The Self-Avoiding Walk

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Abstract

This lecture will give an overview of some of what is known about the self-avoiding walk, including some old and some more recent results, using methods that touch on combinatorics, probability, and statistical mechanics.

Papers and preprints at http://www.math.ubc.ca/∼slade.

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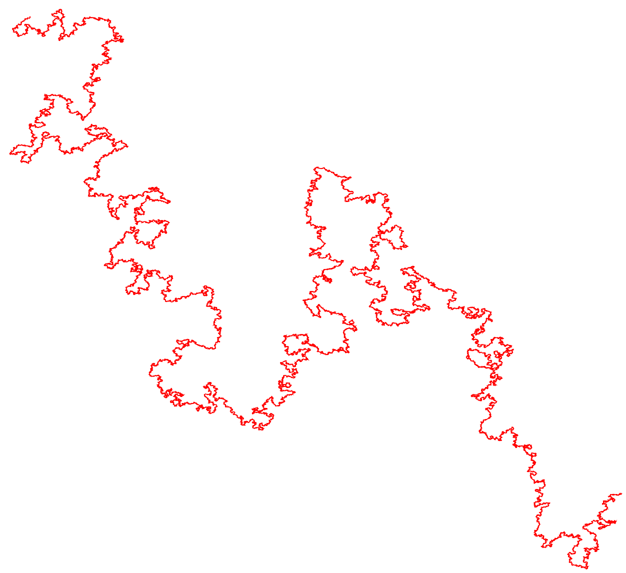
Self-avoiding walk

Let $\mathcal{S}_n(x)$ be the set of $\omega : \{0, 1, \ldots, n\} \to \mathbb{Z}^d$ with:
$\omega(0) = 0$, $\omega(n) = x$, $|\omega(i+1) - \omega(i)| = 1$, and $\omega(i) \neq \omega(j)$ for all $i \neq j$.
Let $\mathcal{S}_n = \cup_{x \in \mathbb{Z}^d} \mathcal{S}_n(x)$.

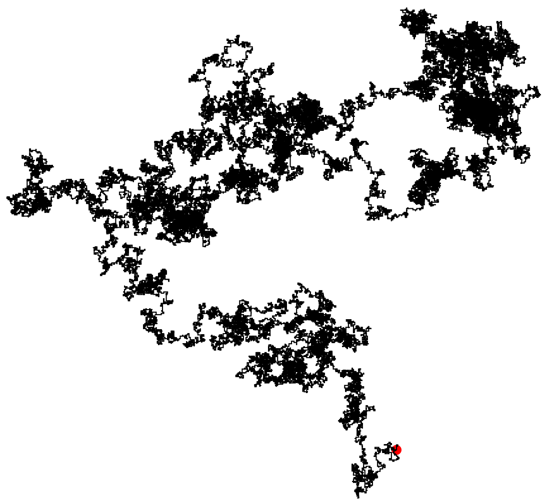
Let $c_n(x) = |\mathcal{S}_n(x)|$. Let $c_n = \sum_x c_n(x) = |\mathcal{S}_n|$.
Declare all walks in $\mathcal{S}_n$ to be equally likely: each has probability c_n^{-1}.



A random SAW on $\mathbb{Z}^2$ with 10^6 steps. (T. Kennedy)

What it is not

It is not like simple random walk:



