Coxeter Lecture Series

Fields Institute

Thematic Program on Asymptotic Geometric Analysis

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“How very little determines a lot”

- Lecture 1: Abstract duality, the Legendre transform and a new duality transform.

- Lecture 2: Order isomorphisms and the fundamental theorem of affine geometry.

- Lecture 3: Multiplicative transforms and characterization of the Fourier transform.

Lecture 2:

Order isomorphisms and the fundamental theorem of affine geometry

Based on joint work with Dan Florentin and Vitali Milman and on joint work with Boaz Slomka.

The fundamental theorem of affine geometry:

$$F : R^n \to R^n$$

collinear points to collinear + non-deg (say, bijection) implies that the mapping is linear

In this talk we'll discuss two variants (one known, one new) which come up when studying order isomorphism.

The first one, which was alluded to in the first talk, regards mappings $F : K_1 \to K_2$.
The second regards $F : R^n \to R^n$ where lines are assumed to be mapped to lines only in a fixed number of directions.

Consider a partially ordered set $(S, \leq)$.

Call a bijection $T : S \to S$ an "order reversing isomorphism" if

(1) $x \leq y$ if and only if $Tx \geq Ty$

Call it an "abstract duality" if also

(2) $T \circ T = Id_S$

We became interested in characterizing such transforms for various classes connected with convexity

Note that if you know one order-reversing isomorphism on the class, the question becomes the same as that of characterizing "order preserving isomorphisms":

$$x \leq y \quad \text{if and only if} \quad Tx \leq Ty$$

Part 1: Order isomorphisms on windows

Notation: The class of lower-semi-continuous convex functions on K will be denoted by $Cvx(K)$. (We call this a "window".)

Let $0 \in K$.

Notation: The subclass of $Cvx(K)$ consisting of non-negative functions satisfying $f(0) = 0$ will be denoted by $Cvx_0(K)$
(later referred to as "geometric convex functions on the window")

We wish to characterize order preserving transforms

$$T : Cvx(K_1) \to Cvx(K_2)$$

(the partial order given by point-wise inequality)

Which do we expect?

Say, $K_1 = K_2$.

Linear ones, of course, but what else?

Start by looking at delta-functions

$$\delta_{x_0,c}(x) = \begin{cases} c, & x = x_0 \\ +\infty, & x \neq x_0 \end{cases}$$

They satisfy an extremal property:

$$f, g \geq \delta_{x_0,c}$$

implies that the two functions are comparable, whereas above any other function there lie two incomparable functions.

This implies that delta-functions are mapped to delta-functions, and thus we have a point map from R^{n+1} to itself.

(note that we are in the simpler case of ALL convex functions. This would NOT apply in the geometric case as the delta functions are not members in the class)

It is not difficult (routine methods by now) to show that:

(a) this point map maps intervals to intervals

(b) the transform is induced by it (via epigraphs)

If a bijective map maps lines to lines on all of R^n, it must be affine-linear. (The fundamental theorem of affine geometry) (n at least 2). This gave:

Theorem [A-M] :

There is a unique duality on the class of all convex functions, up to a linear term it is the Legendre transform.

$$(\mathcal{L}f)(x) = \sup_{y \in R^n} (\langle x, y \rangle - f(y))$$

The three ingredients:

$T' = T \circ \mathcal{L}$ is an order preserving bijection

$f \leq l$ implies $f = l - c$, so
$f, g \leq l$ implies f, g comparable

Note that linear functions are not extremal in the bounded case!

F.T.A.G. = the fundamental theorem of affine geometry

It is not difficult (routine methods by now) that

(a) this point map maps intervals to intervals

(b) the transform is induced by it (via epigraphs)

If $F : K_1 \to K_2$ maps intervals to intervals, what must it look like?

Convexity Preserving Maps

A Fundamental Theorem for subsets:

If $F : K_1 \to K_2$ maps intervals to intervals, it must be fractional-linear:

$$F(x) = \frac{Ax + v_1}{\langle x, v_2 \rangle + c}$$

In the same way the Fundamental Theorem of Affine Geometry worked in the case of all convex functions, these transformations turn up when we talk about windows.

Theorem [Florentin's thesis]: All order isomorphisms

$$T : Cvx(K_1) \to Cvx(K_2)$$

are induced by fractional linear maps.

$$epi(Tf) = F(epi(f))$$

Theorem [Florentin's thesis]: All order isomorphisms

$$T : Cvx_0(K_1) \to Cvx_0(K_2)$$

are induced by fractional linear maps.

$$epi(Tf) = F(epi(f))$$

Of course, not all fractional linear maps are applicable, as they must preserve "epigraph-ity", but this is the only restriction.

