Stability and instability for near-linear Hamiltonian systems

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$$\begin{cases} \dot\theta = \partial_I H(\theta, I), \\ \dot I = -\partial_\theta H(\theta, I), \end{cases} \qquad \begin{cases} H(\theta, I) = h(I) + f(\theta, I), \\ |f| \le \varepsilon \ll 1. \end{cases} \qquad (*)$$

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(2) the tori $\mathbb{T}^n \times \{I_0\}$ are invariant and the dynamic is quasi-periodic, if $\omega_0 = \nabla h(I_0)$, the flow is $\Phi_t^h : (\theta_0, I_0) \longmapsto (\theta_0 + t\omega_0\, [\mathbb{Z}^n], I_0)$.

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For $\varepsilon > 0$ small, the system $H = h + f$ is near-integrable, what happens to (1) and (2) ?

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$$\lim_{\varepsilon \to 0} \left(\sup_{t \in \mathbb{R}} |I(t) - I_0| \right) = 0,$$

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$$|I(t) - I_0| \lesssim \delta, \quad |t| \leq \delta \varepsilon^{-1},$$

so taking $\delta = \varepsilon^c$, $c > 0$ arbitrarily small, *a priori* one has only

$$\lim_{\varepsilon \to 0} \left(\sup_{|t| < \varepsilon^{-1}} |I(t) - I_0| \right) = 0.$$

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Theorem
h is effectively stable $\Longleftrightarrow$ h is rationally steep.

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h is not rationally steep $\iff$ ω is resonant: there exists $k \in \mathbb{Z}^n \setminus \{0\}$ such that $k.\omega = 0$.
Then the system $H(\theta, I) = \omega.I + \varepsilon \cos(k.\theta)$ has solutions for which

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If ω is (γ, τ)-Diophantine, that is $|k.\omega| \geq \gamma |k|^{-\tau}$ for all $k \in \mathbb{Z}^n \setminus \{0\}$ and for some $\gamma > 0$ and $\tau \geq n - 1$, it has been proved

$$|I(t) - I_0| \lesssim (\gamma^{-1}\varepsilon)^{\frac{1}{1+\tau}}, \quad |t| \lesssim (\gamma\varepsilon^{-1})^{\frac{1}{1+\tau}} \exp\left((\gamma\varepsilon^{-1})^{\frac{1}{1+\tau}}\right).$$

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Let $|\,.\,|_{\mathbb{Z}} = d(., \mathbb{Z})$, and define $\Psi = \Psi_\omega$ by

$$\Psi(K) = \max\left\{ |k.\alpha|_{\mathbb{Z}}^{-1} \mid k \in \mathbb{Z}^{n-1},\, 0 < |k| \leq K \right\}, \quad K \in \mathbb{N}^*.$$

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For instance, if ω is (γ, τ)-Diophantine, then we can choose

$$\Psi(x) = \gamma^{-1} x^{\tau}, \quad \Lambda(x) = \gamma^{-1} x^{1+\tau}, \quad \Delta(x) = (\gamma x)^{\frac{1}{1+\tau}}.$$

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Theorem 1

For any sufficiently small perturbation f of size ε, all solutions $(\theta(t), I(t))$ of $H = h + f$ satisfy the estimates

$$|I(t) - I_0| \lesssim \delta, \quad |t| \lesssim \delta\varepsilon^{-1} \exp\left(\Delta(\varepsilon^{-1})\right).$$

for any $\left(\Delta\left(\varepsilon^{-1}\right)\right)^{-1} \lesssim \delta \lesssim 1$.

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Taking $\delta \simeq \varepsilon^c$ with $c > 0$ arbitrarily close to zero, we obtain

Corollary 1

If h is linear, h is effectively stable $\iff$ h is rationally steep.

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Theorem 2

There exists a sequence of perturbation $(f_j)_{j\in\mathbb{N}}$, $|f_j| \leq \varepsilon_j \longrightarrow 0$ *when* $j \longrightarrow +\infty$, *and solutions* $(\theta(t), I(t))$ *of* $H_j = h + f_j$ *for which*

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For some arbitrarily small perturbation and for some solutions of the perturbed system, Theorem 2 says that for $\delta > 0$,

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Therefore Theorem 1 and Theorem 2 cannot be improved in general.

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For $n \geq 2$, the proof of Theorem 2 is the same as for $n = 2$.

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The system can be written

$$H = h_v + f_v, \quad f_v = h - h_v + f, \quad |f_v| \lesssim \varepsilon.$$

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$$f_j^1(I) = h(I) - h_{v_j}(I), \quad |f_j^1| \lesssim \varepsilon_j.$$

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