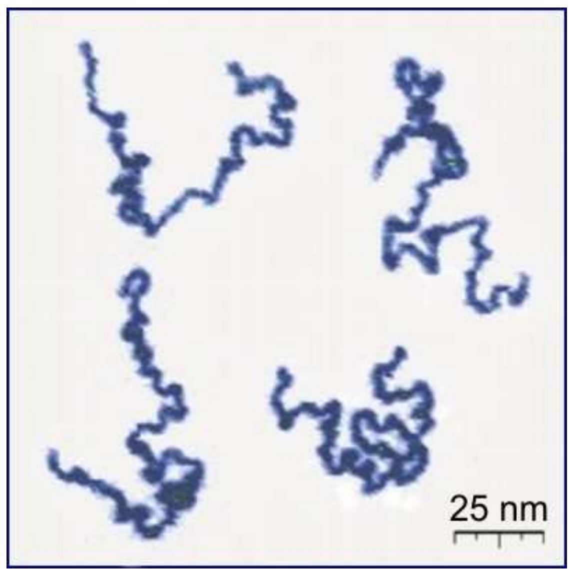
It is not the so-called "true" or "myopic" self-avoiding walk, i.e., the stochastic process which at each step looks at its neighbours and chooses uniformly from those visited least often in the past.

It is not Markovian.

It is not a stochastic process: the uniform measures on $\mathcal{S}_n$ do not form a consistent family.

Motivations

- interesting problem in enumerative combinatorics
- challenging problem in probability
- connection with critical phenomena in equilibrium statistical mechanics: $N \to 0$ limit of the N-vector model (de Gennes 1972)
- standard model of linear polymer chains — universality, excluded volume (Flory 1949):



Appearance of real linear polymer chains as recorded using an atomic force microscope on surface under liquid medium. Chain contour length is 204 nm; thickness is 0.4 nm. (Roiter and Minko, 2005).

Basic questions

Connective constant $\mu = \lim_{n\to\infty} c_n^{1/n}$ exists because $c_{n+m} \le c_n c_m$.
Easy: $d \le \mu \le 2d - 1$.

Determine behaviour of:

- c_n = number of n-step SAWs,
- $\mathbb{E}_n|\omega(n)|^2 = \frac{1}{c_n}\sum_{\omega\in\mathcal{S}_n} |\omega(n)|^2$ = mean-square displacement,
- two-point function $G_z(x) = \sum_{n=0}^{\infty} c_n(x) z^n$ (radius of convergence is $z_c = \mu^{-1}$),
- the scaling limit, i.e., find ν and X such that $n^{-\nu}\omega(\lfloor nt \rfloor) \Rightarrow X(t)$.

Predicted asymptotic behaviour:

$$c_n \sim A\mu^n n^{\gamma-1}, \quad \mathbb{E}_n|\omega(n)|^2 \sim Dn^{2\nu}, \quad G_{z_c}(x) \sim c|x|^{-(d-2+\eta)},$$

with universal critical exponents γ, ν, η obeying $\gamma = (2-\eta)\nu$

(and $(\log n)^{1/4}$ corrections to c_n and $\mathbb{E}_n|\omega(n)|^2$ for $d = 4$).

Dimension $d = 1$

Problem is trivial: $c_n = 2$ and $\mathbb{E}_n|\omega(n)|^2 = n^2$.

However, the 1-dimensional problem is interesting for *weakly* self-avoiding walk. Let $g > 0$. For an n-step *simple random walk* $\omega = (0, \omega(1), \ldots, \omega(n))$, let

$$Q_n(\omega) \propto \exp\left[-g \sum_{i,j=0, i\neq j}^{n} \delta_{\omega(i),\omega(j)}\right].$$

Theorem (Greven, den Hollander 1993; König 1996). For every $g \in (0, \infty)$ there exists $\theta(g) \in (0, 1)$ and $\sigma(g) \in (0, \infty)$ such that, under Q_n, as $n \to \infty$,

$$|\omega(n)| \sim \theta n + \sigma\sqrt{n}Z,$$

where Z is a standard normal random variable.

Note ballistic behaviour for all $g > 0$: weakly self-avoiding walk is in the universality class of strictly self-avoiding walk; $\nu = 1$ (in contrast to $\nu = \frac{1}{2}$ for $g = 0$).
The proof is based on large deviation methods.
The natural conjecture that $g \mapsto \theta(g)$ is (strictly) increasing remains unproved.