One can then compute the formula for the transform. For example:

$$epi(Tf) = F(epi(f))$$

$$F(x,y) = (\frac{x}{y}, \frac{1}{y})$$

$$(\mathcal{J}f)(x) = \inf\{r > 0 : rf(\frac{x}{r}) \leq 1\}$$

$$F(x,y) = (\frac{x}{x_1+1}, \frac{y}{x_1+1})$$

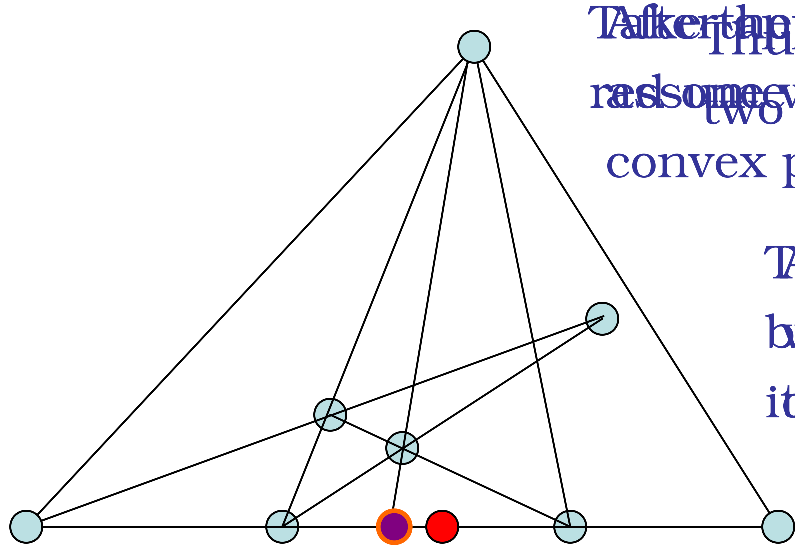
$$(\mathcal{T}f)(x) = (1 - x_1)f(\frac{x}{1-x_1})$$

$$F(x,y) = (\frac{Ax}{Lx+b}, \frac{y}{Lx+b})$$

$$(\mathcal{T}f)(x) = (1 - LA^{-1}x)f(F_1^{-1}x)$$

A few words about this “fundamental theorem”:
It is essentially a theorem in projective geometry, saying that if a mapping preserves projective intervals on an open subset, it can be extended to all the projective space. [Shiffman – using Desargues thm]

We reprove this in a different manner;



Thus the intersection of two such lines is fixed.

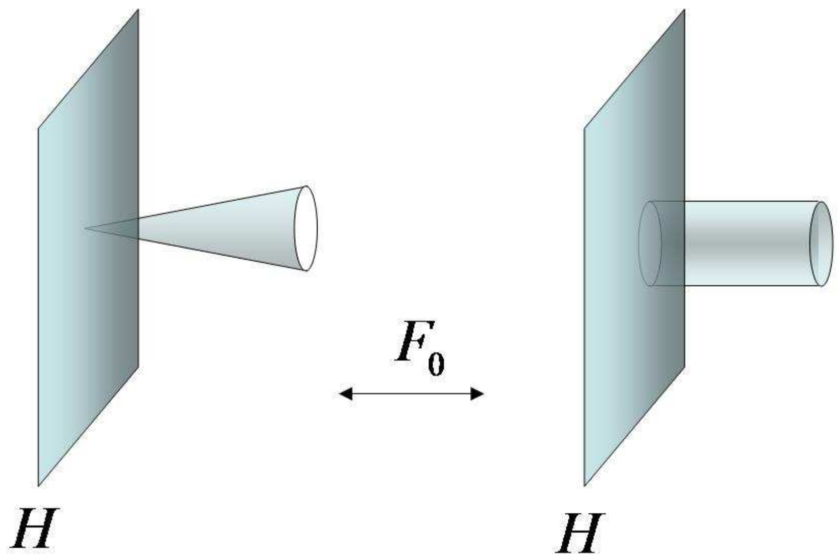
convex position are fixed.

Assume the red point is not fixed

How do fractional linear maps behave?

There's a "defining hyperplane" where they are not defined; cones emanating from it are mapped to cylinders and vice-versa

It is linear on hyperplanes parallel to the defining one.



