Arnold diffusion via normally hyperbolic cylinders

Instability in Hamitonian systems
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Arnold diffusion

- We consider a close-to-integrable Hamiltonian system

$$H(\theta, p, t) = H_0(p) + \epsilon H_1(\theta, p, t), \theta \in \mathbb{T}^n, p \in \mathbb{R}^n, t \in \mathbb{T}.$$

- We say that the system exhibits Arnold diffusion if there exists $c > 0$ such that the following hold: for arbitrarily small ϵ, there exists an orbit $(\theta(t), p(t))$ and $T > 0$ with

$$|p(T) - p(0)| > c > 0.$$

- The first example of Arnold diffusion was constructed by Arnold in 1964.
- Conjecture(Arnold): There is Arnold diffusion for a “typical” system.

Resonances

- We say a vector $\omega \in \mathbb{R}^n$ is resonant if there exists $k \in \mathbb{Z}^n$ and $l \in Z$ such that $k \cdot \omega + l = 0$.
- Denote $\omega(p) = (\omega_1, \cdots, \omega_n)(p) = \partial_p H_0(p)$.

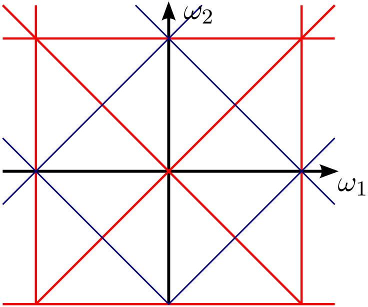

Figure: Resonances for $n = 2$

Averaging near a single resonance

- For simplicity, consider $n = 2$ and the resonance $\omega_1 = 0$.
- Consider the model example

$$H(\theta, p, t) = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \epsilon \cos\theta_1 + \epsilon H_2(\theta, p, t),$$

where $\int_{\mathbb{T}^2} H_2 d\theta_2 dt = 0$.

- There exists a symplectic change of coordinates, under which the Hamiltonian takes the form

$$H(\theta, p, t) = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \epsilon \cos\theta_1 + h.o.t.$$

- Consider ϵ fixed and the higher order terms as a perturbation. This inspires the definition of the *a priori* unstable system:

$$H(\theta, p, t) = h(p_1, p_2, \theta_1) + \delta P(p, \theta, t).$$

Prior work on Arnold diffusion

- A priori unstable system:
  - Geometric methods: Chierchia and Gallavotti '95, Treschev '04, Delshams, de la Llave and Seara '06 (Delshams and Huguet '09).
  - Variational methods: Bernard '08, Cheng and Yan '04, '09.
- A priori stable system: Mather 03 (announcement), various manuscripts by Mather.

A version of Arnold diffusion for a priori stable system

Theorem

Assume that the Hamiltonian is C^r with $r \geq 4$. For a "typical" ϵH_1 (with $\|H_1\|_{C^r} = 1$), there exists $l(H_1) > 0$, an orbit $(\theta, p)(t)$ of the the Hamiltonian system $H_\epsilon = H_0 + \epsilon H_1$ and $T > 0$ such that

$$|p(T) - p(0)| > l(H_1).$$

Remark

- *The "diffusion distance" depends only on the projection of the perturbation on the unit sphere of C^r functions.*
- *Here "typical" means Mather's cusp residue condition.*

The Arnold mechanism

- System:

$$H(\theta_1,\theta_2,p_1,p_2,t) = \frac{1}{2}p^2+\epsilon(\cos\theta_1-1)-\epsilon\mu(\cos\theta_1-1)\cos(\theta_2+t).$$

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- This may be viewed as an a priori unstable system as we can fix ϵ and let $\mu \to 0$.
- There exists a $3-$dimensional-normally hyperbolic invariant cylinder (NHIC) $\Lambda = \{\theta_1 = p_1 = 0\}$.
- Λ is foliated by invariant tori $T_c = \{p_2 = c_2, p_1 = 0, \theta_1 = 0\}$, where $c = (0, c_2)$.
- The stable and unstable manifold of T_c intersect transversally. In particular, T_c admits a homoclinic orbit that has isolated intersections with the section $\{\theta_1 = \frac{\pi}{2}\}$.