Dimension $d = 2$

It was predicted by Nienhuis (1982) that $\gamma = \frac{43}{32}$, $\nu = \frac{3}{4}$, and $\mu_{\mathrm{Hex}} = \sqrt{2+\sqrt{2}}$.

Theorem (Duminil-Copin, Smirnov 2010). $\mu_{\mathrm{Hex}} = \sqrt{2+\sqrt{2}}$.

The exponent values have been verified by extensive Monte Carlo experiments, and by exact enumeration plus series analysis.
For the latter, determine c_n exactly for $n = 1, 2, \ldots, N$ and analyse the sequence to determine μ, A, γ.
For $d = 2$ the "finite lattice method" is remarkable (Jensen 2004):
$c_{71} = 4\,190\,893\,020\,903\,935\,054\,619\,120\,005\,916 \approx 4.2 \times 10^{30}$.

Theorem (Lawler, Schramm, Werner 2004). *If* the scaling limit exists and has a certain conformal invariance property, *then* the scaling limit must be $\mathrm{SLE}_{8/3}$, $\ldots$ $\gamma = \frac{43}{32}$, $\nu = \frac{3}{4}$.

However, existence of the scaling limit remains an open problem.

Some sobering facts for $d = 2, 3, 4$:

Best bound: $\mu^n \le c_n \le \mu^n e^{C\sqrt{n}}$. Hammersley–Welsh ('62), Kesten ('64) $e^{Cn^{2/(d+2)}\log n}$

Not proved: $cn^{2/d} \le \mathbb{E}_n|\omega(n)|^2 \le Cn^{2-\epsilon}$. Madras proved $\mathbb{E}_n|\omega(n)|^2 \ge cn^{4/3d}$.

Dimension $d = 3$

For $d = 3$: no rigorous results for critical exponents.

Flory values (1949), nonrigorous: $\nu = \frac{3}{d+2}$ for $1 \le d \le 4$.
Correct for $d = 1, 2, 4$, close for $d = 3$.

Three methods to compute the exponents:

1. Field theory (physics: $N \to 0$ limit with expansion in $\epsilon = 4 - d$ at $\epsilon = 1$; Guida, Zinn-Justin (1998)).
2. Monte Carlo (walks of length 33,000,000 now being simulated, Clisby (2010))
3. Exact enumeration plus series analysis.
   Currently best method for enumeration in dimensions $d \ge 3$ is the lace expansion (Clisby, Liang, Slade, 2007). $c_{30} = 270\,569\,905\,525\,454\,674\,614$.

Estimates for critical parameters for $d = 3$ via enumeration up to $n = 30$ steps:
$c_n \sim A\mu^n n^{\gamma-1}$, $\quad \mathbb{E}_n[|\omega(n)|^2] \sim D n^{2\nu}$.

$\mu = 4.684043(12)$, $\gamma = 1.1568(8)$, $\nu = 0.5876(5)$,

$A = 1.216(5)$, $D = 1.220(12)$.

Monte Carlo and field theory yield similar estimates, e.g., Monte Carlo: $\nu = 0.587597(7)$.

Dimensions $d \geq 5$

Theorem (Hara, Slade 1992). For $d \geq 5$, $\gamma = 1$ and $\nu = \frac{1}{2}$, in the sense that

$$c_n = A\mu^n[1 + O(n^{-\epsilon})],$$
$$\mathbb{E}|\omega(n)|^2 = Dn[1 + O(n^{-\epsilon})],$$

and, moreover,

$$\frac{\omega(\lfloor nt \rfloor)}{\sqrt{Dn}} \Rightarrow B_t.$$

Hara (2008). Also $\eta = 0$:

$$G_{z_c}(x) = c|x|^{-(d-2)}[1 + O(|x|^{-\epsilon})].$$

Upper critical dimension is $4 = 2 + 2$: Ranges of two independent BMs do not intersect each other iff $d \geq 4$.