All of them are affinely equivalent to the canonical

$$F(x) = \frac{x}{x_1 - 1}$$

Fractional linear maps turn out very naturally in convexity when one considers duality of translations of a convex body

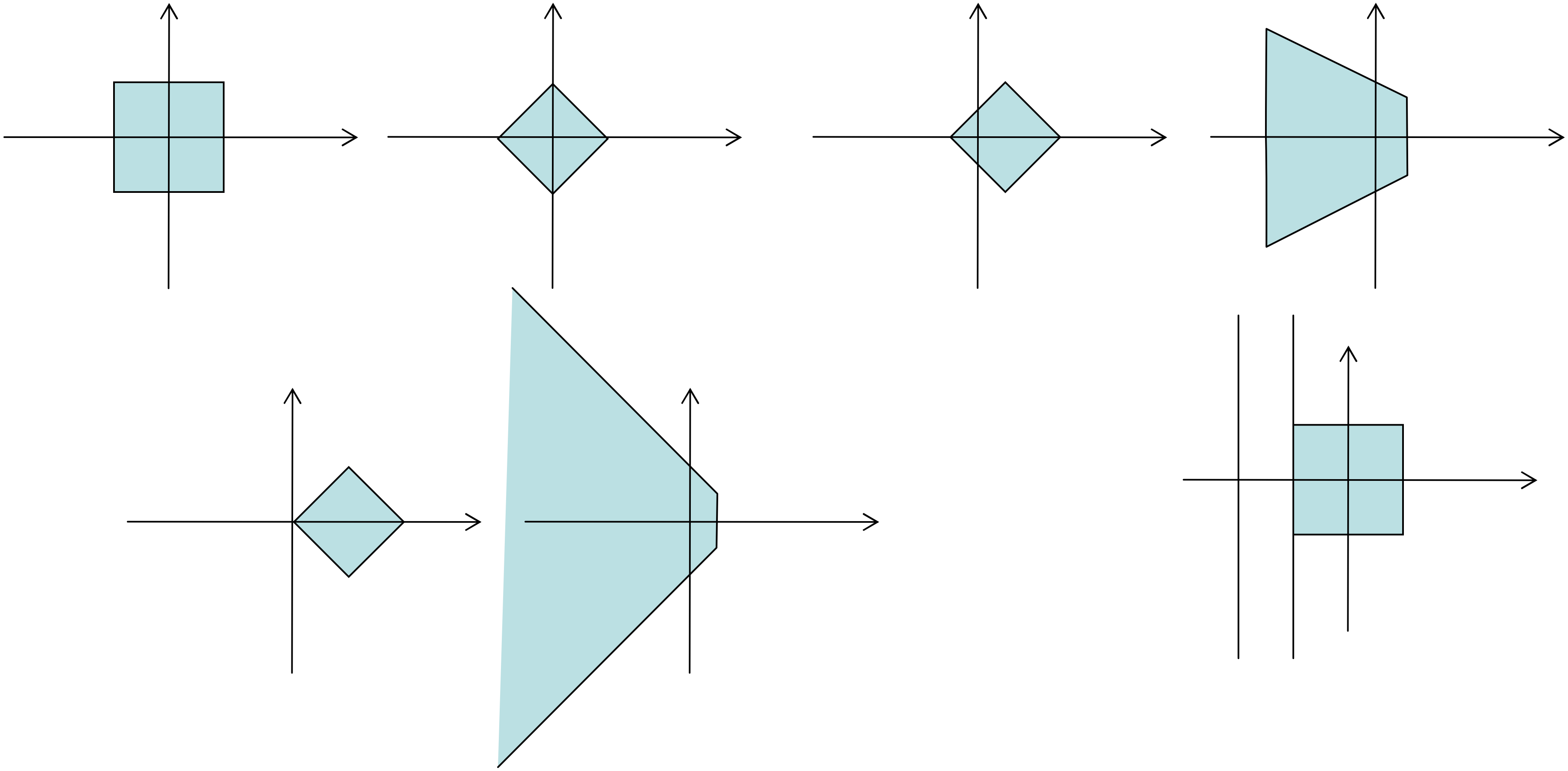
Letting $$F(x) = \frac{x}{1 + \langle x_0, x \rangle}$$

one has $$(x_0 + K)^\circ = F(K^\circ)$$

and so $$(x_0 + K^\circ)^\circ = F(K)$$

$$(x_0 + K^\circ)^\circ = F(K)$$

$$F(x) = \frac{x}{1 + \langle x_0, x \rangle}$$



One may ask “which bodies are fractional linear images of a given body”?

..Clearly combinatorial structure is preserved..

..Conical sections are mapped to conical sections..

And when are the only fractional linear maps of a body onto itself linear ones?

