Integrability by means of variational methods

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Tonelli Lagrangian

Let M be a smooth closed manifold, Throughout the talk, $M = \mathbb{T}^n$.

Let $L = L(x, \dot{x}) : T\mathbb{T}^n \to \mathbb{R}$ be a Tonelli's Lagrangian with respect to the Hamitonian with the following standard assumptions throughout the whole paper:

1. *Smoothness*: $L : T\mathbb{T}^n \to \mathbb{R}$ is of class at least C^2.
2. *Convexity*: The Hessian $\frac{\partial^2 L}{\partial \dot{x}^2}(x, \dot{x})$ is positively definite on each fibre $T_x\mathbb{T}^n$
3. *Superliearity*:

$$\lim_{|\dot{x}|\to\infty} \frac{L(x, \dot{x})}{|\dot{x}|} = \infty, \quad \text{uniformly on } x \in \mathbb{T}^n.$$

Minimal action

Let $\Phi_t : T\mathbb{T}^n \hookrightarrow$ be the Euler-Lagrange flow defined by $\Phi_t(x_0, v_0) = (x(t+t_0), \dot{x}(t+t_0) \text{ mod } \mathbb{Z})$, where $x : \mathbb{R} \to \mathbb{T}^n$ be the solution of the Euler-Lagrange equation with initial conditions $x(t_0) = x_0$ and $\dot{x}(t_0) = v_0$.

Let $\mathscr{M}(L)$ the set of Φ_t-invariant Borel probability measure on $T\mathbb{T}^n$. For every $\mu \in \mathscr{M}(L)$, we can define its *average minimal action*

$$A(\mu) = \int L \, d\mu.$$

The integral is defined since L is bounded below.
A Borel measure μ is said to be a minimal measure if

$$A(\mu) = \inf_{\mu \in \mathscr{M}(L)} \int L \, d\mu.$$

A minimal measure is E-L flow Φ_t-invariant.

β-function and α-function

If $A(\mu) < +\infty$, we may associate to μ its *rotation vector* $\rho(\mu) \in H_1(\mathbb{T}^n, \mathbb{R}) = \mathbb{R}^n$. The rotation vector $\rho(\mu)$ is uniquely characterized by

$$\langle c, \rho(\mu)\rangle = \int \eta_c \, d\mu, \quad \text{for all} \quad c \in H^1(\mathbb{T}^n, \mathbb{R})$$

where $[\eta_c] = c \in H^1(\mathbb{T}^n, \mathbb{R}) = \mathbb{R}^n$.

For every $h \in H_1(\mathbb{T}^n, \mathbb{R})$, we define Mather's β-function, $\beta : H_1(\mathbb{T}^n, \mathbb{R}) \to \mathbb{R}$, as

$$\beta(h) = \inf\{A(\mu) : \mu \in \mathscr{M}(L),\ \rho(\mu) = h\}.$$

$\beta(h)$ is a convex function on $H_1(\mathbb{T}^n, \mathbb{R})$ with superlinear growth.

We define Mather's α-function, $\alpha : H^1(\mathbb{T}^n, \mathbb{R}) \to \mathbb{R}$, the Fenchel's transformation of β-function, i. e.,

$$\alpha(c) = \max\{\langle c, h\rangle - \beta(h) : h \in H_1(\mathbb{T}^n, \mathbb{R})\}, \qquad c \in H^1(\mathbb{T}^n, \mathbb{R}).$$

More on α-function

From the basic facts in convex analysis, $\alpha(c)$ is also a convex function on $H^1(\mathbb{T}^n, \mathbb{R})$ with superlinear growth.

Some useful description of the α-function

- $$\alpha(c) = -\inf_{\mu} \int L - c \, d\mu,$$
- $$\alpha(c) = \inf_{u \in C^1(\mathbb{T}^n)} \max_{x \in \mathbb{T}^n} H(x, du(x) + c).$$

Actually, $\alpha(c)$ is the average of the Hamiltonian on the support of the c-minimal measures.