Proof: Lace expansion (Brydges, Spencer 1985; different formulation with "laces").

Method has been highly developed and extended to several other models:
Percolation ($d > 6$), oriented percolation ($d + 1 > 4 + 1$), contact process ($d > 4$), lattice trees and lattice animals ($d > 8$), Ising model ($d > 4$); also to long-range models in lower dimensions.

The lace expansion

Identifies a function $\pi_m(x)$ such that for $n \geq 1$,

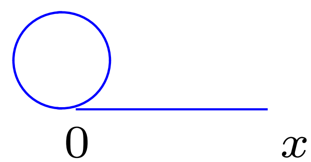
$$c_n(x) = \sum_{y \in \mathbb{Z}^d} c_1(y) c_{n-1}(x-y) + \sum_{m=2}^{n} \sum_{y \in \mathbb{Z}^d} \pi_m(y) c_{n-m}(x-y).$$

Start with

$$c_n(x) = \sum_{y} c_1(y) c_{n-1}(x-y) - R_n^{(1)}(x)$$

where

$$R_n^{(1)}(x) =$$

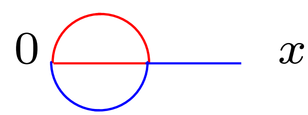


Inclusion-exclusion again:

$$R_n^{(1)}(x) = \sum_{m=2}^{n} u_m c_{n-m}(x) - R_n^{(2)}(x)$$

where

$$R_n^{(2)}(x) =$$

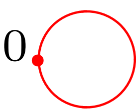
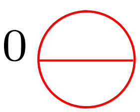
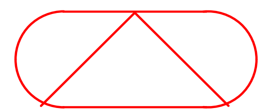


The lace expansion continued

Repetition leads to

$$c_n(x) = \sum_y c_1(y)c_{n-1}(x-y) + \sum_{m=2}^{n} \sum_y \pi_m(y)c_{n-m}(x-y)$$

with

$$\pi_m(y) = -\delta_{0,y}$$ 0  $+$ 0  y $-$ 

$+\cdots$

Then

$$G_z(x) = \sum_{n=0}^{\infty} c_n(x)z^n$$

$$= \delta_{0,x} + z\sum_y c_1(y)G_z(x-y) + \sum_y \Pi_z(y)G_z(x-y),$$

where

$$\Pi_z(y) = \sum_{n=2}^{\infty} \pi_n(y)z^n.$$

Idea of proof of Theorem

Fourier transformation ($\hat{f}(k) = \sum_x f(x) e^{ik\cdot x}$) gives

$$\hat{G}_z(k) = \frac{1}{1 - z\hat{c}_1(k) - \hat{\Pi}_z(k)} \equiv \frac{1}{\hat{F}_z(k)}.$$

Use: $\hat{G}_z(0) = \sum_n c_n z^n$ has radius of convergence $z_c = \mu^{-1}$ and $\hat{F}_{z_c}(0) = 0$.
Taylor expansion:

$$\hat{F}_z(k) = \hat{F}_z(k) - \hat{F}_{z_c}(0) \approx a\frac{|k|^2}{2d} + b\Big(1 - \frac{z}{z_c}\Big)$$

with $a = \nabla_k^2 \hat{F}_{z_c}(0)$ and $b = -z_c \partial_z \hat{F}_{z_c}(0)$, assuming a, b finite. Then

$$\hat{G}_z(k) \approx \frac{1}{a\frac{|k|^2}{2d} + b(1 - \frac{z}{z_c})}, \quad \text{for } k \approx 0,\ z \approx z_c,$$

which is essentially the corresponding generating function for simple random walk.