..a cube; the cross-polytope..
..if $f(0) = 0$, all symmetric bodies..

Part 2: The fundamental theorem of affine geometry for a fixed number of directions.

[Joint with Boaz Slomka]

Motivation: the simplest setting of order isomorphims which is n-dimensional space with a vector ordering.

These questions were studied for specific cones (light cones: Alexandrov ['50] and ['75], non-angular cones Zeeman and Rothaus [‘64-‘66]).

Theorem [A-Slomka]: Fix a cone K (non deg., closed)
And let $T: R^n \to R^n$ be a bijective order isomorphism

$$x \leq_K y \Longleftrightarrow Tx \leq_K Ty$$

Example: $K = (R_+)^3, Tx = (f_1(x_1), f_2(x_2), f_3(x_3))$

Then:

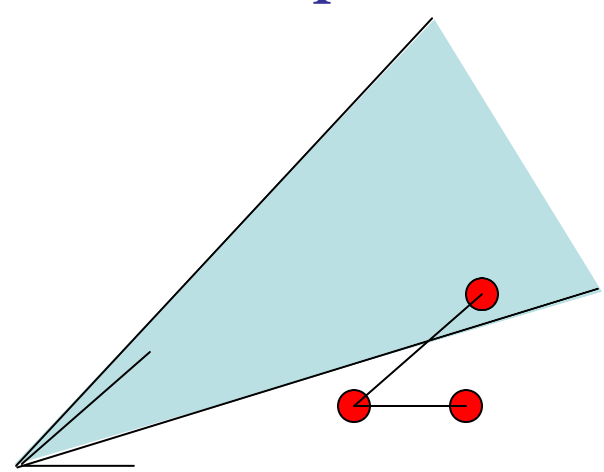
(1) If K has "more than n" extremal rays, T is linear

(2) Otherwise, T is "diagonal" as in the example above.

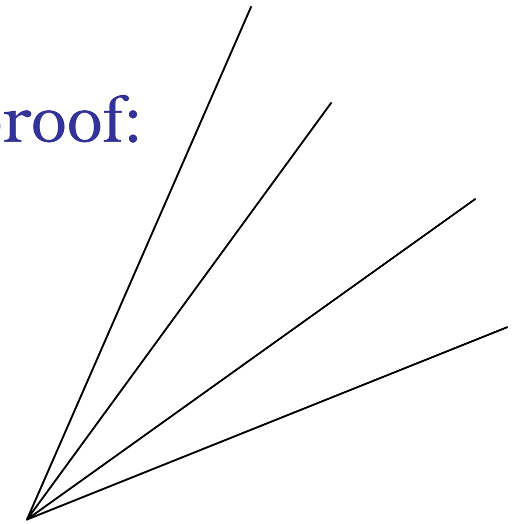
$$x \leq_K y$$

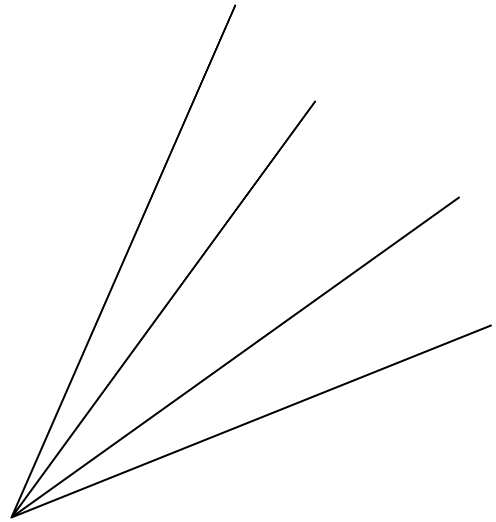
means simply that

$$y - x \in K$$



In the proof:





extremal rays are mapped to extremal rays,

+ translations:

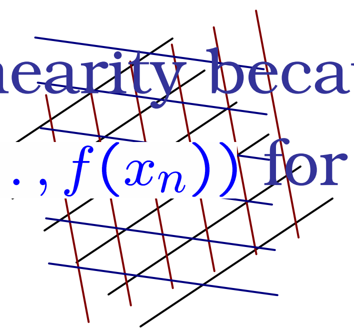
need “a Fundamental theorem for a finite number of directions”

New Fundamental Theorem of Affine Geometry:

Theorem [A-Slomka] : Let $F : R^n \to R^n$ be a bijection and fix (n+1) generic lines $L_1, \dots, L_{n+1}$. Assume $F(x + L_i)$ is a line for all $x \in R^n, i = 1, \dots, n+1$. Then $F - F(0)$ is additive.