Arnold mechanism: picture

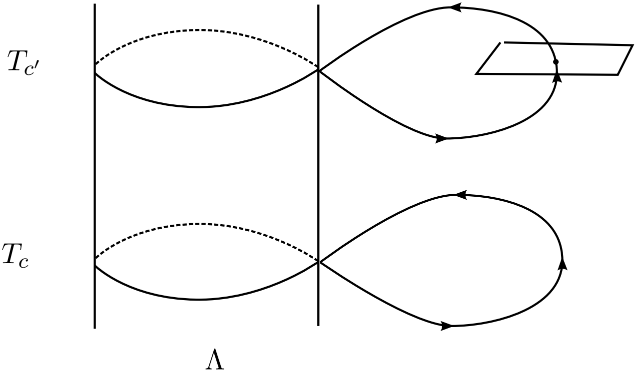


Figure: Arnold mechanism

Arnold mechanism: variational explanation

- Each T_c consists of orbits $\gamma : \mathbb{R} \to \mathbb{T}^n \times \mathbb{R}^n$ that minimize the action $\int_a^b L_c(\theta, \dot{\theta}, t)$, where

$$L_c = \frac{1}{2}(\dot{\theta} - c)^2 + \epsilon(1 - \cos\theta_1) - \epsilon\mu(1 - \cos\theta_1)\cos(\theta_2 + t).$$

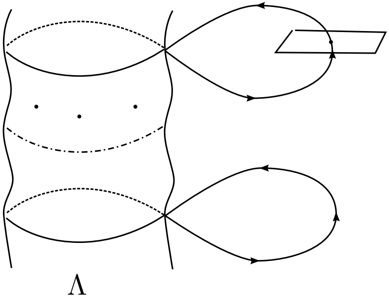
  We call the union of such minimal orbits the Mañe set $\mathcal{N}_c$.
- The homoclinic orbits to T_c are minimal orbits that makes at least one round in the θ_1 direction. We call this set the *lifted* Mañe set $\hat{\mathcal{N}}_c$, for they corresponds to the Mañe set of the double cover.
- If the lifted Mañe set $\hat{\mathcal{N}}_c$ is isolated, then for c' close to c, there is an orbit connecting $\mathcal{N}_c$ and $\mathcal{N}_{c'}$.
- Proof: Mather's method of changing Lagrangian.

NHIC for a priori unstable system

Consider

$$H(\theta_1, \theta_2, p_1, p_2, t) = \frac{1}{2}p^2 + (\cos\theta_1 - 1) + \mu P(\theta, p, t).$$

- There exists NHIC Λ close to $\theta_1 = p_1 = 0$.
- The Mañe set $\mathcal{N}_c$ are contained in Λ, but they are not necessarily tori. They could be:
  - Invariant tori.
  - Periodic orbits.
  - Cantori.



Anrold diffusion for a priori unstable system

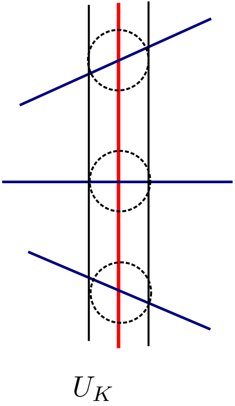
Theorem (Bernard, Cheng and Yan)

Assume that: 1. All $\mathcal{N}_c$ that is an invariant torus has irrational rotation number. 2. If $\mathcal{N}_c$ is an invariant torus, then $\hat{\mathcal{N}}_c$ is isolated. Then there is Arnold diffusion along the NHIC.

Theorem (Cheng and Yan)

For a residue set of perturbations, condition 2 is satisfied for all $\mathcal{N}_c$ simultaneously.

A priori stable system: normal form



- Consider $n = 2$ for simplicity. Let

$$H(\theta_1, \theta_2, p_1, p_2, t) = \frac{1}{2}p^2 + \epsilon H_1(\theta, p, t)$$

and consider the resonance $p_1 = 0$.