Idea of proof continued

For this to work, need $z_c \partial_z \hat{\Pi}_{z_c}(k)$ finite. Leading term is

$$\sum_n u_n n z_c^n \leq \sum_x G_{z_c}(x)^2 = \int_{[-\pi,\pi]^d} \hat{G}_{z_c}(k)^2 \frac{d^d k}{(2\pi)^d}.$$

Reason this might work: insert SRW behaviour on RHS:

$$\int_{[-\pi,\pi]^d} \hat{G}_{z_c}(k)^2 \frac{d^d k}{(2\pi)^d} \approx \int_{[-\pi,\pi]^d} \frac{1}{|k|^4} \frac{d^d k}{(2\pi)^d} < \infty \quad \text{iff } d > 4.$$

Proof finds a way to exploit this.

Proof is computer-assisted to handle all $d \geq 5$; easier for $d \geq d_0$ for some larger d_0.

Dimension $d = 4$

A prediction going back to Brézin, Le Guillou, Zinn-Justin (1973) is that

$$c_n \sim A\mu^n (\log n)^{1/4}, \quad \mathbb{E}_n|\omega(n)|^2 \sim Dn(\log n)^{1/4}, \quad G_{z_c}(x) \sim c|x|^{-2},$$

and that the scaling limit is again Brownian motion (scale by $(Dn)^{-1/2}(\log n)^{-1/8}$).

Theorem (Brydges, Slade 2010). For the continuous-time weakly self-avoiding walk on $\mathbb{Z}^4$, there exists $g_0 > 0$ such that for $0 < g \leq g_0$,

$$G_{z_c}(x) \sim c|x|^{-2}.$$

Proof uses a rigorous renormalisation group method.

Related results, including log corrections, on a 4-dimensional hierarchical lattice, for:
continuous-time weakly SAW (Brydges, Evans, Imbrie 1992; Brydges, Imbrie 2003),
discrete-time weakly SAW (Hara, Ohno 2010).
The hierarchical lattice is a replacement of $\mathbb{Z}^4$ by a recursive structure which is well-suited to the renormalisation group, and which has a long tradition of use for development of renormalisation group methodology.

Continuous-time weakly SAW on $\mathbb{Z}^4$

Let $g \in (0, \infty)$. Let E_0 denote expectation for continuous-time nearest-neighbour simple random walk X (with $\mathrm{Exp}(1)$ holding times), started at 0. Let

$$L_{z,T} = \int_0^T \delta_{z,X(s)} ds, \qquad I(0,T) = \sum_{z \in \mathbb{Z}^4} L_{z,T}^2.$$

Then

$$I(0,T) = \int_0^T \int_0^T \delta_{X(s),X(t)} ds\, dt.$$

On $\mathbb{Z}^4$, or on a discrete torus $\Lambda = \mathbb{Z}^4 / R\mathbb{Z}^4$, let

$$G_\nu(x) = \int_0^\infty E_0 \left(e^{-gI(0,T)} \; \delta_{X(T),x} \right) e^{-\nu T} dT.$$

(We have switched from z to ν via $z = e^{-\nu}$.)

An adaptation of standard arguments (Simon–Lieb inequality) gives

$$G_{\nu_c}(x) = \lim_{\nu \downarrow \nu_c} \lim_{\Lambda \uparrow \mathbb{Z}^4} G_{\Lambda,\nu}(x).$$

So it suffices to consider $\nu > \nu_c$ and finite volume, with sufficient uniformity.

Functional integral representation

Let $\Lambda = \mathbb{Z}^4/R\mathbb{Z}^4$. Given $\varphi : \Lambda \to \mathbb{C}$, let

$$\psi_x = \frac{1}{\sqrt{2\pi i}} d\varphi_x, \quad \bar{\psi}_x = \frac{1}{\sqrt{2\pi i}} d\bar{\varphi}_x,$$

$$\tau_x = \varphi_x \bar{\varphi}_x + \psi_x \wedge \bar{\psi}_x,$$

where $\wedge$ is the wedge product.