So, one need not check ALL lines are mapped to lines, only a very small subfamily

Note that one cannot deduce linearity because of the example: $f(x_1, \dots, x_n) = (f(x_1), \dots, f(x_n))$ for an additive f



Continuous version:

Theorem [A-Slomka] : Let $F : R^n \to R^n$ be a bijection and fix $n+1$ generic directions $v_1, \dots, v_{n+1}$. Assume F is a collineation <u>in these directions</u>. Assume it is also continuous. Then F is affine.

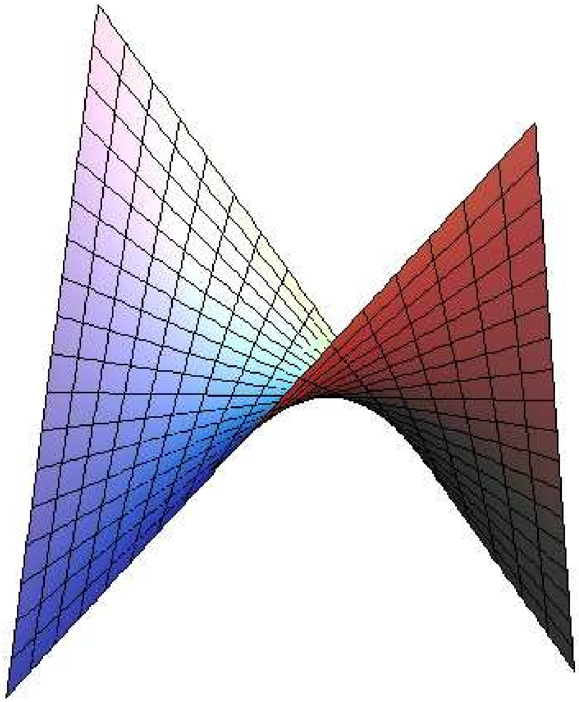
There is a big difference whether one knows parallel lines (in these directions) are mapped to parallel lines, or this is not known in advance.

In the projective setting, this is **forced**, as we will see later.

In many applications, this is known a-priori,
in which case, things more-or-less work by induction.

When this is not assumed, the proof is more interesting.

Proposition : Let $F : R^2 \to R^n$ be an injective mapping such that lines of the form $\{(t_0, s) : s \in R\}$ and $\{(t, s_0) : t \in R\}$ are mapped onto lines.
Then essentially the image is either a plane or



$$F(s,t) = \begin{pmatrix} f(s) \\ g(t) \\ f(s)g(t) \end{pmatrix}$$

Proposition : Let $F : R^k \to R^n$ be an injective mapping such that lines of the form $\{x + se_i : s \in R\}$ for $i = 1, \ldots, k$ are mapped onto lines.
Then up to linear terms the mapping is given by

$$F(x_1, \ldots, x_k) = \sum_{\delta \in \{0,1\}^k} u_\delta \prod_{i=1}^{k} f_i(x_i)^{\delta_i}$$

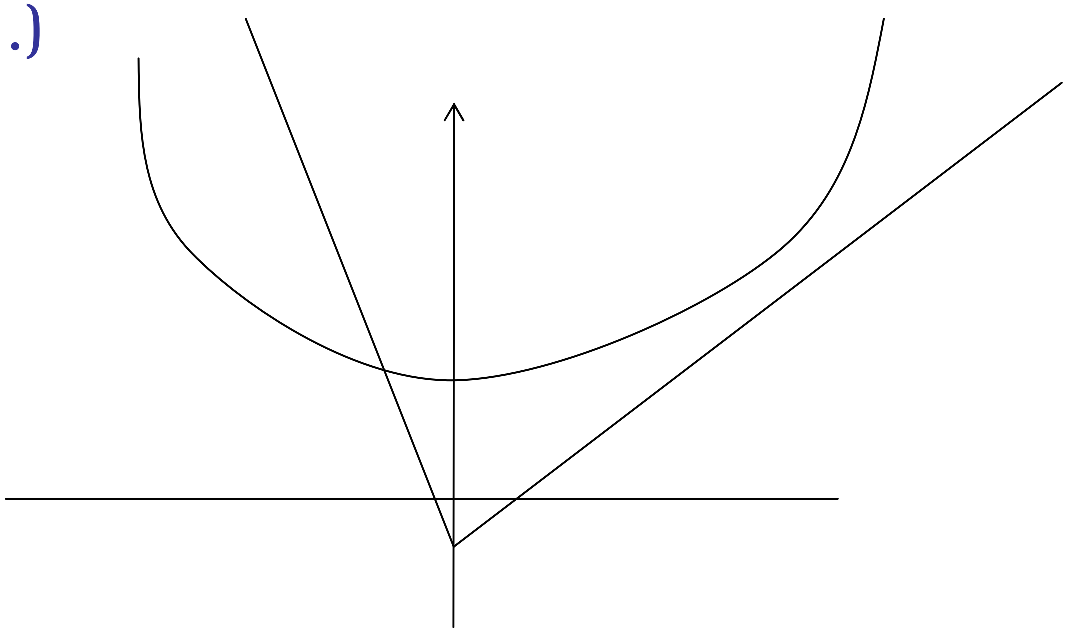
The next step: showing that adding one direction, only the linear terms survive, getting:

Theorem [A-Slomka] : Let $F : R^n \to R^m$ be an injection mapping the lines $x + sp(e_1), \ldots, x + sp(e_n)$ $x + sp(e)$ onto lines. Then it is of the form

$$F(x) = F(0) + \sum_{i=1}^{n} f(x_i) w_i$$

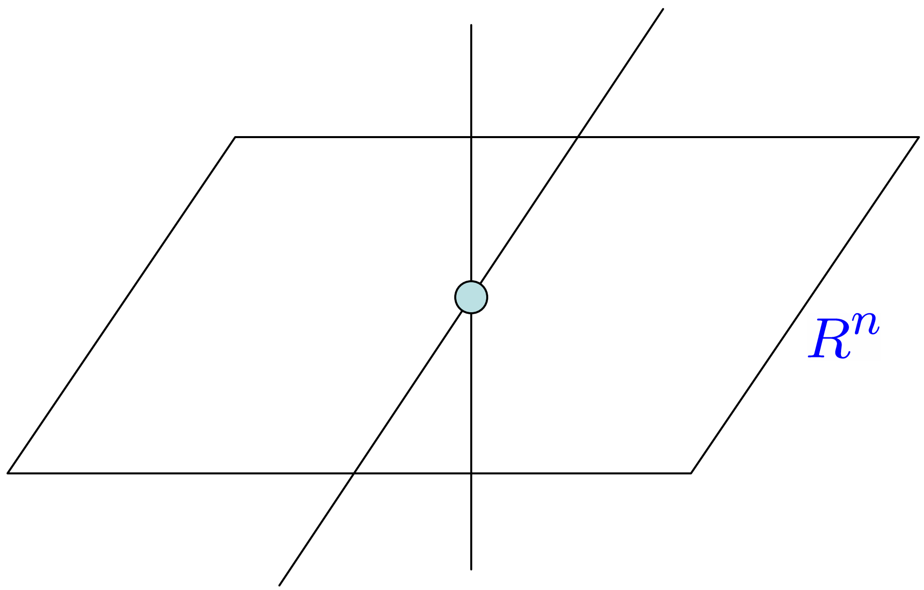
for additive $f : R \to R$.

(Here $e = (1, \ldots, 1)$.)

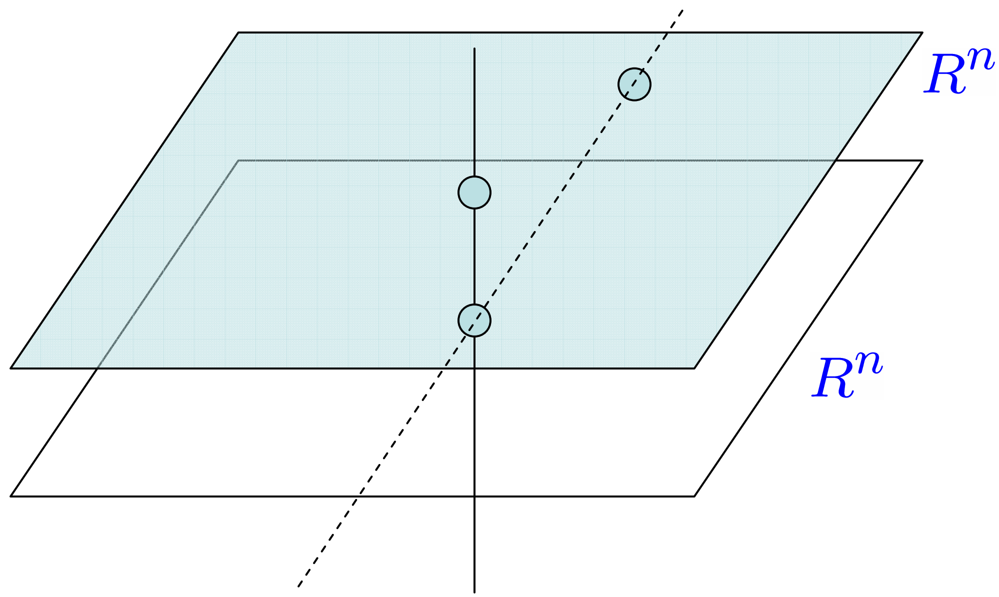


Projective point of view:

Living in projective space, a point is a line through the origin, and a line is a plane through the origin.