Mañé's critical potential and Peierls' barrier

For $t > 0$, $x, y \in \mathbb{T}^n$ and $c \in \mathbb{R}^n$, define

$$h_t^c(x,y) = \inf \int_0^t (L - c)(\xi(s), \dot{\xi}(s))\, ds,$$

where the infimum is taken over of the piecewise C^1 curve $\xi : [0,t] \to \mathbb{T}^n$ such that $\xi(0) = x$ and $\xi(t) = y$.

Define the Mañé's critical potential and Peierls' barrier respectively as

$$\phi_c(x,y) = \inf_{t>0} h_t^c(x,y) + \alpha(c)t,$$
$$h_c(x,y) = \liminf_{t\to\infty} h_t^c(x,y) + \alpha(c)t.$$

Weak KAM solution

Before going on, let us introduce the weak KAM solutions of the Hamilton-Jacobi equations

$$H(x, c + d_x u) = \alpha(c), \quad x \in \mathbb{T}^n,$$

Now we introduce the weak KAM solution from some type of a Generalized Maupertuis' principle.

For given $c \in \mathbb{R}^n$, define the projected Aubry set

$$\mathscr{A}_c = \{x \in \mathbb{T}^n | h_c(x, x) = 0\}.$$

It is well known that for any c, $\mathscr{A}_c$ is nonempty. For any $y \in \mathscr{A}_c$, $\phi_c(x, y) = h_c(x, y)$, this let us define the *weak KAM solution* of the H-J equation as $\phi_c(x, y)$ for any y.

Jacobi-Finsler metric

Let us denote by $\bar{L} = L - c$ for given $c \in \mathbb{R}^n$, and $\bar{H}(x,p) = H(x, p+c)$ its dual.
For any fixed $x \in \mathbb{T}^n$ and $c \in \mathbb{R}^n$, Here is some notations.

- $$\bar{Z}_c(x) = \{p \in \mathbb{R}^n : \bar{H}(x,p) \leqslant \alpha(c)\}, \quad c \in \mathbb{R}^n,$$
- $$\delta_c(x,v) = \sigma_{\bar{Z}_c(x)}(v), \quad x \in \mathbb{T}^n,\ v \in \mathbb{R}^n,$$
- $$S_c(x,y) = \inf \int_0^1 \delta_c(\xi(t), \dot{\xi}(t))\, dt, \quad x, y \in \mathbb{T}^n$$

  where the infimum is taken over of the piecewise C^1 curve $\xi : [0,1] \to \mathbb{T}^n$ such that $\xi(0) = x$ and $\xi(1) = y$.

Theorem

$\phi_c(x,y) = S_c(x,y)$.

A Generalized Maupertuis' principle

The following formulae from convex analysis is useful: for any coercive convex function f on $\mathbb{R}^n$, $C = \{p \in \mathbb{R}^n : f(p) \leqslant a\}$ for $a \geqslant \min_{p\in\mathbb{R}^n} f(p)$,

$$\sigma_C(v) = \inf_{t>0}(tf^*(v/t) + at),$$

where f^* is the Legendre-Fenchel dual of f.

It is not hard to prove the equality $\phi_c(x,y) = S_c(x,y)$ by the formulae above.

The quantity $\delta_c(x,v)$ is just the Jacobi-Finsler metric for the generalized Maupertuis' principle with the restriction of the energy $\alpha(c)$. If the kinetic energy function is of the form of Riemannian metric $g_x(v,v) = \langle v,v\rangle_x$, $\delta_c(x,v)$ is the usual Jacobi metric $\sqrt{E - U(x)}$, where E is the energy not less than $\min_c \alpha(c)$.

An application

For one-dimensional Tonelli Lagrangian $L = \ell(v) - U(x)$, $x \in \mathbb{T}^1$ and $v \in T_x\mathbb{T}^1$, and $H = h(p) + U(x)$ the Hamiltonian.

