log-concave probability measure on $\mathbb{R}^n$, then there exists $\theta \in S^{n-1}$ such that

$$\mu\left(\{x \in \mathbb{R}^n : |\langle x, \theta\rangle| \geq ct\mathbb{E}|\langle \cdot, \theta\rangle|\}\right) \leq e^{-\frac{t^2}{\log(t+1)}}$$

for all $1 \leq t \leq \sqrt{n \log n}$, where $c > 0$ is an absolute constant.

The $\Psi_2(K)$ body

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$ and $p \geq 1$. We define the $Z_p(K)$ as the symmetric convex body with support function

$$h_{Z_p(K)}(\theta) := \left(\int_K |\langle x, \theta \rangle|^p dx \right)^{\frac{1}{p}}.$$

Also we define the $\Psi_2(K)$ as the symmetric convex body with support function

$$h_{\Psi_2(K)}(\theta) := \sup_{1 \leq p \leq n} \frac{h_{Z_p(K)}(\theta)}{\sqrt{p}}.$$

The $\Psi_2(K)$ body

From the definitions, one has $Z_p(K) \subseteq \sqrt{p}\Psi_2(K)$ for all $1 \leq p \leq n$. In particular,

$$Z_2(K) \subseteq \sqrt{2}\Psi_2(K),$$

which implies that

$$|\Psi_2(K)|^{1/n} \geq c\frac{L_K}{\sqrt{n}}.$$

Conjecture: $|\Psi_2(K)|^{1/n} \leq c'\frac{L_K}{\sqrt{n}}$.

Since $Z_2(K)$ is an ellipsoid contained in $\Psi_2(K)$ with volume $\frac{cL_K}{\sqrt{n}}$, the conjecture implies that $\Psi_2(K)$ is a "bounded volume ratio" body.

The $\Psi_2(K)$ body

Theorem Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then,

$$|\Psi_2(K)|^{1/n} \leq c\frac{\sqrt{\log n}}{\sqrt{n}} L_K.$$

Note that $\Psi_2(K) = \mathrm{conv}\left\{\frac{Z_p(K)}{\sqrt{p}}, p \in [1, n]\right\}$,
and using the fact that $Z_{2p}(K) \simeq Z_p(K)$, we may write

$$\Psi_2(K) \simeq \mathrm{conv}\left\{\frac{Z_p(K)}{\sqrt{p}}, p = 2^k,\ k = 1, \ldots, \log_2 n\right\}.$$

Known (P., L-Y-Z) : $|Z_p(K)|^{\frac{1}{n}} \leq c\sqrt{\frac{p}{n}} L_K$, $|Z_p(K)|^{\frac{1}{n}} \geq c\sqrt{\frac{p}{n}}$.

The $\Psi_2(K)$ body

For any A, B, $|A| \leq N(A, B)|B|$.

Lemma

Let $A_1, \ldots, A_s$ be subsets of RB_2^k. For every $t > 0$,

$$N(\mathrm{conv}(A_1 \cup \cdots \cup A_s), 2tB_2^k) \leq \left(\frac{cR}{t}\right)^s \prod_{i=1}^{s} N(A_i, tB_2^k).$$

"Regularity" of the covering numbers of $Z_q(K)$: (G-P-P)

$$\log N(Z_q(K), t\sqrt{q}L_K B_2^n) \leq \frac{c_1 n}{t} + \frac{c_2\sqrt{n}\sqrt{q}}{\sqrt{t}}, \ t \geq 1.$$

Regularity of the covering numbers of $Z_q(K)$

Proposition Let K be an isotropic convex body in $\mathbb{R}^n$, let $1 \leq q \leq n$ and $t \geq 1$. Then,

$$\log N\left(Z_q(K), c_1 t\sqrt{q} L_K B_2^n\right) \leq c_2 \frac{n}{t^2} + c_3 \frac{\sqrt{n}\sqrt{q}}{t},$$

where $c_1, c_2, c_3 > 0$ are absolute constants.
Corollary Let K be an isotropic convex body in $\mathbb{R}^n$ and let $1 \leq q \leq n$. Define $\beta \geq 1$ by the equation $q = n^{1/\beta}$. Let $\alpha := \min\{\beta, 2\}$. Then,

$$N\left(Z_q(K), c_1 t\sqrt{q} L_K B_2^n\right) \leq \exp\left(c_2 \frac{n}{t^\alpha}\right),$$

where $c_1, c_2 > 0$ are absolute constants.

Sudakov-type estimates

Let $C \subseteq \mathbb{R}^n$ be a symmetric convex body. We define

$$W(C) := \int_{S^{n-1}} h_C(\theta) d\sigma(\theta).$$

[Talagrand] Let γ_n be the n-dimensional Gaussian measure . Then, for any $s, t > 0$ we have:

$$N(B_2^n, tC^\circ) \leq e^{(2s/t)^2} [\gamma_n(sC^\circ)]^{-1}.$$

Let m_1 such that $\gamma_n(m_1 C^\circ) = \frac{1}{2}$. Also

$$m_1 \simeq I_1(\gamma_n, C^\circ) = \int_{\mathbb{R}^n} \|x\|_{C^\circ} d\gamma_n(x) \simeq \sqrt{n} W(C).$$

Then $\log N\left(B_2^n, tC^\circ\right) \leq c \frac{nW(C)^2}{t^2}$.