For Δ the discrete Laplacian on Λ, defined by $\Delta\varphi_x = \sum_{y:|y-x|=1}(\varphi_y - \varphi_x)$, let

$$\tau_{\Delta,x} = \frac{1}{2}\Big(\varphi_x(-\Delta\bar{\varphi})_x + (-\Delta\varphi)_x\bar{\varphi}_x + \psi_x \wedge (-\Delta\bar{\psi})_x + (-\Delta\psi)_x \wedge \bar{\psi}_x\Big).$$

Theorem.

$$G_{\Lambda,\nu}(x) = \int_{\mathbb{C}^\Lambda} e^{-\sum_{u\in\Lambda}(\tau_{\Delta,u} + g\tau_u^2 + \nu\tau_u)} \bar{\varphi}_0 \varphi_x.$$

(Parisi, Sourlas 1980; McKane 1980; Luttinger 1983; Le Jan 1987; Brydges, Evans, Imbrie 1992; for survey and extensions: Brydges, Imbrie, Slade 2009).

Meaning of the integral

Meaning of an integral such as

$$\int_{\mathbb{C}^\Lambda} e^{-\sum_{u\in\Lambda}(\tau_{\Delta,u}+g\tau_u^2+\nu\tau_u)}\bar{\varphi}_0\varphi_x$$

is as follows:

- expand entire integrand in power series about degree-zero part (*finite* sum), e.g.,

$$e^{-\tau_x} = e^{-\varphi_x\bar{\varphi}_x-\psi_x\bar{\psi}_x} = e^{-\varphi_x\bar{\varphi}_x}\left(1-\psi_x\bar{\psi}_x\right) = e^{-\varphi_x\bar{\varphi}_x}\left(1-\frac{1}{2\pi i}d\varphi_x d\bar{\varphi}_x\right),$$

- keep only terms with one factor $d\varphi_x$ and one $d\bar{\varphi}_x$ for each $x \in \Lambda$,
- write $\varphi_x = u_x + iv_x$, $\bar{\varphi}_x = u_x - iv_x$, $d\varphi_x = du_x + idv_x$, $d\bar{\varphi}_x = du_x - idv_x$,
- use anti-commutativity of $\wedge$ to rearrange the differentials to $\prod_{x\in\Lambda} du_x dv_x$,
- and finally perform Lebesgue integral over $\mathbb{R}^{2|\Lambda|}$.

Now we study the integral and forget about the walks.

External field and change of variables

Introducing an *external field* $\sigma \in \mathbb{C}$, we obtain

$$G_{\Lambda,\nu}(x) = \int_{\mathbb{C}^\Lambda} e^{-\sum_{u\in\Lambda}(\tau_{\Delta,u}+g\tau_u^2+\nu\tau_u)}\bar{\varphi}_0\varphi_x$$
$$= \frac{\partial}{\partial\sigma}\frac{\partial}{\partial\bar{\sigma}}\Big|_0 \int_{\mathbb{C}^\Lambda} e^{-\sum_{u\in\Lambda}(\tau_{\Delta,u}+g\tau_u^2+\nu\tau_u)-\sigma\bar{\varphi}_0-\bar{\sigma}\varphi_x}.$$

The change of variables $\varphi_x \mapsto \sqrt{1+z_0}\varphi_x$ (with $z_0 > -1$) gives

$$G_{\Lambda,\nu}(x) = (1+z_0)\,\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\bar{\sigma}}\Big|_0 \int e^{-S(\Lambda)-V_0(\Lambda)},\quad \text{where}$$

$$S(\Lambda) = \sum_{u\in\Lambda}\left(\tau_{\Delta,u}+m^2\tau_u\right),\quad V_0(\Lambda) = \sum_{u\in\Lambda}\left(g_0\tau_u^2+\nu_0\tau_u+z_0\tau_{\Delta,u}\right)+\sigma\bar{\varphi}_0+\bar{\sigma}\varphi_x,$$