Without loss of generality, we assume that $\min_p h(p) = h(p_0) = 0$ and such a minimizer is unique. When $p \geqslant p_0$ or $p \leqslant p_0$, h is a strictly monotone function, define h_+^{-1} and h_-^{-1} the inverse of h on the intervals $[p_0, +\infty)$ and $(-\infty, p_0]$ respectively. in this case,

$$\bar{Z}_c(x) = \{p \in \mathbb{R} : h_-^{-1}(-U(x)) - c \leqslant p \leqslant h_+^{-1}(-U(x)) - c\}$$

and

$$\begin{aligned} \delta_c(x, v) &= \sigma_{\bar{Z}_c(x)}(v) = \max\{pv : p \in \bar{Z}_c(x)\} \\ &= \begin{cases} (h_-^{-1}(-U(x)) - c)v, & v \leqslant 0; \\ (h_+^{-1}(-U(x)) - c)v, & v \geqslant 0. \end{cases} \end{aligned}$$

The explicit representation of the one-dimensional H-J equation is deduced from the observation that $u(x) = S_c(x, x_0)$, where $x_0 \in \mathscr{A}_c$.

Choose $\bar{x} \in [x_0, x_0 + 1]$ such that

$$\int_{x_0}^{\bar{x}} h_+^{-1}(-U(x)) - c\, dx = \int_{\bar{x}}^{1+x_0} -h_-^{-1}(-U(x)) + c\, dx.$$

Then the weak KAM solution can deduced directly by the Generalized Maupertuis' principle above.

$$u(x) = \begin{cases} \int_{x_0}^{x} h_+^{-1}(-U(s)) - c\, ds, & x_0 \leqslant x \leqslant \bar{x}; \\ \int_{x}^{1+x_0} -h_-^{-1}(-U(s)) + c\, ds, & \bar{x} \leqslant x \leqslant 1 + x_0. \end{cases}$$

Note that this implies that the α-function has a flat part on the closed interval $[\int h_-^{-1}(-U(x))\, dx, \int h_+^{-1}(-U(x))\, dx]$.

Problem

Now we turn to the problem of the relations between the regularity property of the α-function and the integrability of the systems.

This problem appeared firstly in [Burago, D., Ivanov, S., Kleiner, B.: On the structure of the stable norm of periodic metrics. *Math. Res. Lett.*, **4**, 791–808 (1997)] in the context of roundness of stable norm in geodesic flows.

The similar problem introduced by P. Bernard can also be found in `http://www.aimath.org/WWN/dynpde/articles/html/20a/`.

Problem: How does the variational structure of the system determine the integrability of the system?

For the case of twist maps or geodesic flows on 2-torus, the answer is affirmative. [Bangert and Mather]

Suppose $L = \ell(p) - U(x)$, here ℓ is strictly convex, $U(x) \leqslant \max_{x\in\mathbb{T}^n} U(x) = 0$.

Lemma

Let $L_{U,\ell}(x,\dot{x}) = \ell(\dot{x}) - U(x)$ be the mechanical Tonelli Lagrangian. Suppose $U(x) \leqslant \tilde{U}(x)$ for any $x \in \mathbb{T}^n$ and $\ell(\dot{x}) \geqslant \tilde{\ell}(\dot{x})$ for any $\dot{x} \in \mathbb{R}^n$, then the relation between the α-function of systems $L_{U,\ell}$ and $L_{\tilde{U},\tilde{\ell}}$ satisfies $\alpha_{U,\ell} \leqslant \alpha_{\tilde{U},\tilde{\ell}}$.

Theorem

Let $L(x,v) = \ell(v) - U(x)$ be a mechanical system, $E = \{x \in \mathbb{T}^n : U(x) = \max_{x\in\mathbb{T}^n} U(x) = 0\}$. If the natural lift of the set E to the universal covering space $\mathbb{R}^n$ of $\mathbb{T}^n$ does not contain n straight lines in linear independent rational directions, then the α-function has a flat part near c_0, $\min_c \alpha(c) = \alpha(c_0)$.