Sudakov-type estimates

Let $0 < p$ and let m_p such that $\gamma_n(m_p C^\circ) = \frac{1}{2^p}$. Also we write I_{-p} and $W_{-p}(C)$ for

$$I_{-p} \equiv I_{-p}(\gamma_n, C) := \left(\int_{\mathbb{R}^n} \|x\|_K^{-p} \, d\gamma_n(x) \right)^{-1/p},$$

$$W_{-p}(C) := \left(\int_{S^{n-1}} h_C^{-p}(\theta) d\sigma(\theta) \right)^{-\frac{1}{p}}$$

Then (Markov's inequality) $m_p \geq \frac{1}{2} I_{-p}$.
Assume that for some p and some $\alpha > 1$ we have the following "regularity" condition:

$$I_{-p} \leq \alpha I_{-2p}.$$

Then, by applying the Paley-Zygmund inequality we get

$$\gamma_n(x : \|x\|_C^\circ \leq 2I_{-p}) \geq 2^{-2p \log(2\alpha)}.$$

It follows that $m_{2p\log(2\alpha)} \leq 2I_{-p}$ and $I_{-p}(\gamma_n, C^\circ) \simeq \sqrt{n} W_{-p}(C)$.

Sudakov-type estimates

Corollary Let C be a symmetric convex body in $\mathbb{R}^n$ and let $1 \leq p \leq n/2$ be such that $W_{-2p}(C) \simeq W_{-p}(C)$. Then,

$$\log N\left(C, c_1\sqrt{n/p}W_{-p}(C)B_2^n\right) \leq c_2 p,$$

where $c_1, c_2 > 0$ are absolute constants.
Proof: Choose $s := m_p \simeq \sqrt{n}W_{-p}(C)$. Then

$$N(B_2^n, tK^\circ) \leq e^{\frac{nW_{-p}(C)^2}{t^2}}[\gamma_n(sK^\circ)]^{-1} \leq e^{\frac{nW_{-p}(C)^2}{t^2}}e^p.$$

Choose $t = \sqrt{n/p}W_{-p}(C)$, then $\log N(B_2^n, tK^\circ) \leq cp$. Then "duality of entropy" (A-M-S). □

Back to $Z_q(K)$

Let $-n < q \le n$, $I_q(K) = \left(\int_K \|x\|_2^q dx\right)^{\frac{1}{q}}$. Let $q > 0$, then (P.),

$$I_q(K) \simeq \sqrt{\frac{n}{q}} W_q(Z_q(K)),$$

$$I_{-q}(K) \simeq \sqrt{\frac{n}{q}} W_{-q}(Z_q(K)).$$

Then

$$c_1\sqrt{q} \le W_{-n}(Z_q(K)) \le W_{-q}(Z_q(K)) \le c_2\sqrt{q} L_K.$$

Convolutions

Proposition Let K be an isotropic convex body in $\mathbb{R}^n$. There exists an isotropic convex body K_1 in $\mathbb{R}^n$ with the following properties:

1. $L_{K_1} \leq c_1$.
2. $c_2 Z_p(K_1) \subseteq \frac{Z_p(K)}{L_K} + \sqrt{p} B_2^n \subseteq c_3 Z_p(K_1)$ for all $1 \leq p \leq n$.
3. $c_4 \Psi_2(K_1) \subseteq \frac{\Psi_2(K)}{L_K} \subseteq c_5 \Psi_2(K_1)$.

The constants c_i, $i = 1, \ldots, 5$ are absolute positive constants.

Back to $Z_q(K)$

Proposition Let K_1 be an isotropic convex body as before , let $1 \leq q \leq n/2$ and $1 \leq t \leq \sqrt{n/q}$. Then,

$$\log N\left(Z_q(K_1), c_1 t\sqrt{q} B_2^n\right) \leq c_2 \frac{n}{t^2}.$$

Proof: For $q \leq r \leq n$, $W_{-r}(Z_q(K)) \simeq \sqrt{q}$. Then

$$\log N\left(Z_q(K_1), \sqrt{\frac{n}{r}} W_{-r}(Z_q(K)) B_2^n\right) \leq r.$$

Set $t = \sqrt{\frac{n}{r}}$. □

Back to $\Psi_2(K)$

Let $2^{k_1} = \frac{n}{\log n}$. Let $V_1 := \mathrm{conv}\left\{ \frac{Z_p(K_1)}{\sqrt{p}}, p = 2^k, k = 1, \dots, k_1 \right\}$ and $V_2 := \mathrm{conv}\left\{ \frac{Z_p(K_1)}{\sqrt{p}}, p = 2^k, k = k_1, \dots, \log_2 n \right\}$.
Note that $\Psi_2(K_1) \simeq \mathrm{conv}\left\{ V_1, V_2 \right\}$.
Then $\log N\left(V_1, \sqrt{\log n} B_2^n\right) \leq n$ and $\log N\left(V_2, \log\log n B_2^n\right) \leq n$.
So, $\log N\left(\Psi_2(K_1, c\sqrt{\log n} B_2^n\right) \leq n$ and

$$|\Psi_2(K_1)|^{\frac{1}{n}} \leq c\frac{\sqrt{\log n}}{\sqrt{n}}$$

So,

$$|\Psi_2(K)|^{\frac{1}{n}} \leq c\frac{\sqrt{\log n}}{\sqrt{n}} L_K$$