$$g_0 = (1+z_0)^2 g,\quad \nu_0 = (1+z_0)\nu_c,\quad m^2 = (1+z_0)(\nu-\nu_c).$$

Want to show that first part of V_0 is a small perturbation and use

$$\lim_{m^2\downarrow 0}\lim_{\Lambda\uparrow\mathbb{Z}^4}\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\bar{\sigma}}\Big|_0 \int e^{-S(\Lambda)-\sigma\bar{\varphi}_0-\bar{\sigma}\varphi_x} = (-\Delta_{\mathbb{Z}^4})^{-1}(0,x) \sim \text{const}|x|^{-2}.$$

Gaussian "expectation"

For a positive definite $\Lambda \times \Lambda$ matrix C, and $A = C^{-1}$, let

$$S_A(\Lambda) = \sum_{x,y\in\Lambda} \Big(\varphi_x A_{xy} \bar{\varphi}_x + \psi_x A_{xy} \bar{\psi}_y \Big)$$

and, for a form F,

$$\mathbb{E}_C F = \int_{\mathbb{C}^\Lambda} e^{-S_A(\Lambda)} F.$$

With $C = (-\Delta + m^2)^{-1}$, our goal is to compute

$$\lim_{m^2\downarrow 0} \lim_{\Lambda\uparrow\mathbb{Z}^4} G_{\Lambda,\nu}(x,y) = \lim_{m^2\downarrow 0} \lim_{\Lambda\uparrow\mathbb{Z}^4} (1+z_0) \frac{\partial}{\partial\sigma}\frac{\partial}{\partial\bar{\sigma}}\Big|_0 \mathbb{E}_C e^{-V_0(\Lambda)}.$$

These integrals have much in common with standard Gaussian integrals. However, this is not ordinary probability theory and and in general $\mathbb{E}_C$ will be a Grassmann integral that take values in a space of differential forms.

Finite-range decomposition of covariance

Theorem (Brydges, Guadagni, Mitter 2004). Fix a large L and suppose $|\Lambda| = L^{Nd}$. Let $C = (m^2 - \Delta)^{-1}$. It is possible to write:

$$C = \sum_{j=1}^{N} C_j$$

with

$$C_j(x,y) = 0 \quad \text{if} \quad |x-y| \geq L^j$$

and, with $[\phi] = \frac{1}{2}(d-2)$ (this is $[\phi] = 1$ for $d = 4$),

$$|C_{j+1}(x,x)| \leq \text{const} L^{-2[\phi]j}, \qquad \left|\nabla_x^2 C_{j+1}(x,x)\right| \leq \text{const} L^{-2[\phi]j-2j}.$$

The covariance decomposition induces a field decomposition (write $\phi = (\varphi, \bar{\varphi})$, $d\phi = (d\varphi, d\bar{\varphi})$) and allows the expectation to be done iteratively:

$$\phi = \sum_{j=1}^{N} \xi_j, \qquad d\phi = \sum_{j=1}^{N} d\xi_j, \qquad \mathbb{E}_C = \mathbb{E}_{C_N} \circ \cdots \circ \mathbb{E}_{C_2} \circ \mathbb{E}_{C_1}.$$

The RG map

Write $\phi_j = \sum_{i=j+1}^{N} \xi_i$, with $\phi_0 = \phi$, $\phi_N = 0$. Then $\phi_j = \phi_{j+1} + \xi_{j+1}$. Let

$$Z_0 = Z_0(\phi, d\phi) = e^{-V_0(\Lambda)},$$

and

$$Z_j(\phi_j, d\phi_j) = \mathbb{E}_{C_j} \cdots \mathbb{E}_{C_1} Z_0.$$

Our goal is to compute

$$Z_N = \mathbb{E}_C Z_0 = \mathbb{E}_C e^{-V_0(\Lambda)}$$

and we are led to study the RG map:

$$Z_{j+1} = \mathbb{E}_{C_{j+1}} Z_j.$$

Role of $d = 4$: Typically $|\xi_{j+1,x}| \approx L^{-j[\phi]}$ and is approximately constant on scale L^j, so

$$\sum_{|x| \leq L^j} |\xi_{j+1,x}|^4 \approx L^{dj-4[\phi]j} = L^{(4-d)j}$$

is *irrelevant* if $d > 4$, *marginal* if $d = 4$, *relevant* if $d < 4$.