Key idea:

- If $\alpha(c) \leqslant \hat{\alpha}(c)$, $\alpha(c_0) = \hat{\alpha}(c_0)$, then if $\hat{\alpha}(c)$ has a flat part near c_0, then $\alpha(c)$ does too.
- one dimensional systems $L^i = \ell^i - U_i$, $i = 1, \dots, n$ can be chosen such that $L \geqslant \hat{L} = \sum L_i$, and $\alpha(c) \leqslant \hat{\alpha}(c)$.
- $\hat{\alpha}$, the α-function of $\hat{L}$, has (at least one-dimensional) flat part near c_0.

In particular, For Morse functions U, the α-function have flat part.

Theorem

Let L be the mechanical Lagrangian, suppose that $\tilde{S}$ contains a straight line given by $h'(\xi(r)) + x_1$, $x_1 \in \mathbb{R}^n$ in the rational direction, where $\xi(r)$ is the lift of a smooth curve on $\mathbb{T}^n$, then $\alpha(\xi(r)) = h(\xi(r))$, $r \in \mathbb{R}$.

The strict convex assumption on h implies that $h' : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism. Here we ignore the emphasis on the fact that h' is actually a map from a vector space to its dual. Thus if the straight line in the covering space is $\{r\mathbf{n} + x_1 : r \in \mathbb{R}\}$ for some $\mathbf{n} \in \mathbb{Z}^n$, then $\xi(r) = (h')^{-1}(r\mathbf{n})$.

Key idea:

- for any $c \in \mathbb{R}^n$,

$$\begin{aligned} -\alpha(c) &= \inf \int \ell(\dot{x}) - \langle c, \dot{x} \rangle - U(x)\ d\mu \\ &\geqslant \inf \int -h(c)\ d\mu = -h(c) \end{aligned}$$

  by Young's inequality and $U \leqslant 0$. So $\alpha(c) \leqslant h(c)$.
- $\alpha(\xi(r)) \geqslant h(\xi(r))$ since $x(t) = rt\mathbf{n} + x_1$ and $\dot{x}(t) \equiv h'(\xi(r))$, and use Young-Fenchel inequality.

An alternative approach

Now let $L(x,v) = \frac{1}{2}|v|^2 - U(x)$ with the potential $U(x) \leqslant 0$ and $\max_{x\in\mathbb{T}^n} U(x) = 0$, we try to give an alternative approach of the problem. Here we use some idea from Novikov's work on multiple integral (topology of closed 1-forms there) of Hamiltonian systems.

Now let us recall some basic facts on the construction of the closed 1-form on a closed smooth manifold M.

Let $f : M \to \mathbb{T}^1$ be a smooth map, the circle $\mathbb{T}^1$ is equipped with canonical angular form $d\theta$, $d\theta$ is a closed 1-form, which cannot be represented as a differential of a smooth function on $\mathbb{T}^1$. The pullback $f^*(d\theta)$ is a closed 1-form on M.

Theorem (see e.g book of Farber)

A closed 1-form ω on M can be represented by this form if and only if the de Rham cohomology class $[\omega] \in H^1(M,\mathbb{Z})$.

Let $\kappa(x) = \sqrt{2(-U(x))}$, We want to find a C^1 vector field $X(x)$ as a function $X : \mathbb{T}^n \to \mathbb{R}^n$, such that

$$|X(x)| = \kappa(x) \tag{1}$$

and

$$dX(x) = dX^*(x). \tag{2}$$

Condition (2) means X is a *gradient-like vector field* corresponding to a closed 1-form on $\mathbb{T}^n$ in the following sense: the closed 1-form ω is defined by

$$\omega(x)(v) = \langle X(x), v\rangle, \quad v \in T_x\mathbb{T}^n \cong \mathbb{R}^n.$$

By Mañé's Lagrangian

$$L^1(x,v) = \frac{1}{2}|v - X(x)|^2 = L^0(x,v) - \langle X(x), v\rangle. \qquad (3)$$

Denote by α_0 and α_1 the α-function of L^0 and L^1 respectively.