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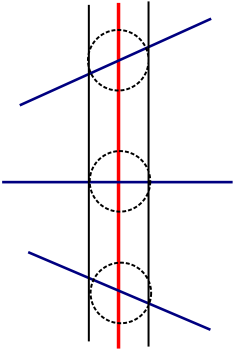

U_K

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and consider the resonance $p_1 = 0$.

- There exists a normal form

$$H = \frac{1}{2}p^2 + \epsilon Z(\theta_1, p) + \epsilon R(\theta, p, t),$$

where $\|R\|_{C^2} \leq \delta$ on the set $U_K = \{|p_1| \leq \epsilon^{\frac{1}{6}}, |p_2 k_2 + l| > \epsilon^{\frac{1}{6}}/K,$ for all $|k_2|, |l| \leq K\}$. K depend on δ.

- Diffusion distance depends on δ. Hence, it is important that δ does not go to zero as $\epsilon \to 0$.

Existence of crumpled cylinders

- For the normal form system with sufficiently small δ, generically, there exists finitely many NHIC's $\Lambda_j = \{(\theta_1, p_1) = X_j(\theta_2, p_2, t)\}$. We also have

$$\left\| \frac{\partial \theta_1}{\partial p_2} \right\| = O(\delta/\sqrt{\epsilon}), \quad \left\| \frac{\partial \theta_1}{\partial(\theta_2, t)} \right\| = O(\delta).$$

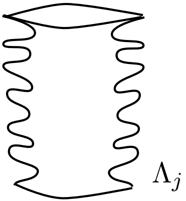

Figure: Crumpled cylinder

The properties of the Mañe set

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- Assume 1. All $\mathcal{N}_c$ that is an invariant torus has irrational rotation number. 2. If $\mathcal{N}_c$ is an invariant torus, then $\hat{\mathcal{N}}_c$ is isolated. Then there is Arnold diffusion along the NHIC.
- The genericity theorem of Cheng and Yan applies.
- It is also possible to jump from Λ_j to Λ_{j+1}.

Double resonance

Work-in-progress with V. Kaloshin:

- Study
$$H(\theta_1, \theta_2, p_1, p_2, t) = \frac{1}{2}p^2 + \epsilon H_1(\theta, p, t)$$
near $\{p_1 = p_2 = 0\}$.

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- Normal form:

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on $|p_1|, |p_2| \leq \sqrt{\epsilon}$. Rescale:

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$$H(\theta, I, t) = \frac{1}{2}I^2 + Z(\theta, 0) + \tilde{R}.$$
- Study of double resonance related to the 2 degrees of freedom mechanical system $H = K(I) - U(\theta)$.

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Mañe sets for the mechanical system

- By Maupertuis principle, the minimal orbits corresponds to minimal geodesics of the Jacobi metric $K(E+U)$, for $E > -\min U$.
- If $E = -\min U$, the minimal geodesics could be a concatenation of simple curves.
- Assume that $U(\theta_0) = \min U$, generically $(0, \theta_0)$ is a hyperbolic saddle with distinct eigenvalues. Minimal geodesics are homoclinic orbits to $(0, \theta_0)$.

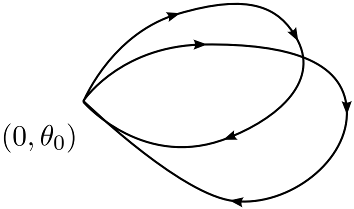

Figure: Homoclinics

Flower cylinder

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- Shil'nikov-Tureav '89 (Bolotin-Rabinowitz '01) Under some additional assumptions, for each E sufficiently close to 0, there exists an orbit shadowing $\Gamma_1 * \Gamma_2$.
- With some work, we can see that the family of periodic orbits form a normally hyperbolic cylinder with the "figure eight" as boundary.

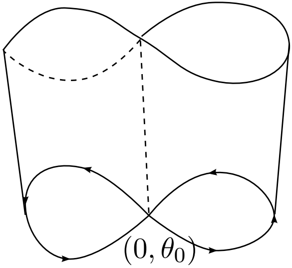

Figure: Flower cylinder