The $I \circ K$ representation

Let $\mathcal{B}_j$ represent the blocks in a paving of Λ by blocks of side L^j, and let $\mathcal{P}_j$ denote the set of finite unions of such blocks. Given even forms F, G defined on $\mathcal{P}_j$, let

$$(F \circ G)(\Lambda) = \sum_{X \in \mathcal{P}_j} F(X) G(\Lambda \setminus X).$$

This defines an associative and commutative product. For $X \in \mathcal{P}_0$, let

$$I_0(X) = e^{-V_0(X)}, \qquad K_0(X) = \delta_{X,\varnothing}.$$

Then

$$Z_0 = I_0(\Lambda) = (I_0 \circ K_0)(\Lambda).$$

Main part of proof: There exists an inductive parametrisation

$$Z_j = (I_j \circ K_j)(\Lambda), \qquad Z_{j+1} = \mathbb{E}_{C_{j+1}} Z_j = (I_{j+1} \circ K_{j+1})(\Lambda),$$

with I_j and I_{j+1} parametrised by V_j and V_{j+1} respectively where

$$V_{j,x} = g_j \tau_x^2 + \nu_j \tau_x + z_j \tau_{\Delta,x},$$

and with K_j third-order in g_j and *irrelevant*.

Flow of coupling constants

This creates a dynamical system in a suitable Banach space of forms, the *flow of the coupling constants* $(g_j, \nu_j, z_j, K_j) \mapsto (g_{j+1}, \nu_{j+1}, z_{j+1}, K_{j+1})$:

$$\begin{aligned} g_{j+1} &= g_j - c g_j^2 + r_{g,j} \\ \nu_{j+1} &= \nu_j + 2 g_j C_{j+1}(0,0) + r_{\mu,j} \\ z_{j+1} &= z_j + r_{z,j} \\ K_{j+1} &= r_{K,j}. \end{aligned}$$

Additional equations track the evolution of coupling constants for terms $\sigma\bar{\varphi}_0 + \bar{\sigma}\varphi_x$ and $\sigma\bar{\sigma}$ that must be added into V_j.

With suitably chosen initial conditions z_0, ν_0, the flow is to $(0,0,0,0)$ and the two-point function behaves like $(-\Delta)^{-1}(0,x) \sim \text{const}|x|^{-2}$.

The choice of ν_0 ensures that we are at the critical point. The choice of z_0 affects the constant in the asymptotic formula const$|x|^{-2}$ for the critical two-point function.

Conclusions

- $d = 1$: ballistic behaviour ($\nu = 1$) obvious for nearest-neighbour strictly self-avoiding walk, interesting for weakly self-avoiding walk.

- $d = 2$: *if* the scaling limit can be proved to exist and to be conformally invariant, *then* the scaling limit is $\mathrm{SLE}_{8/3}$, . . . $\gamma = \frac{43}{32}$, $\nu = \frac{3}{4}$.

- $d = 3$: no rigorous results, numerically $\gamma \approx 1.157$, $\nu \approx 0.5875$. No idea how to describe the scaling limit.

- $d = 4$: renormalisation group methods give $\eta = 0$ for weakly SAW and possibly more; on a *hierarchical* lattice it is proved that $\gamma = 1$ and $\nu = \frac{1}{2}$ with log corrections.

- $d \geq 5$: problem is solved using the lace expansion: $\gamma = 1$, $\nu = \frac{1}{2}$ and scaling limit is Brownian motion.