Lemma

If for the potential U of L^0, there exists a C^1 vector field X satisfying (1) and (2) and $c = \int_{\mathbb{T}^n} X(x)dx$, then

$$\alpha_0(c) = \alpha_1(0) = 0 \qquad (4)$$

It is well known that $L-\lambda$ has the same Euler-Lagrange equation as L does if the 1-form λ is closed. It is clear from (3), if $c = \int_{\mathbb{T}^n} X(x)dx$, then $\alpha_1(0) = \alpha_0(c) = 0$. $|c| \leqslant \int_{\mathbb{T}^n} \kappa(x)dx$ follows easily from the definition of the vector field X. If $c \neq 0$, then (4) implies $\alpha_0(rc) = 0$ for $0 \leqslant r \leqslant 1$ since $\alpha_0(0) = \min_{c\in\mathbb{R}^n} \alpha_0(c)$ and α-function is convex.

Theorem

If there exist k vector fields X_i on $\mathbb{T}^n$ independently, $1 \leqslant k \leqslant n$ such that X_i's satisfy (1) and (2) with $\int_{\mathbb{T}^n} X_i(x)dx \neq 0$, $i = 1, \ldots, k$, then the α-function has k-dimensional flat part near $c = 0$.

This is a direct consequence of Lemma.

Theorem

If the critical set E of U of the system $L(x, v) = \frac{1}{2}|v|^2 - U(x)$ does not contain a simple closed homotopically nontrivial smooth curve, the α-function has fully dimensional flat part near $c = 0$.

Example 1 when $n=1$, and $\max_{x\in\mathbb{T}^n} U(x)=0$, the flat part $|c|\leqslant\int_{\mathbb{T}^1}\kappa(x)dx$ of the α-function is well known, see e.g. the famous preprint of Lions-Papanicolaou-Varadhan. Let $V_\varepsilon(x)=U(x)-\varepsilon$ for $\varepsilon>0$, then

$$L^0_\varepsilon(x,v)-\langle X_\varepsilon(x),v\rangle=\frac{1}{2}|v|^2-V_\varepsilon(x)-\langle X_\varepsilon(x),v\rangle=L^1_\varepsilon(x,v),$$

where X_ε satisfies (1) and (2) for the potential V_ε. The existence of such a X_ε can be easily obtained by $X_\varepsilon=\sqrt{-V_\varepsilon}$. Denote by α^0_ε and α^1_ε the α-function of L^0_ε and L^1_ε respectively, then we have $\alpha^1_\varepsilon(0)=\alpha^0_\varepsilon(c_\varepsilon)=0$ by Lemma , where $c_\varepsilon=\int_{\mathbb{T}^1}X_\varepsilon(x)dx$. This implies $\alpha^0(c_\varepsilon)-\varepsilon=0$ for any $\varepsilon>0$. So $\alpha(c_0)=0$ by the continuity of c_ε with respect to ε, and $c_0\neq 0$ if $U\not\equiv 0$. Thus the α-function has flat part on $[0,c_0]$, and the case of $[-c_0,0]$ is similar by choosing $X_\varepsilon=-\sqrt{-V_\varepsilon}$.

Proof of Theorem.

If the critical set E of U does not contain a simple closed homotopically nontrivial smooth curve, then there exist n independent gradient-like vector fields $\{X_{i,\varepsilon}\}_{i=1}^{n}$ for $V_\varepsilon(x) = U(x) - \varepsilon$ as in Example 1. Apply the argument in Example 1 to each $X_{i,\varepsilon}$, we can see there exists $c_{0,i} \neq 0$ such that the α-function has flat part in the direction of $c_{0,i}$. The independency of $X_{i,\varepsilon}$ means the independency of such $c_{0,i}$'s, thus we get the conclusion.

Thank you.