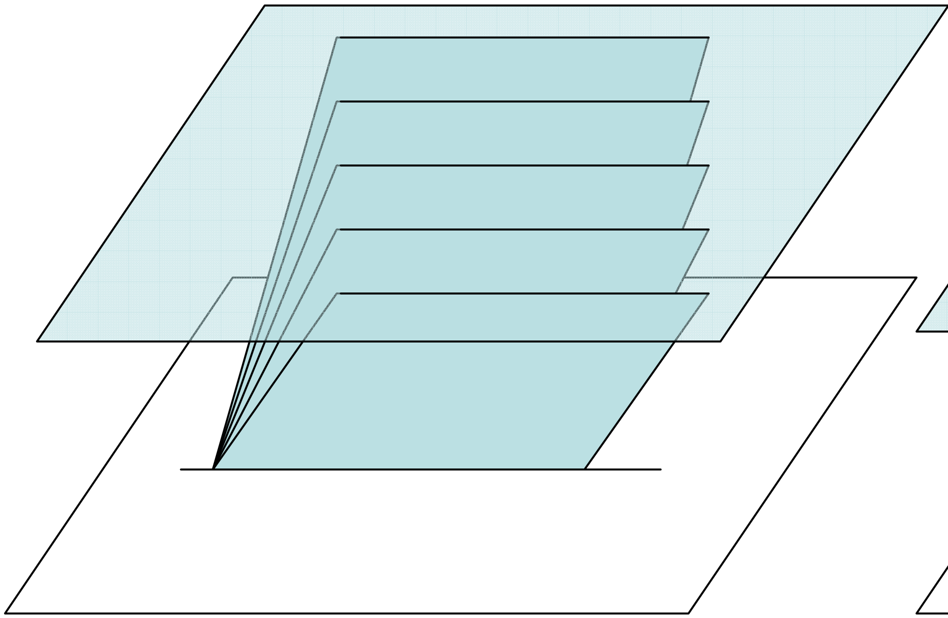
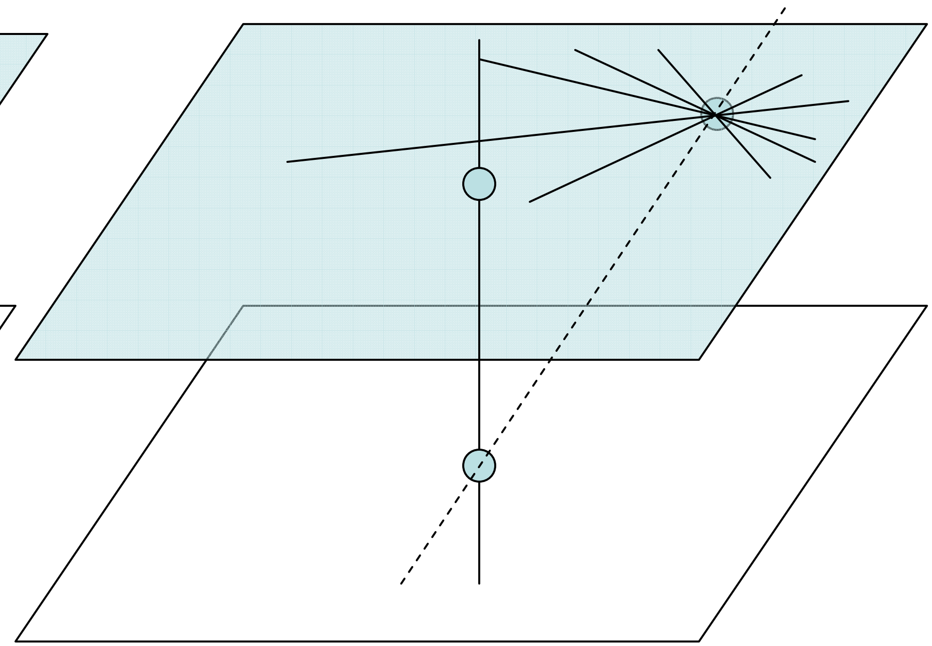
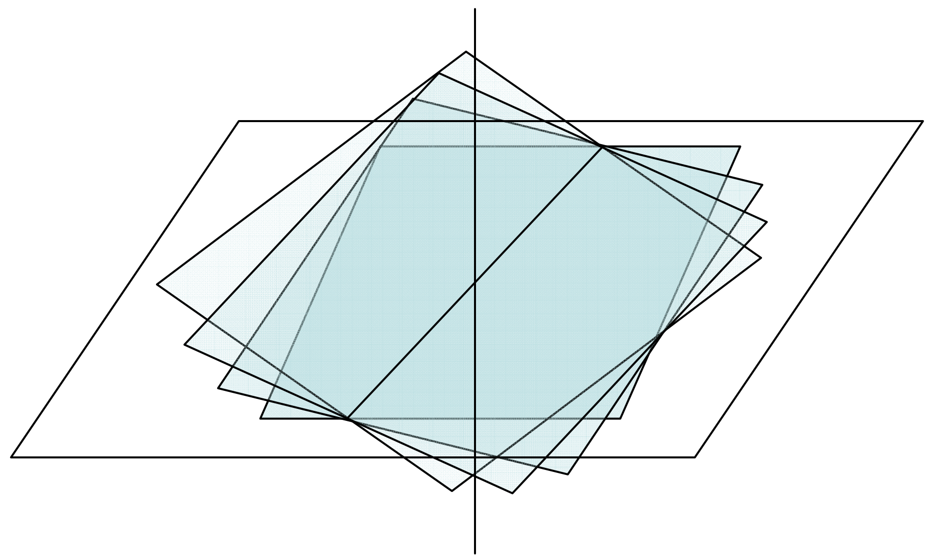


Fundamental theorem: as usual.



Any two projective lines intersect at a projective point.

Parallel lines in n-space are projective lines all passing through one point, i.e., a “star”.

In the projective context, for the plane, results in this spirit have been obtained by Prenowitz.
For an open set of directions: by Shiffman [also for bounded domains].

Theorem [A-Slomka]: Let $n \geq 2$ and let $p_0, \ldots, p_{n+1} \in RP^n$ satisfy one of the following non-degeneracy conditions:

(a) $p_0, \ldots, p_n$ are generic in an n-dimensional subspace and $p_1, \ldots, p_{n+1}$ are linearly independent

(b) $p_0, \ldots, p_n$ are linearly independent, so are $p_1, \ldots, p_{n+1}$ and $p_0 \neq p_{n+1}$

Assume $F : RP^n \to RP^n$ maps projective-lines through these points onto projective lines.
Then the mapping is linear.

We remark that all results apply to any field which is not $\mathbb{Z}_2$. (Finite fields are of special interest.)

Also that it simplifies the proof of Alexandrov's "fundamental theorem of chronogeometry" and its generalizations by Pfeffer.

Let us state a simple application based on a result by Gardner and Mauldin, a particular case of which is:

Theorem: No non-affine bijection exists sending circles into circles.
Proof: take three collinear points in the image, if their pre-images are not collinear, they are contained in a circle, and so their image is too.

Theorem: Assume a bijection $F : R^n \to R^n$ is given, and a class of geometric objects $\mathcal{C}$. (An element in $\mathcal{C}$ is a subset of R^n, say circles, triangles, etc).

Assume the class satisfies that every three non-collinear points are included in some element of the class, and no element of the class contains three collinear points in one of $n+1$ fixed directions.

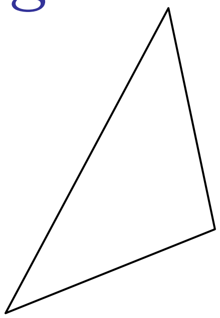
If F maps elements in $\mathcal{C}$ onto elements in $\mathcal{C}$, and the induced mapping on $\mathcal{C}$ is a bijection, then F is linear.

Corollary: No non-affine bijection exists which maps triangles "in directions" other than a pre-chosen n+1 onto such triangles, bijectively.

Conjecture [Gardner]: Same for ALL triangles.
Proof [Li and Wang 09]

Easy: circles to circles, ellipses, surfaces of strictly convex bodies, etc. **Gardner-Mauldin**: Circles into boundaries of strictly convex bodies; triangles into circles.

Note: triangle means the <u>boundary</u> and not the solid



In the next lecture: replace "order" with preservation of "product" or "sum"

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THANK YOU FOR YOUR ATTENTION